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BESOV SPACES ON LOCAL FIELDS

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This is a survey of the author's results on Besov spaces on non-archimedean local fields and their applications to Haar bases in usual Besov spaces. Proofs will appear in [3].

The main theorem of this paper, which establishes an isomorphism between Besov spaces on $F_2[[t]]$ and on R, was proved by the author during a stay at the Banach Center from March to May 1986. I use this opportunity to thank the Banach Center for its great hospitality. Furthermore, I thank Dr. W. Sickel, who directed my attention to the problem of comparing usual and Walsh–Besov spaces.

1. Notation

Let K be a locally compact non-discrete field (i.e., a field equipped with a locally compact non-discrete topology such that the field operations are continuous). We denote the Haar measure on the additive group of K by m_K. The modular function μ_K on K is defined by the property

$$m_K(kM) = \mu_K(k)\, m_K(M) \tag{1}$$

for every Borel-measurable subset $M \subset K$ and every $k \in K$. It is clear that

$$\mu_K(k_1 k_2) = \mu_K(k_1)\, \mu_K(k_2).$$

K satisfies exactly one of the following two properties:

(i) K is archimedean, i.e., $\mu_K(nx) \to \infty$ if $x \in K^\times = K - \{0\}$ and if the natural number n tends to infinity.

(ii) K is non-archimedean:

$$\mu_K(x+y) \leqslant \mu_K(x) \quad \text{if } \mu_K(y) \leqslant \mu_K(x). \tag{2}$$

An archimedean field K is isomorphic either to the field R of real numbers or to the field of complex numbers. In the sequel we consider only non-archimedean fields. Since confusion is impossible, we put for the sake of brevity $\mu_K(x) = x$.

By $\mathfrak{O} = \{x \mid |x| \leqslant 1\}$ we denote the maximal compact subring of K. It is a local ring with maximal ideal $\mathfrak{p} = \{x \mid |x| < 1\}$. The residue field $\mathfrak{k} = \mathfrak{O}/\mathfrak{p}$ is finite, we denote by Q the number of its elements. For every integer n we put

$$(3) \qquad \mathfrak{p}^n = \{x \mid |x| \leqslant Q^{-n}\}.$$

We have

$$(4) \qquad \mathfrak{p}^0 = \mathfrak{O}, \quad \mathfrak{p}^1 = \mathfrak{p}, \quad \mathfrak{p}^k \cdot \mathfrak{p}^l = \mathfrak{p}^{k+l}.$$

We normalize the Haar measure by $m_K(\mathfrak{O}) = 1$. Then

$$(5) \qquad m_K(\mathfrak{p}^n) = Q^{-n}.$$

Let Ψ be a unitary character of K^+ (the additive group of K) such that $\Psi|_{\mathfrak{O}}$ is trivial and $\Psi|_{\mathfrak{p}^{-1}}$ is non-trivial. We denote the Fourier transform on K by $Ff = \hat{f}$:

$$(6) \qquad (Ff)(x) = \int_K \Psi(xy) f(y)\, dy.$$

It extends to a unitary operator F: $L_2(K) \to L_2(K)$. Let S be the space of all locally constant functions with compact support on K. F maps S onto itself and can therefore be extended to a mapping on the space S' of all distributions on K.

Let E_n be the σ-field generated by $\mathfrak{p}^n$ and its translates. It is easy to see that the elements of S' can be identified with martingales for the sequence $\{E_n\}$.

The Fourier transform has the property

$$(7) \qquad f \text{ is } E_j\text{-measurable if and only if } \operatorname{supp} Ff \subseteq \mathfrak{p}^{-j}.$$

Let ω_j be the characteristic function on $\mathfrak{p}^j$. We have

$$(8) \qquad F\omega_j = Q^{-j} \omega_{-j}.$$

Finally, we mention the well-known property

$$(9) \qquad F(f * g) = (Ff) \cdot (Fg),$$

where $*$ denotes convolution on K.

The main references for the above facts are [6] and [9].

2. Besov spaces on K

Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. We introduce the sequence spaces

$$(10) \qquad L_p(l_q) = \{\{f_k\}_{k=0}^{\infty} \mid \|\{f_k\}|L_p(l_q)\| = \Big(\int_K \Big(\sum_{k=0}^{\infty} |f_k(x)|^q\Big)^{p/q} dm_K(x)\Big)^{1/p} < \infty\},$$

$$(11) \qquad l_q(L_p) = \{\{f_k\}_{k=0}^{\infty} \mid \{\|f_k|L_p\|\}_{k=0}^{\infty} \in l_q\}$$

(with an obvious modification in (10) if $\max(p, q) = \infty$).

Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and $-\infty < s < +\infty$. We define

$$(12) \qquad B^s_{pq}(K) = \{f \in S'(K) \mid \|f|B^s_{pq}\| = \|\{Q^{ks} F^{-1} \omega'_k Ff\}_{k=0}^{\infty}|l_q(L_p)\| < \infty\},$$

where

$$(13) \qquad \omega'_0 = \omega_0, \quad \omega'_j = \omega_{-j} - \omega_{1-j} \quad \text{if } j > 0.$$

$F^s_{pq}(K)$ with $p < \infty$ is defined by replacing $l_q(L_p)$ in (12) by $L_p(l_q)$, but this definition does not work for $p = \infty$. In this case we put for $1 < q \leqslant \infty$

$$(14) \qquad F^s_{\infty q} = \{f \mid f \text{ admits a representation } f = \sum_{j=0}^{\infty} F^{-1} \omega'_j Ff_j Q^{-js}$$
$$\text{with } \{f_j\}_{j=0}^{\infty} \in L_\infty(l_q)\}.$$

The spaces $B^s_{pq}(K)$ and $F^s_{pq}(K)$ are quasi-Banach spaces of distributions (or martingales). We have

$$(15) \qquad S(K) \subset B^s_{p:\min(p,q)}(K) \subseteq F^s_{pq}(K) \subseteq B^s_{p:\max(p,q)}(K) \subset S'(K).$$

Bounded E_j-measurable functions are pointwise multipliers for B^s_{pq} and F^s_{pq}.

We mention the following relations to classical function spaces on K, they are consequences of [1] and [4]:

(i) $F^0_{p2} = L_p$ for $1 < p < \infty$, and there exists a constant c_p with

$$(16) \qquad \|f|L_p(K)\| \leqslant c_p \|f|F^0_{p2}(K)\| \leqslant c_p^2 \|f|L_p(K)\| \quad \text{for every local field } K.$$

(ii) $F^0_{p2}(K)$ with $0 < p < \infty$ is a martingale Hardy space:

$$(17) \qquad \|f|F^0_{p2}(K)\| \sim \|\{Q^j \omega_j * f\}_{j=0}^{\infty}|L_p(L_\infty)\|,$$

the constants in (17) are independent of K if $p \geqslant 1$ and depend only on Q if $p < 1$.

(iii) $F^0_{\infty 2}$ is a martingale BMO space:

$$(18) \qquad \|f|F^0_{\infty 2}\| \sim \|f|L_\infty\| + \sup_{\substack{j>0 \\ x \in K}} Q^j \int_{\mathfrak{p}^j} |f(x+y) - \int_{\mathfrak{p}^j} Q^j f(x+z)\,dz|\,dy,$$

the constant is independent of K.

(iv) If $s > 0$, then

$$\|f|B^s_{\infty\infty}\| \sim \|f|F^s_{\infty\infty}\| \sim \|f|L_\infty\| + \left\|\left|\frac{f(x)-f(y)}{|x-y|^s}\right| L_\infty(K\times K)\right\|.$$

The Jackson norm for F^s_{pq} has the following form:

$$\|f|L_p\| + \inf\|\{Q^j(f-f_j)\}_{j=0}^{\infty}|L_p(l_q)\|, \tag{19}$$

where the infimum is taken over all sequences $\{f_j\}$ where f_j is an E_j-measurable function. We can prove that $B^s_{pq}(K)$ with $s > \max(1/p-1, 0)$ and $F^s_{pq}(K)$ with $s > \max(1/p-1, 1/q-1, 0)$ satisfy the Jackson theorem (i.e., the Jackson norm is an equivalent quasi-norm). If N: $C \to C$ satisfies the Lipschitz condition, then $f \to N(f)$ is a continuous mapping on all B- or F-spaces on K that satisfy the Jackson theorem.

Let $p' = 0$ for $p = 1$ and $p' = p/(p-1)$ if $1 < p < \infty$. The dual spaces of B^s_{pq} and F^s_{pq} with $1 \leqslant p$; $q < \infty$ are given by

$$(B^s_{pq}(K))' = B^{-s}_{p'q'}(K), \quad (F^s_{pq}(K))' = F^{-s}_{p'q'}(K). \tag{20}$$

Let $\tilde{p}' > p$ and $\tilde{s}-1/\tilde{p} = s-1/p$. Then

$$B^s_{pq}(K) \subset B^{\tilde{s}}_{\tilde{p}q}(K), \tag{21}$$

$$F^s_{pq}(K) \subset B^{\tilde{s}}_{\tilde{p}p}(K), \tag{22}$$

$$B^s_{p\tilde{p}}(K) \subset F^{\tilde{s}}_{\tilde{p}q}(K). \tag{23}$$

3. Application to Besov spaces on R

We recall the definition of Besov spaces on R (cf. [7]). Let

$$P_j = \begin{cases} \{\xi|\ |\xi| < 2\} & \text{if } j = 0, \\ \{\xi|\ 2^{j-1} < |\xi| < 2^{j+1}\} & \text{if } j = 1, 2, \ldots \end{cases}$$

There exist sequences $\{\psi_j\}_{j=0}^{\infty}$ of C^∞-functions on R with the following properties:

(i) $\operatorname{supp}\psi_j \subset P_j$,

(ii) $\sum_{j=0}^{\infty} \psi_j(\xi) = 1$,

(iii) $|(d/dt)^k \psi_j| \leqslant c_k 2^{-kj}$.

