COVERING THEOREMS FOR UNIQUENESS
AND EXTENDED UNIQUENESS SETS

BY

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Introduction. A covering theorem for a class of sets $C$ asserts that every set in $C$ can be covered by a countable union of sets in some (somehow simpler) class $C'$.

In the theory of sets of uniqueness on the unit circle $T$ the first result of this kind is Piatetski-Shapiro's theorem in [PS], which states that every closed set of uniqueness can be covered by countably many closed sets in the class $U_1'$, consisting of those closed sets $E \subseteq T$ for which there exists a sequence of functions in $A(T)$, vanishing on $E$, which converges to the function 1 in the weak*-topology.

Recently, attention has been focused on such covering results for uniqueness sets ($\mathcal{U}$-sets) as well as extended uniqueness sets ($\mathcal{U}_0$-sets) in an attempt to overcome the difficulty of characterizing these sets, revealed by the theorem of Solovay [S] (see also [KL] and Kaufman [Ka2]) that the sets

$$U = \mathcal{U} \cap K(T) \quad \text{and} \quad U_0 = \mathcal{U}_0 \cap K(T)$$

of closed $\mathcal{U}$- and $\mathcal{U}_0$-sets are coanalytic but not Borel subsets of the space $K(T)$ of closed subsets of $T$. Perhaps there is a simply characterizable (in particular, Borel) class of $U$- (resp. $U_0$-) sets such that every $U$- (resp. $U_0$-) set is a countable union of sets in this class.

In 1986, we proved such a covering result for $U_0$, namely that every closed set of extended uniqueness can be written as a countable union of sets in the so-called class $U_0'$, consisting of those closed sets $E$ for which there is $a > 0$ such that, for every probability measure $\mu$ on $E$,

$$\lim |\mu(n)| \geq a.$$

Our original proof of this result has not been published and used a good deal of descriptive set theory.

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Using our result and some more descriptive set theory, especially results from [KLW] as well as a strengthening of the Kaufman—Solovay Theorem due to Kaufman [Ka3], Debs and Saint Raymond [DStR] extended this to Borel and even analytic sets in \( \mathcal{U}_0 \): Every analytic \( \mathcal{U}_0 \)-set can be covered by countably many \( U_0 \)-sets. This gives a complete description of analytic \( \mathcal{U}_0 \)-sets and has several consequences in the theory of sets of uniqueness, like the solution of the Category Problem (see [B1], and also [KL]): Every (extended) uniqueness set with the property of Baire is of the first category in \( T \).

In the first part of this paper, we give a direct new proof of these two covering results, using only tools from functional analysis and a Baire category-type argument. This direct approach has also the advantage to adapt to the case of sets of uniqueness (as opposed to extended uniqueness), where the situation is more complicated. The point here is that although \( U_0 \) is a subclass of \( U \), \( U_1 \) is not a subclass of \( U \), by Körner’s Theorem on Helson sets of multiplicity [Kö]. In fact, in [DStR] Debs and Saint Raymond prove that there is no family \( B \subseteq U \) which is Borel and forms a basis for \( U \) in the sense that every \( U \)-set can be covered by countably many sets in \( B \) or, equivalently, \( U = B_\sigma \), which by definition is the \( \sigma \)-ideal of closed sets generated by \( B \). On the other hand, \( U_0 \) is Borel and a basis for \( U_0 \). (We prove also in [KL] that if \( B \subseteq U \) is such that \( U = B_\sigma \), then there is an extension \( B_1 \) of \( B \) with \( B = B_1 \cap U \) and \( (B_1)_\sigma = (U_1)_\sigma \), i.e., \( B_1 \) generates the \( \sigma \)-ideal generated by \( U_1 \), so that in some sense the only way to improve the Piatetski-Shapiro basis \( U_1 \cap U \) for \( U \) is to get a better basis for \( (U_1)_\sigma \) itself, not \( U \).

Despite these negative results, which prevent the descriptive set theoretic approach of [DStR] to be used in the context of \( U \)- and \( U_1 \)-sets, the methods developed in Section 1 allow us to extend, in the second part of the paper, Piatetski-Shapiro’s result to analytic \( \mathcal{U} \)-sets: Every analytic \( \mathcal{U} \)-set, in fact every analytic set which contains no closed set of multiplicity, can be covered by countably many sets in \( U_1 \). We show moreover that this result is best possible: The \( \sigma \)-ideal \( (U_1)_\sigma \equiv U_1^* \) is the least \( \sigma \)-ideal of closed sets \( I \) with the property that every \( G_\delta \) set which contains no closed set of multiplicity can be covered by countably many sets in \( I \). Whereas \( U_1^* \) was defined indirectly by weakening the definition of \( U^* \), the subclass of closed sets of uniqueness consisting of those \( E \) for which there is a sequence \( f_n \in A(T) \) of functions vanishing in an open neighborhood (nbhd) of \( E \) which weak* -converges to 1, the above result gives surprisingly an “intrinsic” characterization of the \( \sigma \)-ideal \( U_1^* \) in terms of \( U \). Another intrinsic characterization of \( U_1^* \) in terms of \( U \) has been recently obtained by Dougherty and Kechris [DK], who showed that \( U_1^* \) consists of exactly those closed sets within which, for some \( h \), Hausdorff \( \mu_h \)-measure 0 implies uniqueness.

Finally, we relate the class \( U_1^* \) with questions of harmonic synthesis of pseudomeasures. The class \( U_1^* \) can be characterized as consisting of those closed sets which support no pseudofunction satisfying synthesis or, equivalently,
those closed sets for which uniqueness holds for all trigonometric series $\sum S(n)e^{inx}$ with $S$ (viewed as a pseudomeasure) satisfying synthesis. Extending this definition to arbitrary subsets of $T$ produces a new type of “uniqueness” set $U^*_1$ lying strictly between $U$ and $U_0$ ($U \not\subseteq U^*_1 \not\subseteq U_0$). We show moreover that every set which contains no closed set of multiplicity is also contained in $U^*_1$, i.e., it is a set of “uniqueness” as far as trigonometric series satisfying synthesis are concerned. (Whether it is actually a uniqueness set is the so-called Interior Problem—see [B1] and [KL]—and remains open.) Combining this with earlier results of Kechris and Louveau [KL] and independently Debs and Saint Raymond [DStR], one has a further corollary on the Union Problem for Borel $U$-sets. The union of countably many $G_{\delta_0}$ $U$-sets is of uniqueness for trigonometric series satisfying synthesis. So a possible counterexample to the Union Problem for $G_\delta$ sets has to incorporate a construction of non-synthesis sets (Malliavin’s Theorem, see [KS] or [GMcG]); in fact, even more, the existence of sets in $U_1' - U$ (Körner’s Theorem [Kö]).

