ANNALES
POLONICI MATHEMATICI
XLVI (1985)

Some remarks on univalence criteria for functions meromorphic in the exterior of the unit disc

by Zdzisław Lewandowski (Lublin)

Franciszek Leja in memoriam

Abstract. The paper contains the following main result.

Theorem 2. *Let $E^0 = \{\zeta \in \bar{C}\colon |\zeta| > 1\}$. Suppose that $g(\zeta) = \zeta + b_0 + b_1\zeta^{-1}$ and $h(\zeta) = 1 + c_2\zeta^{-2} + \ldots$ are regular in $E^0 \setminus \{\infty\}$ and E^0, respectively, with $g'(\zeta) \neq 0$ for $\zeta \in E^0$. Let for some numbers $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbf{R}$, $\frac{1}{2} < a \leqslant \alpha$ the inequality*

$$\left|\frac{\zeta g'(\zeta)}{g(\zeta)h(\zeta)} - \frac{as}{\alpha}\right| \leqslant \frac{a|s|}{\alpha}$$

holds in E^0. If the inequality

$$\left||\zeta|^{2\varkappa}\frac{\zeta g'(\zeta)}{g(\zeta)h(\zeta)} + (1-|\zeta|^{2\varkappa})\left[\frac{\zeta g'(\zeta)}{g(\zeta)} + s\frac{\zeta h'(\zeta)}{h(\zeta)}\right] - \frac{as}{\alpha}\right| \leqslant \frac{a|s|}{\alpha}$$

holds for $\zeta \in E^0$ and $\varkappa = a/\alpha$, then g is univalent in E^0.

The paper contains also some corollaries about sufficient conditions of univalence.

1. Introduction. The purpose of the paper is to establish a theorem representing a univalence criterion of a meromorphic function g (Theorem 2). In Section 2 we will give a proof of this theorem. It is an application of a result of Pommerenke (Theorem 1) to a parametrized family of functions generated by g. Section 3 contains two corollaries which extend an earlier result of Ruscheweyh.

We begin with some notation: $\mathbf{C}$ is the complex plane; $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$; $\bar{A}$ is the closure of the set $A \subset \bar{\mathbf{C}}$; $\mathbf{R} = (-\infty, \infty)$; $K(S, R)$ is an open disc of centre S and radius R; $E_r = \{z\colon |z| < r\}$, $r \in (0, 1]$, $E_1 = E$; $E_r^0 = \{\zeta\colon |\zeta| > r \geqslant 1\}$, $E_1^0 = E^0$; Σ^0 is the class of functions g that are regular in $E^0 \setminus \{\infty\}$ and such that $g(\zeta) = \zeta + b_0 + b_1\zeta^{-1} + \ldots$ for $\zeta \in E^0$; G^0 is the class of functions h that are regular in E^0 and such that $h(\infty) = 1$ and $h(\zeta) \neq 0$ for $\zeta \in E^0$.

We will now cite the above-mentioned

THEOREM 1. *Let* $0 < r_0 \leqslant 1$ *and let* $f(z, t) = a_1(t)z + \ldots$, $a_1(t) \neq 0$, *be regular in* E_{r_0} *for each fixed* $t \in [0, \infty)$ *and locally absolutely continuous in* $[0, \infty)$, *local uniformly with respect to* E_{r_0}.

For almost all $t \in [0, \infty)$ *suppose that*

$$\frac{\partial}{\partial t} f(z, t) = zf'(z, t)p(z, t), \qquad z \in E_{r_0},$$

where $p(z, t)$ *is regular in* E *and satisfies the condition* $\operatorname{Re} p(z, t) > 0$, $z \in E$. *If* $|a_1(t)| \to \infty$ *as* $t \to \infty$ *and if* $\{f(z, t)/a_1(t)\}$ *forms a normal family in* E_{r_0}, *then, for each* $t \in [0, \infty)$, $f(z, t)$ *has a regular and univalent extension to the whole disc* E.

This result of Pommerenke was formulated in his paper as Corollary 3, [1].

2. Before the formulation of our main result we will make a simple but useful remark:

Remark 1. Let $D \subset C$ be a convex domain whose boundary does not contain any rectilinear segment. Suppose that $A \in \bar{D}$ and $w(\lambda_0) = \lambda_0 A + (1-\lambda_0)B \in \bar{D}$, where A, B, λ_0 are fixed points with $\lambda_0 \geqslant 1$ and $A \neq B$. Then, for each $\lambda \in (1, \lambda_0)$, $w(\lambda) \in D$.

