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Determination of differential concomitants of the first class of a pair of covariant vectors in a two-dimensional space

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The problem mentioned in the title was solved by S. Gołąb in paper [4] with the aid of an analytic method involving the theory of partial differential equations and thus of some assumptions of the regularity of the unknown functions appearing in the functional equation of the problem. It is possible to resolve this equation by the method of J. Makai [15]. This method, however, is artificial and can be applied if the solution is known a priori.

We shall now present a new method [17] which shows that the result can be derived in a shorter way. This method, moreover, is more general; it leads to the differential concomitants of different kinds (not only covariant or contravariant vectors) and above all it has the advantage of not making any assumptions of regularity.

Consider a pair of fields of covariant vectors $u_i(\xi^1, \xi^2)$, $v_i(\xi^1, \xi^2)$, $i = 1, 2$, defined in a neighbourhood of a point $\underset{0}{\xi} = (\underset{0}{\xi^1}, \underset{0}{\xi^2})$. We assume that at the point $\underset{0}{\xi}$

$$\det(u_i, v_i) \neq 0 \tag{1}$$

and that there exist derivatives $\partial_j u_i, \partial_j v_i$.

Definition 1 (see [1]). A geometric object σ^λ with the transformation formula

$$\sigma^{\lambda'} = F^{\lambda'}(\sigma^\lambda; A^i_{i'}, A^i_{i'j'}) \qquad (\lambda' = 1', \ldots, m') \tag{2}$$

is called the *differential concomitant of the first class of the object* (u_i, v_i) if there exist functions φ^λ satisfying the functional equations

$$\varphi^{\lambda'}(u_{i'}, v_{i'}, \partial_{j'}u_{i'}, \partial_{j'}v_{i'}) = F^{\lambda'}\big(\varphi^\lambda(u_i, v_i, \partial_j u_i, \partial_j v_i); A^i_{i'}, A^i_{i'j'}\big) \tag{3}$$

where

$$A^i_{i'} = \frac{\partial \xi^i}{\partial \xi^{i'}}, \quad A^i_{i'j'} = \frac{\partial^2 \xi^i}{\partial \xi^{i'} \partial \xi^{j'}}, \quad \det(A^i_{i'}) \neq 0 \tag{4}$$

and

$$\begin{aligned} u_{i'} &= A^i_{i'} u_i , \\ v_{i'} &= A^i_{i'} v_i , \\ \partial_{j'} u_{i'} &= A^i_{i'j'} u_i + A^i_{i'} A^j_{j'} \partial_i u_j , \\ \partial_{j'} v_{i'} &= A^i_{i'j'} v_i + A^i_{i'} A^j_{j'} \partial_i v_j . \end{aligned} \tag{5}$$

In this paper we shall give the general solution of (3) with given $F^{\lambda'}$ and unknown $\varphi^{\lambda'}$. We shall use the method described in [17], where we are given a theory of functional equations of the type

$$\varphi\big(f(x, \vartheta)\big) = F\big(\varphi(x), \vartheta\big), \qquad x \in X,\ \vartheta \in \Theta \tag{6}$$

with given f, F and unknown φ. A function φ which satisfies (6) is called the general homogeneous function with respect to the given functions f and F. We see that our functional equations (3) are of type (6). [17] gives a necessary and sufficient condition for the existence of solutions of (6), which, in our case, may be translated into the following statement: A necessary and sufficient condition for the existence of solutions of (3) is that object σ^λ (with the transformation formula (2)) be of the type $[m, 2, k], k \leqslant 2$ (see [1]).

Let us introduce the set

$$X = \{(u_i, v_i, \partial_j u_i, \partial_j v_i)\colon \det(u_i, v_i) \neq 0\} . \tag{7}$$

In the set X formulas (5) define an abstract object with the fibre X and the transformation formula (5) (see [10], [11], [12]). If in (5), instead of the variables $(u_i, v_i, \partial_j u_i, \partial_j v_i)$, we put an arbitrary point x of the set X and treat the variables $A^i_{i'}, A^i_{i'j'}$ as parameters, then we shall obtain the parametric equations of a transitive fibre, corresponding to the point x, of the above-mentioned geometric object. Every transitive fibre of our geometric object represents a 10-dimensional surface in a 12-dimensional space.

The method of solving (3) will be based on the following observation: If the value of function $\varphi = (\varphi^\lambda)$, which is a solution of (3), at a point $x_0 \in X$ is known, then its values at the points belonging to the transitive fibre determined by that point x_0 are uniquely defined by the right-hand-side expressions of equations (3).

Definition 2 (see [17]). A set $\underset{0}{X} \subset X$ which has exactly one point in common with each transitive fibre of a geometric object with the transformation formula (5), is called the *generator of the fibre X*.