In the sequel we use standard terminology and notation in harmonic analysis; see, e.g., [KS], [GMcG] or [KL].

1. The case of $U_0$-sets. For every closed set $E \subseteq T$, let us denote, respectively, by $M(E)$, $M^+(E)$, $\text{PROB}(E)$ the spaces of (complex) measures, positive measures and probability measures supported by $E$.

A measure $\mu \in M(T)$ is a Rajchman measure if
\[
\hat{\mu}(n) \to 0, \quad \text{i.e.,} \quad R(\mu) = \lim |\hat{\mu}(n)| = 0.
\]

We denote by $\mathcal{R}$ the set of Rajchman probability measures and for each $\varepsilon > 0$ by $\mathcal{R}_\varepsilon$ the set
\[
\mathcal{R}_\varepsilon = \{ \mu \in \text{PROB}(T): R(\mu) < \varepsilon \},
\]
so that
\[
\mathcal{R} = \bigcap_{\varepsilon > 0} \mathcal{R}_\varepsilon.
\]

A set $P \subseteq T$ is an extended uniqueness set (a $U_0$-set) if, for every $\mu \in \mathcal{R}$, $\mu(P) = 0$. Otherwise, $P$ is called of restricted multiplicity (an $\mathcal{M}_0$-set). As is well-known, if $P$ is analytic, this is equivalent to
\[
\mu(P) < 1 \quad \text{for all } \mu \in \mathcal{R}
\]
or to
\[
\forall \mu \in M(T) \left( \sum \hat{\mu}(n)e^{int} = 0, \forall t \notin P \Rightarrow \mu = 0 \right).
\]

In particular, an analytic set $P$ is in $U_0$ iff every closed subset $E$ of $P$ is in $U_0$. We denote by $K(T)$ the space of closed subsets of $T$ and we let
\[
U_0 = U_0 \cap K(T)
\]
be the family of closed $U_0$-sets. Let also

\[ M_0 = \mathcal{M}_0 \cap K(T). \]

Then for $E \in K(T)$

\[ E \in U_0 \iff \text{PROB}(E) \cap \mathcal{R} = \emptyset. \]

We define a subclass $U'_0$ of $U_0$ by

\[ E \in U'_0 \iff \exists \varepsilon > 0 (\text{PROB}(E) \cap \mathcal{R}_\varepsilon = \emptyset). \]

The family $U'_0$ is a Borel (in fact $G_\delta$) subset of $K(T)$, equipped with the standard Hausdorff metric under which it is compact. On the other hand, $U_0$ is co-analytic but not Borel ([S], [Ka2]; see also [KL]).

1.1. Theorem. The class $U_0$ is the $\sigma$-ideal generated by $U'_0$, i.e. every closed set of extended uniqueness is a countable union of sets in $U'_0$.

Proof. Let $E \in K(T)$ be a set which cannot be covered by countably many sets in $U'_0$. We want to show that $E \notin U_0$. Let $F \subseteq E$ be defined by

\[ x \in F \iff x \in E \& \forall V [V \text{ (open) nbhd of } x \Rightarrow E \cap V \text{ cannot be covered by countably many sets in } U'_0]. \]

The set $F$ is closed and non-empty by the hypothesis on $E$. Moreover, for any open $V$,

\[ F \cap V \neq \emptyset \Rightarrow \overline{F \cap V} \notin U'_0. \]

We show now that

\[ \mathcal{R}(F) = \mathcal{R} \cap \text{PROB}(F) \]

is non-empty, a contradiction.

First we claim that for each $\varepsilon > 0$

\[ \mathcal{R}_\varepsilon(F) = \mathcal{R}_\varepsilon \cap \text{PROB}(F) \]

is dense in $\text{PROB}(F)$ equipped with the usual $w^*$-topology for which it is compact and metrizable. To see this, note that $\mathcal{R}_\varepsilon(F)$ is convex. Hence it is enough to show that the Dirac measures $\delta_x, x \in F$, are in $\overline{\mathcal{R}_\varepsilon(F)}^{w^*}$. Let $V_n$ be a sequence of nbhds of $x$ with $\text{diam}(V_n) \to 0$ and (as $\overline{V_n \cap F} \notin U'_0$) let

\[ \mu_n \in \text{PROB}(V_n \cap F) \]

be in $\mathcal{R}_\varepsilon(F)$. Clearly,

\[ \mu_n \overset{w^*}{\to} \delta_x \]

and we are done.

We construct now inductively a sequence $\{\mu_n\}$ in $\text{PROB}(F)$ and an increasing sequence $\{N_n\}$ of positive integers such that
\[ (*) \text{ If } i \leq n \text{ and } N_i \leq |k| \leq N_{i+1}, \text{ then } |\hat{\mu}_n(k)| < 2^{-i-1}. \]

Then if \( \mu \) is a w*-limit of a subsequence of \( \{\mu_n\} \), clearly
\[ |\hat{\mu}(k)| \leq 2^{-i-1} \text{ for } |k| \geq N_i; \]

hence \( \mu \in \mathcal{R}(F) \).

To construct \( \{\mu_n\}, \{N_n\} \) we choose \( \mu_n \) to satisfy \((*)\) for \( i \leq n-1 \) and
\[ (**) \quad \forall |k| \geq N_n, |\hat{\mu}_n(k)| \leq 2^{-n-4}. \]

For \( n = 0 \), pick \( \mu_0 \in \mathcal{R}_\varepsilon(f) \) with \( \varepsilon = 2^{-4} \), and then \( N_0 \) so that
\[ |\hat{\mu}_0(k)| \leq 2^{-4} \text{ for } |k| \geq N_0. \]

Suppose now that \( \mu_0, \ldots, \mu_n, N_0, \ldots, N_n \) have been defined satisfying \((*)\) and \((**)\). Let \( m \) be any number \( \geq N_n \). There are a measure \( \mu(m) \) and an integer \( \varphi(m) > m \) such that
\[ (i) \quad \mu(m) \text{ satisfies } (*) \text{ for } i \leq n-1; \]
\[ (ii) \quad |\hat{\mu}(m)(k)| \leq 2^{-n-3} \text{ for } N_n \leq k < m; \]
\[ (iii) \quad |\mu(m)(k)| \leq 2^{-n-5} \text{ for } |k| \geq \varphi(m). \]

To see this, note that the measure \( \mu_n \) satisfies \((i), (ii)\), so by the density of \( \mathcal{R}_\varepsilon(F) \), with \( \varepsilon = 2^{-n-5} \), we can find \( \mu(m) \in \mathcal{R}_\varepsilon(F) \) satisfying \((i), (ii)\). Then choose \( \varphi(m) \) to make \((iii)\) true.