We will now give the proof of

THEOREM 2. *Suppose that* $g(\zeta) = \zeta + b_0 + b_1\zeta^{-1} + \ldots \in \Sigma^0$, $g'(\zeta) \neq 0$ *for* $\zeta \in E^0$, $h(\zeta) = 1 + c_2\zeta^{-2} + \ldots \in G^0$. *For some fixed numbers* $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \boldsymbol{R}$, $\frac{1}{2} < a \leqslant \alpha$, *let the following inequality hold*:

$$\left|\frac{\zeta g'(\zeta)}{g(\zeta)h(\zeta)} - \frac{as}{\alpha}\right| \leqslant \frac{a|s|}{\alpha}, \qquad \zeta \in E^0. \tag{1}$$

If

$$\left| |\zeta|^{2\varkappa} \frac{\zeta g'(\zeta)}{g(\zeta)h(\zeta)} + (1-|\zeta|^{2\varkappa})\left[\frac{\zeta g'(\zeta)}{g(\zeta)} + s\frac{\zeta h'(\zeta)}{h(\zeta)}\right] - \frac{as}{\alpha}\right| \leqslant \frac{a|s|}{\alpha} \tag{2}$$

holds for $\zeta \in E^0$ *and* $\varkappa = a/\alpha$, *then* g *is univalent in* E^0.

Proof. For $t \in [0, \infty)$ let us put formally

$$f(z, t) = \frac{1}{g(e^{st}z^{-1})}[1-(1-e^{-2at})h(e^{st}z^{-1})]^{-s}, \qquad z \in E,\ t \in [0, \infty). \tag{3}$$

Then we have

$$g(e^{st}z^{-1}) = \frac{e^{st}}{z} + b_0 + b_1 ze^{-st} + \ldots, \qquad h(e^{st}z^{-1}) = 1 + c_2 z^2 e^{-2st} + \ldots \tag{4}$$

Putting

$$A(z;\ a, s, t) = 1-(1-e^{-2at})h(e^{st}z^{-1}) = e^{-2at} - (1-e^{-2at})(c_2 z^2 e^{-2ts} + \ldots),$$

we infer that there exists a fixed number $r_1 \in (0, 1]$ such that $A(z;\, a, s, t) \neq 0$ for $z \in E_{r_1}$ and for each $t \in [0, \infty)$. From the hypotheses of the theorem and by (1), we have $g(\zeta) \neq 0$ for $\zeta \in E^0$. Hence, for each fixed $t \in [0, \infty)$, each fixed single-valued branch of $f(z, t)$ is regular in E_{r_1}. Further, from (3) we obtain $a_1(t) = [e^{-t} e^{2at}]^s$. In what follows we choose that fixed branch of power in $a_1(t)$ for which $|a_1(t)| = |[e^{-t} e^{2at}]^s| = e^{-\alpha t} e^{2a\alpha t}$. Thus $|a_1(t)| = e^{(2a-1)\alpha t} \to \infty$ as $t \to \infty$ because $a > \frac{1}{2}$ and $\alpha > 0$. By the definition of $A(z;\, a, s, t)$ and (4) we obtain

$$\frac{f(z, t)}{a_1(t)} = \frac{z}{(e^{st} + b_0 z + b_1 z^2 e^{-st} + \ldots) A^s(z;\, a, s, t) e^{-st} e^{2ats}}$$

and ultimately

$$\text{(5)} \qquad \frac{f(z, t)}{a_1(t)} = \frac{z}{(1 + b_0 z + b_1 z^2 e^{-st} + \ldots)[1 - (e^{2at} - 1)(c_2 z^2 e^{-2st} + \ldots)]^s}.$$

If we now apply our previous considerations to (5), we infer that $\{f(z, t)/a_1(t)\}$ forms a normal family in E_{r_0} for each $0 < r_0 < r_1$, $r_0 = \frac{1}{2} r_1$, say, if $e^{2at} |e^{-2st}| = e^{t(2a - 2\alpha)} \leqslant 1$, i.e., if $a \leqslant \alpha$. From the definition of $f(z, t)$ and its regularity in E_{r_1} it follows that $\partial f(z, t)/\partial t$ is uniformly bounded with respect to $\bar{E}_{r_0}$ for $t \in [0, T]$, where $T > 0$ is an arbitrarily chosen fixed number. Thus $f(z, t)$ is absolutely continuous in $[0, T]$, uniformly with respect to E_{r_0}.