One of the generators $\overset{\circ}{X}$ of the fibre X is given by the parametric equations

$$
\begin{aligned}
u_i &= \delta_i^1 ,\\
v_i &= \delta_i^2 ,\\
\partial_j u_i &= \delta_i^1 \delta_j^2 t_1 ,\\
\partial_j v_i &= \delta_i^1 \delta_j^2 t_2 , \qquad (t_1, t_2) \in R^2
\end{aligned} \tag{8}
$$

which results from the following.

The transformation

$$
\begin{aligned}
u_{i'} &= A_{i'}^i \delta_i^1 ,\\
v_{i'} &= A_{i'}^i \delta_i^2 ,\\
\partial_{j'} u_{i'} &= A_{i'j'}^i \delta_i^1 + A_{i'}^i A_{j'}^j \delta_i^1 \delta_j^2 t_1 ,\\
\partial_{j'} v_{i'} &= A_{i'j'}^i \delta_i^2 + A_{i'}^i A_{j'}^j \delta_i^1 \delta_j^2 t_2 ,
\end{aligned} \tag{9}
$$

obtained by putting in (5), instead of the variables $u_i, v_i, \partial_j u_i, \partial_j v_i$, the right-hand-side expressions of (8) is a one-to-one transformation of the set X onto itself (see (4)).

The inverse transformation to (9) is the following:

$$
\begin{aligned}
t_i &= \frac{\partial_{1'} u_{2'} - \partial_{2'} u_{1'}}{\Delta} \delta_i^1 + \frac{\partial_{2'} v_{1'} - \partial_{1'} v_{2'}}{\Delta} \delta_i^2 ,\\
A_{i'}^i &= u_{i'} \delta_1^i + v_{i'} \delta_2^i ,\\
A_{i'j'}^i &= (\partial_{i'} u_{j'} - u_{i'} v_{j'} t_1) \delta_1^i + (\partial_{i'} v_{j'} - u_{i'} v_{j'} t_2) \delta_2^i ,
\end{aligned} \tag{10}
$$

where

$$
\Delta = \det(u_{i'}, v_{i'}) . \tag{11}
$$

Now in (3), instead of the variables $u_i, v_i, \partial_j u_i, \partial_j v_i$, we put the right-hand-side expressions of (8) and obtain

$$
\varphi^{\lambda'}(u_{i'}, v_{i'}, \partial_{j'} u_{i'}, \partial_{j'} v_{i'}) = F^{\lambda'}\big(\varphi^{\lambda}(\delta_i^1, \delta_i^2\, \delta_i^1 \delta_j^2 t_1, \delta_i^1 \delta_j^2 t_2); A_{i'}^i, A_{i'j'}^i\big) . \tag{12}
$$

If we use the notation

$$
\Phi^{\lambda}(t_1, t_2) = \varphi^{\lambda}(\delta_i^1, \delta_i^2, \delta_i^1 \delta_j^2 t_1, \delta_i^1 \delta_j^2 t_2) ,
$$

then we receive from (12)

$$
\varphi^{\lambda'}(u_{i'}, v_{i'}, \partial_{j'} u_{i'}, \partial_{j'} v_{i'}) = F^{\lambda'}\big(\Phi^{\lambda}(t_1, t_2); A_{i'}^i, A_{i'j'}^i\big) . \tag{13}
$$

If in (13), instead of the variables t_i and $A_{i'}^i, A_{i'j'}^i$, we put the right-hand-side expressions of (10), then we get the general solution of the given equations (3), functions Φ^{λ} defined on R^2 being arbitrary.

Remark 1. If we take the following specification for the transformation formula (2)

$$\sigma_{k'} = A^k_{k'} \sigma_k, \quad k = 1, 2,$$

then using formulas (13), (10) and (11) we get

(14) $\varphi_{k'}(u_{i'}, v_{i'}, \partial_{j'} u_{i'}, \partial_{j'} v_{i'}) = (u_{k'} \delta^k_1 + v_{k'} \delta^k_2) \tilde{\Phi}_k(u_{i'}, v_{i'}, \partial_{j'} u_{i'}, \partial_{j'} v_{i'})$,

where

$$\tilde{\Phi}_k = \Phi_k \left(\frac{\partial_{1'} u_{2'} - \partial_{2'} u_{1'}}{u_{1'} v_{2'} - u_{2'} v_{1'}}, \frac{\partial_{2'} v_{1'} - \partial_{1'} v_{2'}}{u_{1'} v_{2'} - u_{2'} v_{1'}} \right).$$

Formulas (14) are identical with that obtained by S. Gołąb.

Remark 2. In [2], [3], [5], [6], [7], [8], [9], [13], [16] the present authors study the problems of the existence and determination of geometric objects of types $[m, n, k]$. Using the results contained in these works and formulas (13), (10), (11), we may immediately write the formulas for the other differential concomitants of the first class of the above mentioned object (u_i, v_i).

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