Define then a sequence \( \{v_j\} \) in PROB\((f)\) and \( \{m_j\} \) by
\[ v_0 = \mu(N_n), \quad m_0 = \varphi(N_n) \]
and
\[ v_{j+1} = \mu(m_j), \quad m_{j+1} = \varphi(m_j). \]

Let for each \( k \)
\[ \theta_k = \frac{1}{k+1} \sum_{j=0}^{k} v_j. \]

Then \( \theta_k \) satisfies \((*)\) for \( i \leq n-1 \) and
\[ |\hat{\theta}_k(m)| \leq 2^{-n-5} \text{ for } |m| \geq m_k. \]

If now \( N_n \leq |m| < m_k \), there is at most one \( j \) (the one for which \( m_j \leq |m| < m_{j+1} \)) such that \( |\hat{v}(m)| \geq 2^{-n-3} \); hence
\[ |\hat{\theta}_k(m)| \leq \frac{k \cdot 2^{-n-3} + 1}{k+1}. \]

Thus if we choose \( k \) large enough and let \( \mu_{n+1} = \theta_k, N_{n+1} = m_k \), we also get \((*)\) for \( i = n \) and \((**)\). This completes the proof.

This result leads to some kind of description of the \( \sigma \)-ideal of closed extended uniqueness sets, similar to the classical case of countable closed sets: Define
\[ M_0^\mathcal{B} = \{ E \in \text{K(T)} : \forall \text{ open } V \ (V \cap E \neq \emptyset \Rightarrow \overline{V \cap E} \notin U_0\} \}. \]
which, by the preceding theorem, is equal to

\[ \{ E \in K(T) : \forall \text{ open } V \ (V \cap E \neq \emptyset \Rightarrow \overline{V \cap E} \notin U_0) \}. \]

Then any closed set \( E \) can be uniquely decomposed into a kernel \( E^* \), the largest \( M_0^b \) set contained in \( E \), and \( E - E^* \) which is a countable union of \( U_0 \)-sets. And \( E^* \) can be obtained by a derivation process, similar to the Cantor–Bendixson one: for each closed set \( F \), let

\[ d(F) = \{ x \in F : \forall \text{ open } V (x \in V \Rightarrow \overline{V \cap F} \notin U_0) \}. \]

Define inductively \( E^{(a)}_0 \) by

\[ E^{(0)}_0 = E, \quad E^{(a+1)}_0 = d(E^{(a)}_0) \]

and

\[ E^{(\lambda)}_0 = \bigcap_{a < \lambda} E^{(a)}_0 \quad \text{for limit } \lambda. \]

Then, for some \( a < \omega_1 \), \( E^{(a+1)}_0 = E^{(a)}_0 = E^* \). And as \( E^* = \emptyset \) iff \( E \in U_0 \), this gives a countable "intrinsic" testing process for membership in \( U_0 \).

Note that although \( U_0 \) is complicated, i.e., co-analytic but not Borel, the class \( M_0^b \) (which is easily the same as the class of supports of Rajchman measures) is Borel, in fact \( F_{\sigma \delta} \). This can be used to show for example that the perfect symmetric sets \( E_{1,2,\ldots} \) of varying ratios of dissection, which are in \( U_0 \), form a Borel (\( G_{\sigma \delta} \)) set in \( K(T) \), suggesting perhaps the possibility of an "explicit" characterization of these sets (see [KL]).

We go now to the case of analytic sets in \( \mathcal{U}_0 \), i.e., the Debs–Saint Raymond Theorem. For simplicity let us consider first the case of \( G_\delta \) sets.

1.2. Theorem (Debs and Saint Raymond [DStR]). Let \( H \) be a \( G_\delta \) set in \( \mathcal{U}_0 \). Then \( H \) can be covered by countably many sets in \( U_0 \) (and hence in \( U_0 \) by Theorem 1.1).

Equivalently, if \( H \) is a \( G_\delta \) set which is dense in a non-empty \( M_0^b \)-set \( E \), then \( H \) is not a \( \mathcal{U}_0 \)-set.

Proof. The first assertion implies the second by the Baire Category Theorem and the fact that \( U_0 \)-sets are nowhere dense in \( M_0^b \)-sets. Conversely, if \( H \) is a \( G_\delta \) \( \mathcal{U}_0 \)-set and cannot be covered by countably many \( U_0 \)-sets, the set

\[ H' = \{ x \in H : \forall \text{ open } V (x \in V \Rightarrow V \cap H \text{ cannot be covered} \} \]

by countably many sets in \( U_0) \}

is \( G_\delta \), non-empty and \( E = H' \) is in \( M_0^b \), so that the second statement fails.

To prove the second assertion let

\[ H = \bigcap_n V_n, \]

\( V_n \) open dense in \( E \). Since \( E \in M_0^b \), we infer that \( \mathcal{R}(E) \) is \( w^* \)-dense in PROB\( \mathcal{E} \) (as in the proof of Theorem 1.1). Let

\[ \mathcal{R}(V_n) = \mathcal{R}(E) \cap \{ \mu : \text{ supp}(\mu) \subseteq V_n \}. \]
Since $V_n$ is dense in $E$, $\mathcal{R}(V_n)$ is $w^*$-dense in PROB($E$), so $\mathcal{R}(V_n)$ is dense in $\text{PF} \cap \text{PROB}(E) (= \mathcal{R}(E))$ for the weak topology of $\text{PF}$ (= the space of pseudofunctions), hence for the norm topology of $\text{PF}$ by Mazur's Theorem, since $\mathcal{R}(V_n)$ is convex.

Let $\mu_0 \in \mathcal{R}(E)$. As $\mathcal{R}(V_0)$ is dense in $\mathcal{R}(E)$, we can find $W_1$ open in $E$ with

$$W_1 \subseteq V_0, \quad \mu_1 \in \mathcal{R}(W_1) \quad \text{and} \quad \|\mu_1 - \mu_0\|_{\text{PF}} \leq 2^{-1}.$$ \n
Now $\bar{W}_1 \in M_0$ and $V_1 \cap W_1$ is open dense in it, so we can find $W_2$ open in $E$ with

$$W_2 \subseteq V_1 \cap W_1, \quad \mu_2 \in \mathcal{R}(W_2) \quad \text{and} \quad \|\mu_1 - \mu_2\|_{\text{PF}} \leq 2^{-2},$$ \n
etc. So we construct a Cauchy sequence in $\text{PF}$ of probability measures $\mu_n$ and a decreasing sequence $W_n$ of open sets in $E$ with

$$\bigcap_n W_n = \bigcap_n \bar{W}_n \subseteq H \quad \text{and} \quad \mu_n \in \mathcal{R}(\bar{W}_n).$$ \n
So $\mu_n \to \mu \in \mathcal{R}$, $\mu \neq 0, \text{ supp } (\mu) \subseteq \bigcap_n \bar{W}_n \subseteq H$, hence $H \notin \mathcal{U}_0$.