The spaces $B^s_{pq}(R)$ and $F^s_{pq}(R)$ are defined in a similar manner to the corresponding spaces on non-archimedean local fields K: The Fourier transform on K is to be replaced by the Fourier transform on R, and ω'_k must be replaced by ψ_j. The relations to classical spaces mentioned in Section 2 remain valid if martingale Hardy and BMO spaces are replaced by the usual ones. The only difference occurs in the case of difference quasi-norms: For the characterization of $B^s_{pq}(R)$ and $F^s_{pq}(R)$ with $s > 1$ by means of difference norms we must use higher differences. This is not necessary if spaces on non-archimedean local fields are considered.

Let F_2 be the unique Galois field with 2 elements. We consider the following local field K:

$\mathfrak{O} = F_2[[t]]$ is the ring of all formal power series with coefficients in F_2.

The quotient field K of $\mathfrak{O}$ is the field of all formal Laurent series $\sum\limits_{j \gg 0} a_j t^j$ with coefficients in F_2.

We consider the following continuous, surjective, and measure-preserving map Φ: $K \to R$:

$$\Phi\Big(\sum_{j \gg 0} a_j t^j\Big) = \sum a_j 2^{-j} \quad \text{if } a_j \in \{0, 1\}. \tag{24}$$

Our main result is:

THEOREM 1. *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $1/p - 1 < s < \min(1/p, 1)$. *In the case of the F-spaces we suppose $p < \infty$ or $q > 1$. Then the map Φ^* defined for $L_{1,\mathrm{loc}}$-functions f on R by $(\Phi^* f)(x) = f(\Phi(x))$ can be extended to an isomorphism between $B^s_{pq}(R)$ and $B^s_{pq}(K)$ and between $F^s_{pq}(R)$ and $F^s_{pq}(K)$.*

Let us recall that Φ identifies the characters of $\mathfrak{O}$ with the system of Walsh functions on $I = [0; 1]$. If X is one of the spaces B^s_{pq} and F^s_{pq}, then $X(I)$ consists of the restrictions of the elements of $X(R)$ to I, equipped with the quasi-norm

$$\|f \,|\, X(I)\| = \inf_{g|_I = f} \|g \,|\, X(R)\|.$$

Let $\{\psi_n\}_{n=0}^{\infty}$ be the orthogonal system of Walsh functions (cf. [6]). A formal Walsh series $\sum\limits_{j=0}^{\infty} a_j \psi_j$ can be identified with a martingale for acertain sequence of σ-fields on I. We define $F^s_{pq,\mathrm{Walsh}}$ as the space of all formal Walsh series for which the quasi-norm

$$|a_0| + \Big\| \Big\{ 2^{js} \sum_{k=2^j}^{2^{j+1}-1} a_j \psi_j \Big\} \,\Big|\, L_p(l_q) \Big\| \tag{25}$$

is finite. A similar definition is possible for B^s_{pq}. The mapping Φ identifies Walsh–Besov spaces with Besov spaces on $\mathfrak{O} = F_2[[t]]$. Theorem 1 implies

THEOREM 2. *If s, p, and q satisfy the assumptions of Theorem* 1, *then*

$$B^s_{pq,\mathrm{Walsh}} = B^s_{pq}(I) \quad \textit{and} \quad F^s_{pq,\mathrm{Walsh}} = F^s_{pq}(I).$$

Let χ_n be the nth Haar function (we use the notation of [7], 2.12.3). It is not hard to check that (25) does not change if we replace ψ_n by χ_{n+1} and a_n

by the $(n+1)$-th Fourier coefficient of $\sum_{j=0}^{\infty} a_j \psi_j$ with respect to $\{\chi_n\}$. But such an exchange considerably simplifies the norm expression:

THEOREM 3. *Let s, p, and q satisfy the assumptions of Theorem* 1. *Then the orthogonal system of Haar functions is an unconditional Schauder basis in* $F^s_{pq}(I)$, *and*

$$\left\| \sum_{n=1}^{\infty} a_n \chi_n \,|\, F^s_{pq}(I) \right\| \sim \| \{ n^s a_n \chi_n \} \,|\, L_p(l_q) \| .$$

In a similar manner we can generalize the B^s_{pq}-part of the results of [8] for Haar functions. Theorem 3 can be interpolated with the results of Oswald [5]:

THEOREM 4. *If* $0 < p < \infty$, $0 < q < \infty$, *and* $1/p - 1 < s < 1/p$, *then the system of Haar functions is a Schauder basis in* $F^s_{pq}(I)$.

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