1.3. Corollary (Debs and Saint Raymond [DStR]). Let $E \in M_0$. If $A \subseteq E$ has the property of Baire in $E$ and $A \in \mathcal{U}_0$, then $A$ is of the first category in $E$. In particular, every $A \subseteq T$ with the property of Baire which is in $\mathcal{U}_0$ is of the first category.

Proof. Since $A$ has the property of Baire in $E$, $A = H \cup P$ with $H \in G_\delta$ and $P$ meager in $E$. Since $H \in \mathcal{U}_0$, $H$ can be covered by countably many (closed) $U_0$-sets, which, since $E \in M_0$, are nowhere dense in $E$. Thus $H$ is meager in $E$, and so is $A$.

1.4. Corollary (Menshov, see [B2]). There is a (closed) $M_0$-set of (Lebesgue) measure 0.

Proof. Let $G \subseteq T$ be a dense $G_\delta$ set of measure 0. Then $G \notin \mathcal{U}_0$, so there is a closed $E \subseteq G$, $E \in M_0$.

1.5. Corollary (Ivashev-Musatov [IM], Kaufman [Ka1]). Let $E \in M_0$. Let $h \colon [0, \infty) \to [0, \infty)$ be non-decreasing with $h(0+) = 0$, $h(t) > 0$ for $t > 0$. Then there is a closed set $F \subseteq E$, $F \in M_0$ of $h$-Hausdorff measure 0.

Proof. We can assume that $E \in M_0^\circ$, $E \neq \emptyset$. Let $\{x_n\}$ be dense in $E$ and choose, for each $n$, open intervals $I_n^m$ with

$$x_m \in I_n^m \quad \text{and} \quad \sum_m h(|I_n^m|) \leq 1/n.$$ \n
Then

$$H = \bigcap_n (E \cap \bigcup_m I_n^m)$$ \n
is dense $G_\delta$ in $E$ and of $h$-Hausdorff measure 0. But $H \notin \mathcal{U}_0$, hence $H$ contains $F \in M_0$ which is still of $h$-Hausdorff measure 0.

1.6. Corollary (Kechris and Louveau [KL]). Let $E \in M_0$ and let $\gamma$ be
a continuous capacity on $T$ (i.e., a capacity such that $\gamma(K \cup \{x\}) = \gamma(K)$ for all $K \in K(T)$, $x \in T$). Then there is a closed $F \subseteq E$ in $M_0$ with $\gamma(F) = 0$.

Proof. As before, we may assume that $E \in M_0^g$ and it is enough to construct a $G_\delta$ set $H$ dense in $E$ with $\gamma(H) = 0$. For that we construct as before open intervals $I_n^m$ with

$$\gamma(I_n^m \cup \ldots \cup I_n^m) < 1/n$$

for all $m$, using the continuity of $\gamma$.

1.7. Corollary (Lyons [L]). The set $W^*(\{q^k\})$ of non-normal numbers to the base $q \geq 2$ is in $M_0$. 

Proof. $W^*(\{q^k\})$ is the set of $x \in [0, 2\pi]$ for which the sequence $q^kx \pmod{2\pi}$ is not uniformly distributed, i.e., by Weyl's Criterion

$$\lim_{N} \frac{1}{N} \sum_{k=1}^{N} \exp\{iq^kmx\} \neq 0 \quad \text{for some } m \in \mathbb{Z}, \ m \neq 0.$$

Let

$$P = \left\{ x \in [0, 2\pi]: \lim_{N} \left| \frac{1}{N} \sum_{k=1}^{N} \exp\{iq^kx\} \right| = 1 \right\}.$$ 

The set $P$ is comeager in $T$ and $P \subseteq W^*(\{q^k\})$.

We prove now the extension of Theorem 1.2 to analytic sets.

1.8. Theorem (Debs and Saint Raymond [DSR]). Let $A$ be an analytic set in $\mathcal{U}_0$. Then $A$ can be covered by countably many sets in $U_0$ (and thus in $U_0'$).

Proof. Assume this fails and let $H \subseteq T \times 2^N$ be $G_\delta$ with $A = \pi(H)$, where $\pi$ denotes projection on $T$. Let $U_0^{ext}$ be the family of subsets of $T$ which can be covered by countably many sets in $U_0$. Let

$$H' = \{ x \in H : \forall \text{ open } V \text{ in } T \times 2^N \ (x \in V \Rightarrow \pi(V \cap H) \notin U_0^{ext}) \}.$$

Then $H' \in G_\delta$ and, since $A \notin U_0^{ext}$, we have $H' \neq \emptyset$. Moreover, $H'$ satisfies

$$\forall \text{ open } V \ (V \cap H' \neq \emptyset \Rightarrow \exists \pi(V \cap H') \notin U_0).$$

For $F$ closed in $T \times 2^N$, let us say that $F$ is “in $M_0^g$” if, for all non-empty open $V$ in $F$, $\pi\overline{V} \in M_0$. Thus $E = H'$ is “in $M_0^g$”, and so is any $\overline{V}$ for $V$ open non-empty in $E$. Note now that if $F$ is “in $M_0^g$” and $V \subseteq F$ is dense open in $F$, then

$$\mathcal{R}^*(F) = \{ \mu \in \mathcal{R}(\pi F) : \exists K \subseteq V \ (K \text{ closed and } \text{supp} \ (\mu) \subseteq \pi(K)) \}$$

is, as in the proof of Theorem 1.2, $w^*$-dense in PROB$(\pi(F))$ and convex, hence norm-dense in $\mathcal{R}(\pi(F))$. Thus the same argument as in Theorem 1.2 allows us to build a measure $\mu \in \mathcal{R}(\pi(E))$ and a closed set $K \subseteq H'$ with $\text{supp}(\mu) \subseteq \pi(H')$. Then $\pi(H') \notin U_0$, so $A \notin \mathcal{U}_0$.

The argument behind Theorems 1.2 and 1.8 looks very much like a Baire category argument. In fact, a category-type largeness property of the class of Rajchman measures can be extracted from it.
Let us say that for \( E \subseteq K(T) \) a set \( X \subseteq \text{PROB}(E) \) is almost comeager if for any non-empty open set \( V \subseteq \text{PROB}(E) \) and any sequence \( \{ V_n \} \) of open dense in \( V \) convex sets in \( \text{PROB}(E) \) we have
\[
X \cap \left( \bigcap_n V_n \right) \neq \emptyset.
\]
The set of Rajchman probability measures on \( T \) is meager in \( \text{PROB}(T) \). But we have

1.9. Theorem. Let \( E \subseteq K(T) \) be non-empty in \( M_0^\circ \). Then
\[
\mathcal{R}(E) = \mathcal{R} \cap \text{PROB}(E)
\]
is almost comeager in \( \text{PROB}(E) \).

Proof. Fix \( \emptyset \neq V \subseteq \text{PROB}(E) \), \( V \) convex open. Let also \( G \subseteq V \) be convex, open and dense in \( V \), and therefore in \( Q = \overline{V}^* \). By the proof of Theorem 1.1, \( \mathcal{R} \cap \text{PROB}(E) \) is dense in \( \text{PROB}(E) \), so \( \mathcal{R}_G = \mathcal{R} \cap G \) is dense in \( Q \) and of course convex. Now \( \mathcal{R}_Q = \mathcal{R} \cap Q \) is a norm-closed subset of PF and
\[
\mathcal{R}_G^* \cap \text{PF} = Q \cap \text{PF} = R_Q.
\]
Since the weak-closure \( \mathcal{R}_G^* \cap \text{PF} \) of \( \mathcal{R}_G \) in PF is the same as its norm-closure, it follows that \( \mathcal{R}_G \) is norm-dense in \( \mathcal{R}_Q \), and since \( \mathcal{R}_G = G \cap \mathcal{R}_Q \), it is clearly open in \( \mathcal{R}_Q \) with its strong topology.

To summarize: For every non-empty open convex \( V \subseteq \text{PROB}(E) \) and every open convex \( G \subseteq V \) which is dense in \( V \), \( \mathcal{R}_G \) is open dense in \( \mathcal{R}_Q \) in the norm-topology. So if \( V \neq \emptyset \) is open in \( \text{PROB}(E) \) and \( V_n \) are open dense in \( V \) and convex, then assuming without loss of generality that \( V \) itself is convex (as it surely contains a non-empty open convex subset) we have, by the Baire Category Theorem in \( \mathcal{R}_Q \), which being closed in PF is Polish,
\[
\bigcap_n \mathcal{R}_{V_n} = \left( \bigcap_n V_n \right) \cap \mathcal{R}_Q \neq \emptyset
\]
and we are done.

Note that Theorem 1.9 also easily implies Theorem 1.2: If the \( V_n \)'s are decreasing open and dense in \( E, E \subseteq M_0^\circ, E \neq \emptyset \), then
\[
V^*_n = \{ \mu \in \text{PROB}(E) : \mu(V_n) > \frac{1}{2} \}
\]
is dense open and convex in \( \text{PROB}(E) \), so Theorem 1.9 gives a Rajchman probability measure \( \mu \) with
\[
\mu \in \bigcap_n V^*_n, \quad \text{i.e.,} \quad \mu(\bigcap_n V_n) \geq \frac{1}{2},
\]
so that
\[
\bigcap_n V_n \notin \mathcal{U}_0.
\]

2. Results on \( U \) and \( U_1 \). A set \( P \subseteq T \) is a set of uniqueness if every trigonometric series converging to 0 off \( P \) is identically 0. Denote by \( \mathcal{U} \) the class
of such sets and by $U = \mathcal{U} \cap K(T)$ the class of closed $\mathcal{U}$-sets. These can be also characterized as the closed sets $E \subseteq T$ for which the ideal

$$J(E) = \{ f \in A(T) : f \text{ vanishes in a nbhd of } E \}$$

is $w^*$-dense in $A(T)$ for the topology of duality with PF. We also let

$$U^\text{int} = \{ P \subseteq T : \forall E \in K(T) (E \subseteq P \Rightarrow E \in U) \}$$

so that $\mathcal{U} \subseteq U^\text{int} \not\subseteq \mathcal{U}_0$. Finally, $\mathcal{M}, \mathcal{M}, M$ are the classes of sets of multiplicity and closed sets of multiplicity, respectively.

Let, for $E \in K(T)$,

$$I(E) = \{ f \in A(T) : f = 0 \text{ on } E \}.$$ 

A set $E \in K(T)$ is a $U_1$-set if $I(E)$ is $w^*$-dense in $A(T)$, and a $U'_1$-set if $I(E)$ is $w^*$-sequentially dense in $A(T)$, i.e., for some sequence $\{ f_n \}$ in $I(E)$,

$$f_n \overset{w^*}{\to} 1.$$ 

We will also let

$$U_1^\sigma = (U_1)_\sigma = (U'_1)_\sigma$$

be the $\sigma$-ideal of closed sets generated by $U_1$ or, equivalently, $U'_1$ by Piatetski-Shapiro’s Theorem in [PS]. Finally, let

$$U_1^{\text{ext}} = (U'_1)^{\text{ext}} = (U^\uparrow)^{\text{ext}}$$

be the $\sigma$-ideal of sets generated by $U_1$ or $U'_1$, i.e.,

$$U_1^{\text{ext}} = \{ P \subseteq T : \exists \{ E_n \} \in U_1 \ (P \subseteq \bigcup_n E_n) \}.$$ 

For a closed set $E$, we also let

$$\text{PM}(E) = J(E)^\perp = \{ S \in \text{PM} : S \text{ is supported by } E \},$$

where PM is the space of pseudomeasures and

$$N(E) = I(E)^\perp = \{ S \in \text{PM} : \forall f \in I(E) \ (\langle f, S \rangle = 0) \}.$$ 

Clearly, $M(E) \subseteq N(E) \subseteq \text{PM}(E)$ and $M(E)$ is $w^*$-dense in $N(E)$ for the topology of duality with $A(T)$.

Piatetski-Shapiro’s Theorem in [PS] implies, in particular, that $U \subseteq (U'_1)_\sigma$. Using the ideas of the first section we can extend this to $\Sigma^1_1$-sets.

**2.1. Theorem.** Let $P \subseteq T$ be $\Sigma^1_1$. If $P \in U^\text{int}$ (in particular, if $P \in \mathcal{U}$), then $P$ can be covered by countably many sets in $U'_1$, i.e., $P \in U_1^{\text{ext}} = (U'_1)^{\text{ext}}$.

We will prove this result in the case $P \in G_\sigma$, the general case being handled by a projection argument, as in Theorem 1.8.

Let us denote by $M^\sigma$ the class of $E \in K(T)$ such that, for every non-empty open set $V$ in $E$, $\overline{V} \not\subseteq U_1$. As usual, Theorem 2.1 is reduced to the following equivalent:
2.2. **Theorem.** If \( E \in M_1 \) and \( H \subseteq E \) is \( G_\delta \) dense in \( E \), then \( H \) contains a set in \( M \) (i.e., \( H \notin U^{\text{im}} \)).

We will need the following standard lemma:

2.3. **Lemma.** Let \( x \in T \), \( \{ V_n \} \) open nbhds of \( x \) with \( \text{diam}(V_n) \to 0 \). Let \( \{ S_n \} \) be pseudomeasures supported by \( \overline{V}_n \) with

\[
S_n(0) = 1 \quad \text{and} \quad \sup_n \| S_n \|_{PM} < \infty.
\]

Then \( S_n \overset{w^*}{\to} \delta_x \).

**Proof.** Let \( f \in A(T) \) and \( \varepsilon > 0 \) be given. Let \( \tilde{f}(t) = f(t) - f(x) \). Then, since \( \tilde{f}(x) = 0 \), we can find a \( g \in A(T) \) with \( \| g \|_A < \varepsilon \) and \( g = \tilde{f} \) in a nbhd of \( x \), hence in a nbhd of \( \overline{V}_n \) for all large enough \( n \). As \( S_n \in \text{PM}(\overline{V}_n) \), it follows that for all large enough \( n \)

\[
| \langle \tilde{f}, S_n \rangle | = | \langle g, S_n \rangle | \leq K \varepsilon \quad \text{with} \quad K = \sup_n \| S_n \|_{PM}.
\]

Since \( \langle \tilde{f}, S_n \rangle = \langle f, S_n \rangle - \langle f, \delta_x \rangle \), we are done.

Let \( V \) be open in \( E \in K(T) \). We define

\[ V = \{ S \in \text{PF}: \exists K \in K(T) \ [ K \subseteq V \& S \in N(K) ] \}. \]

2.4. **Lemma.** Suppose that \( E \in M_1 \) and \( V \subseteq E \) is open dense in \( E \). Then \( V \) is \( w^* \)-dense in \( N(E) \).

**Proof.** It is enough to show that \( \overline{V}^{w^*} \) contains the measures with finite support in \( E \). Since \( V \) is a subspace, it is enough to consider only Dirac measures \( \delta_x, x \in E \), and since \( V \) is dense in \( E \), we can assume that \( x \in V \). Let \( V_n \) be open nbhds of \( x \) in \( E \) with \( \text{diam}(V_n) \to 0 \) and \( \overline{V}_n \subseteq V \). As \( E \in M_1 \), there is

\[ S_n \in N(\overline{V}_n) \cap \text{PF} \quad \text{with} \quad S_n(0) = 1 \quad \text{and} \quad \| S_n \|_{PM} < 2. \]

By Lemma 2.3,

\[ S_n \overset{w^*}{\to} \delta_x \]

and we are done.

In particular, for \( E, V \) as in the lemma, \( V \) is dense in \( N(E) \cap \text{PF} \) for the weak-topology of \( \text{PF} \), hence in the norm-topology by Mazur's Theorem. We can proceed now exactly as in the proof of Theorem 1.2.

**Remark.** Colella [C] has also independently shown that if \( E \in M_1 \) and \( E_n \) is an increasing sequence of closed subsets of \( E \) with \( E_n \in M_1 \) and \( \bigcup_n E_n \) dense in \( E \), then for any \( S \in \text{PF} \cap N(E) \) there are

\[ S_n \in \text{PF} \cap N(E_n) \quad \text{with} \quad \| S_n - S \|_{PM} \to 0. \]

We present now some applications.

Combining Theorem 2.2 with a result of Debs and Saint Raymond in
we obtain first the following characterization of $M^*_1$ and $U^*_1$ in terms of $U$:

2.5. COROLLARY. Let $E \in K(T)$. Then the following are equivalent:

(i) $E \in M^*_1$.

(ii) Every non-meager $G_\delta$ subset of $E$ is not in $U^{\text{int}}$.

In particular, $U^*_1$ can be characterized as the least $\sigma$-ideal of closed sets $I$ such that any $G_\delta$ set in $U^{\text{int}}$ can be covered by countably many sets in $I$.

Proof. (i) $\Rightarrow$ (ii) is a restatement of Theorem 2.2: If $H$ is non-meager $G_\delta$ in $E$, then $\bar{H}$ contains a non-empty set $V$ open in $E$, and $\bar{V}$ is in $M^*_1$ too, so that $H \cap \bar{V}$, and hence $H$, is not in $U^{\text{int}}$.

(ii) $\Rightarrow$ (i). In [DStR], the authors show that if $F \in U'_1$, then $F$ contains a dense $G_\delta$ in $U^{\text{int}}$. Now, if $E \notin M^*_1$, then there is $\emptyset \neq V$ open in $E$ with $F = \bar{V} \in U'_1$, so that there is $G$ dense $G_\delta$ in $F$ with $G \subseteq U^{\text{int}}$. Since $G$ is a non-meager $G_\delta$ in $E$, we are done.

For the last assertion, first note that $U^*_1$ satisfies this property by Theorem 2.1. If $I$ is now any $\sigma$-ideal with this property and $E \in U^*_1 - I$, towards a contradiction, there is $F \neq \emptyset$, $F \in U'_1$ which is $I$-perfect, i.e., $\bar{V} \notin I$ for all non-empty $V$ open in $F$. Then subsets of $F$ in $I$ are nowhere dense in $F$, and so all $G_\delta$ sets in $U^{\text{int}}$ which are contained in $F$ are meager in $F$. This contradicts the result of Debs and Saint Raymond.

We also have analogs of Corollaries 1.4–1.6 with the same proofs.

2.6. COROLLARY. Let $E \notin U^*_1$. Let $\gamma$ be a continuous capacity or a Hausdorff measure. Then there is $F \subseteq E$, $F \in M$, with $\gamma(F) = 0$.

It is not known if in the conclusion of this corollary one can get $F \notin U^*_1$. However, by a recent result of Dougherty and Kechris [DK] one cannot assume only that $E \in M$. In fact, they show that the following is true, thereby giving a different intrinsic characterization of $U^*_1$ in terms of $U$: For $E \in K(T)$, $E \in U^*_1$ $\Rightarrow$ there is a Hausdorff measure $\mu_h$ on $T$ such that, for all $F \subseteq E$, $F \in K(T)$, $\mu_h(F) = 0$ $\Rightarrow$ $F \subseteq U$.

Our next applications relate $U^*_1$ to the Union and Interior Problems for sets of uniqueness, and problems of synthesis of pseudmeasures. (The Union Problem is the question of whether finite or countable unions of Borel sets in $\mathcal{U}$ are in $\mathcal{U}$ and the Interior Problem is the question of whether $(G_\delta)$ $\mathcal{M}$-sets contain $M$-sets. The Union Problem is open even for two $G_\delta$ sets and the Interior Problem is open even for $G_\delta$ $\mathcal{M}$-sets.)

Let $\sum S(n)e^{inx}$ be a non-zero trigonometric series which converges to 0 almost everywhere, so that in particular $S \in \text{PF}$. Let

$$\text{RN}_S = \{x \in T: \sum_{|n| \leq N} S(n)e^{inx} \text{ is unbounded}\}$$

be the so-called reduced nucleus of $S$. It is an old problem of Bary [B1] to find
out if $\text{RN}_S$ is an $\mathcal{M}$-set. Recall that a pseudomeasure $S$ satisfies synthesis if
\[ f \in I(\text{supp}(S)) \Rightarrow  \langle f, S \rangle = 0. \]

2.7. Corollary. The reduced nucleus of any $S$ which satisfies synthesis is an $\mathcal{M}$-set, in fact it contains an $M$-set.

Proof. If $S \in \text{PF}$ satisfies synthesis, then $\text{supp}(S) \subseteq M_1^r$. But $\text{RN}_S$ is a dense $G_\delta$ in $\text{supp}(S)$, so we are done by Theorem 2.2.

2.8. Corollary. Let $G$ be a set in $U^{\text{int}}$. Then there is no non-zero trigonometric series $\sum S(n)e^{inx}$ with $S$ satisfying synthesis such that
\[ \sum S(n)e^{inx} = 0, \forall x \notin G. \]
In particular, if $G_n \in U^{\text{int}}$, $G_n \subseteq \Sigma_3^0 (\equiv G_\delta)$, then there is no such $S$ with
\[ \sum S(n)e^{inx} = 0, \forall x \notin \bigcup_n G_n. \]

Proof. If such an $S$ exists, then $\text{RN}_S \subseteq G$ and $\text{RN}_S \notin U^{\text{int}}$ by Corollary 2.7, so $G \notin U^{\text{int}}$. For the second conclusion use the fact that $U^{\text{int}}$ is closed under countable unions of $G_\delta$ sets (see [KL] and [DStR]).

This corollary says that any possible negative solution of the Union Problem for $G_\delta$ sets must incorporate a construction of non-synthesis sets, in fact sets in $U_1 - U$ (Körner's Theorem [Kö]).

The concept of pseudomeasures satisfying synthesis is actually very closely connected to the concept of $U_1^*\text{-set}$. To see this we need the following lemma:

2.9. Lemma. Let $E \in K(T)$. Then the following are equivalent:
(i) $E \in M_1^r$.
(ii) $E = \text{supp}(S)$, where $S \in \text{PF}$ satisfies synthesis.

Proof. The direction (ii) $\Rightarrow$ (i) is easy. So, conversely, let $E \in M_1^r$. Let $\{V_n\}$enumerate all basic open sets with $V_n \cap E \neq \emptyset$. Then for each $n$ there is a non-zero pseudofunction in $N(E)$ with support contained in $V_n$. We will define inductively on $n$ a pseudofunction $S_n \in N(E)$ with $\|S_n\|_{PM} = 1$, a function $f_n \in A$ with support contained in $V_n$ and a sequence of positive numbers $\varepsilon_{n+1}^{(n)}, \varepsilon_{n+2}^{(n)}, \ldots$ as follows: For $n = 0$, let $S_0 \in \text{PF}$ be such that
\[ \|S_0\|_{PM} = 1, \quad \text{supp}(S_0) \subseteq V_0 \quad \text{and} \quad S_0 \in N(E). \]
Let $f_0 \in A$ be such that
\[ \text{supp}(f_0) \subseteq V_0 \quad \text{and} \quad \langle f_0, S_0 \rangle \neq 0. \]
Choose positive $\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots$ so that
\[ \|f_0\|_A \left( \sum_{m \geq 1} \varepsilon_m^{(0)} \right) < |\langle f_0, S_0 \rangle|. \]
Then for every \( S_1', S_2', \ldots \) with \( \|S_m'\|_{PM} = 1 \) and every \( 0 < \delta_m \leq \varepsilon_m^{(0)} \) we have
\[
\langle f_0, \sum_{m \geq 1} \delta_m S'_m \rangle \neq 0.
\]
Let now \( \delta_1 = \varepsilon_1^{(0)} \) and find \( S_1 \in PF, f \in A \) such that
\[
\|S_1\|_{PM} = 1, \quad \text{supp}(S_1) \subseteq V_1, \quad S_1 \in N(E), \quad \text{supp}(f_1) \subseteq V_2
\]
and
\[
\langle f_1, S_0 + \delta_1 S_1 \rangle \neq 0.
\]
Choose positive \( \varepsilon_2^{(1)}, \varepsilon_3^{(1)}, \ldots \) so that
\[
\|f_1\|_A (\sum_{m \geq 2} \varepsilon_m^{(1)}) < |\langle f_1, S_0 + \delta_1 S_1 \rangle|.
\]
Then for every \( S_2', S_3', \ldots \) with \( \|S_m'\|_{PM} = 1 \) and every \( 0 < \delta_m \leq \varepsilon_m^{(1)} \) we have
\[
\langle f_1, S_0 + \delta_1 S_1 + \sum_{m \geq 2} \delta_m S'_m \rangle \neq 0.
\]
Let now \( \delta_2 = \min\{\varepsilon_2^{(0)}, \varepsilon_2^{(1)}\} \) and proceed as before with \( V_2, \) etc. Letting now, in general,
\[
\delta_n = \min\{\varepsilon_n^{(0)}, \varepsilon_n^{(1)}, \ldots, \varepsilon_n^{(n-1)}\},
\]
we have \( \sum \delta_n \leq \sum \varepsilon_n^{(0)} < \infty \), and thus if
\[
S = S_0 + \sum_{n \geq 1} \delta_n S_n,
\]
then \( S \in PF \) and \( S \in N(E) \). Finally, \( \text{supp}(S) = E \), since \( \langle f_n, S \rangle \neq 0 \) for all \( n \).

We have now

2.10. COROLLARY. Let \( E \in K(T) \). Then the following are equivalent:

(i) \( E \in U_1^\dagger \).

(ii) \( E \) supports no non-zero pseudofunction which satisfies synthesis.

(iii) Every trigonometric series \( \sum S(n) e^{inx} \) for which \( S \) satisfies synthesis and
\[
\sum S(n) e^{inx} = 0, \quad \forall x \notin E,
\]
is identically 0.

Thus \( U_1^\dagger \) appears as an interesting analog of \( U \) and \( U_0 \). Moreover, \( U_1^\dagger \) has a natural extension to arbitrary subsets of \( T \), namely if
\[
P \in \mathcal{U}_1^\dagger \iff \text{every trigonometric series } \sum S(n) e^{inx} \text{ for which } S \text{ satisfies synthesis and } \sum S(n) e^{inx} = 0, \forall x \notin E, \text{ is identically 0,}
\]
then \( \mathcal{U}_1^\dagger \cap K(T) = U_1^\dagger \). Note that, by Corollary 2.8, \( U_{1}^{\text{int}} \subseteq \mathcal{U}_1^\dagger \).

A very interesting question is whether the conclusion of Theorem 2.1 goes through only under the hypothesis that \( P \in (U_1^\dagger)^{\text{int}} \), thus obtaining a full extension of the Piatetski-Shapiro Theorem \( (U_1 \subseteq (U_1)_* \). If so, then one would have for \( \Sigma_1^1 \)-sets
\[
\mathcal{U}_1^\dagger = (U_1^\dagger)^{\text{int}} = (U_1^\dagger)^{\text{ext}}.
\]
We finish by using the preceding ideas to provide a new proof of the Piatetski-Shapiro Theorem itself.

**2.11. Theorem** (Piatetski-Shapiro [PS]). *Every* $U_1$-set *is a countable union of* $U_1$-sets.

**Proof.** As usual it is enough to prove that if $\emptyset \neq E \subset K(T)$ is such that for all non-empty open $V$ in $E$, $\bar{V} \notin U_1$, then $E \in M_1$. For $\varepsilon > 0$, let

$$N^*(E) = \{ S \in N(E) : \lim_{i \to \infty} |S(k)| < \varepsilon \}.$$

First note that $N^*(E)$ is $w^*$-dense in $N(E)$: To see this let

$$\mu = \sum_{i=1}^k \alpha_i \delta_{x_i}, \quad x_i \in E,$$

be a finite support measure. Let $x_i \in V_n'$, $V_n'$ open in $E$, $\text{diam}(V_n') \to 0$. Choose then $S_n' \in N(\bar{V}_n')$ with $R(S_n') < (\varepsilon/k) \max_i |\alpha_i|$, $\| S_n' \|_{PM} < 2$, $S_n'(0) = 1$.

Then

$$S_n' \overset{w^*}{\to} \delta_{x_i},$$

and so if $S_n = \sum \alpha_i S_n'$, we have

$$R(S_n) < \varepsilon \quad \text{and} \quad S_n \overset{w^*}{\to} \mu.$$

Next we need a lemma.

**2.12. Lemma.** Suppose that

$$C \subseteq \text{PM}^* = \{ S \in \text{PM} : \lim_{i \to \infty} |S(k)| < \varepsilon \}$$

is convex and $T \in C^{w^*}$. Then there is a sequence $T_n \in C$,

$$T_n \overset{w^*}{\to} T \quad \text{with} \quad \|T_n\|_{PM} < \|T\|_{PM} + \varepsilon.$$

**Proof.** Let

$$C^{(1)} = \{ S \in \text{PM} : S \text{ is a } w^*-\text{limit of a sequence in } C \}.$$

If we can prove the lemma for $T \in C^{(1)}$, we are done, since this implies that $C^{(1)}$ is closed under $w^*$-limits of sequences, hence $C^{(1)} = \bar{C}^{w^*}$. So let $T \in C^{(1)}$ and let $S_n \in C$ be a sequence $w^*$-converging to $T$, with say $\| S_n \|_{PM} \leq M$. Given any $N$ and $\delta < \varepsilon$ we will find $S \in C$ with

$$|T(m) - S(m)| < \delta \quad \text{for} \quad |m| \leq N \quad \text{and} \quad \| S \|_{PM} < \| T \|_{PM} + \varepsilon.$$

For this note that for $p \geq N$ there are $S \in C$ with $\| S \|_{PM} \leq M$ and $q \geq p$ such that

$$|T(m) - S(m)| < \delta \quad \text{for} \quad |m| \leq p \quad \text{and} \quad |S(m)| < \varepsilon \quad \text{for} \quad |m| \geq q.$$

By iterating and averaging as in the proof of Theorem 1.1 we are done.
Returning to the proof of Theorem 2.11, we see that for any $S \in N(E)$ with $\|S\|_{PM} < \alpha$ there is a sequence $T_n \in N^\epsilon(E)$ with

$$T_n \rightharpoonup^* S \quad \text{and} \quad \|T_n\|_{PM} < \alpha + \epsilon.$$ 

We define then inductively $S_1, S_2, \ldots$ in $N(E)$, and $0 < n_1 < n_2 < \ldots$, such that

$$\|S_1\|_{PM} < 1, \quad \|S_k\|_{PM} < 1 + \sum_{i \leq k} 2^{-i} \quad \text{for } k \geq 2,$$

$$|S_k(n)| < 2^{-i} \quad \text{for } |n| > n_i \quad \text{and} \quad k \geq i \quad \text{and} \quad S_k(0) > \frac{1}{2}.$$ 

Then if $\{S_k\}$ is a $w^*$-converging subsequence, say with limit $S$, we have $S \in N(E)$, $S \neq \emptyset$, $S \in PF$, so that $E \in M_1$.

To define the $S_i, n_i$, choose first $S_1 \in N^{1/2}(E)$ with $\|S_1\|_{PM} < 1$ and $S_1(0) > \frac{1}{2}$, and then $n_1$ with $|S(n)| < \frac{1}{2}$ for $|n| \geq n_1$. By using the above, for each $m \geq n_1$ there are $m' \geq m$ and $S \in N(E)$ with

$$\|S\|_{PM} \leq 1 + \frac{1}{2}, \quad |S(k)| < \frac{1}{2} \quad \text{for } n_1 < |k| < m,$$

$$|S(k)| < \frac{1}{2} \quad \text{for } |k| > m' \quad \text{and} \quad S(0) > \frac{1}{2}.$$ 

Iterating and averaging as before, we get $S_2$, and then $n_2$, etc.

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