Now, by some computations we obtain from (3)

$$\frac{f_t'(z, t)}{z f_z'(z, t)} = p(z, t) = -s + \frac{2ase^{-2at} g(\zeta e^{ts}) h(\zeta e^{ts})}{\zeta e^{ts} g'(\zeta e^{ts})[1 - (1 - e^{-2at}) h(\zeta e^{ts})] - s[(1 - e^{-2at}) \zeta e^{ts} g(\zeta e^{ts}) h'(\zeta e^{ts})]},$$

where $\zeta = z^{-1}$. Thus

$$\text{(6)} \qquad p(z, t) = -s + \frac{2as}{e^{2at} A(\zeta e^{ts}) + (1 - e^{2at}) B(\zeta e^{ts})},$$

where

$$A(\zeta e^{ts}) = \zeta e^{ts} g'(\zeta e^{ts})/[g(\zeta e^{ts}) h(\zeta e^{ts})]$$

and

$$B(\zeta e^{ts}) = [\zeta e^{ts} g'(\zeta e^{ts})/g(\zeta e^{ts})] + s\zeta e^{ts} h'(\zeta e^{ts})/h(\zeta e^{ts}).$$

(1) implies that $A(\zeta e^{ts}) \in \bar{K}(as/\alpha, a|s|/\alpha)$ for each fixed $\zeta \in E^0$ and $t \in [0, \infty)$. Moreover, $A(\zeta) \neq 0$, because $f'(\zeta) \neq 0$ for $\zeta \in E^0$. From (2) it follows that the quantity $|\zeta e^{ts}|^{2\varkappa} A(\zeta e^{ts}) + (1 - |\zeta e^{ts}|^{2\varkappa}) B(\zeta e^{ts})$ lies in $\bar{K}(as/\alpha, a|s|/\alpha)$, and in addition $|\zeta e^{ts}|^{2\varkappa} = |\zeta|^{2a/\alpha} e^{2at} > e^{2at}$. Hence by Remark

1 with $\lambda_0 = |\zeta e^{ts}|^{\alpha}$ and $\lambda = e^{2at}$, we see that the denominator d on the right-hand side of (6) lies in $K(as/\alpha, a|s|/\alpha)$ for each $\zeta \in E^0$ and $t \in (0, \infty)$. Thus $p(z, t)$ is regular in E^0 for each $t \in [0, \infty)$. The inequality $\operatorname{Re} p(z, t) > 0$ and the relation $d \in K(as/\alpha, a|s|/\alpha)$ are equivalent by (6). Then $\operatorname{Re} p(z, t) > 0$ for $z \in E^0$ and $t \in (0, \infty)$. Moreover, $\operatorname{Re} p(z, 0) \geqslant 0$ for $z \in E$. Thus we see from the above considerations that all assumptions of Theorem 1 are fulfilled. Hence $f(z, t)$ is univalent in E for each $t \in [0, \infty)$, and so is g because $f(z, 0) = 1/g(z^{-1})$. The proof of Theorem 2 has been completed.

3. We will now give some corollaries. Theorem 2 implies the following

COROLLARY 1. *Suppose that $g \in \Sigma^0$, $b_0 = 0$ and let a, s be fixed numbers such that $s = \alpha + i\beta$, $\beta \in \mathbf{R}$, $\frac{1}{2} < a \leqslant \alpha$. If the inequality*

$$(7) \qquad \left|(|\zeta|^{2a/\alpha} - 1)\alpha\left[(1-s)\left(1 - \frac{\zeta g'(\zeta)}{g(\zeta)}\right) - s\frac{\zeta g''(\zeta)}{g'(\zeta)}\right] - [(a-1)\alpha + a\beta i]\right| \leqslant a|s|$$

holds for $\zeta \in E^0$, then g is univalent in E^0.

Proof. Taking $h(\zeta) = \zeta g'(\zeta)/g(\zeta)$ in (1) and (2), we see that (1) is fulfilled automatically. Moreover, $\zeta g'(\zeta) g(\zeta) \neq 0$ by the hypotheses of the corollary and (7), and thus $h \in G^0$ and $h(\infty) = 1$. In this case relation (2) is equivalent to the following one:

$$(8) \qquad \left||\zeta|^{2a/\alpha} + (1 - |\zeta|^{2a/\alpha})\left[(1-s)\frac{\zeta g'(\zeta)}{g(\zeta)} + s\left(1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right)\right] - \frac{as}{\alpha}\right| \leqslant \frac{a|s|}{\alpha}.$$

Multiplying both sides of (8) by α, and performing grouping with respect to the factor $(|\zeta|^{2a/\alpha} - 1)$, we obtain (7). The univalence of g follows from Theorem 2. The proof of Corollary 1 has been completed.

In what follows we need the following

Remark 2. Let $\varphi(x; \tau) = (x^2 - 1)/(x^{2\tau} - 1)$ be defined for $x \in [1, \infty)$, where $\tau = (0, 1]$ and $\varphi(1) = \lim_{x \to 1^+} \varphi(x; \tau) = 1/\tau$. It is easy to verify that $\varphi(x; \tau)$ increases in $[1, \infty)$ from $1/\tau$ to infinity provided $\tau \neq 1$. It is evident in the case of $\tau = 1$ that $\varphi(x) \equiv 1$.

From Corollary 1 we deduce the following

COROLLARY 2. *Under the assumptions of Corollary* 1 *with $a = 1$ and $\alpha \geqslant 1$ the inequality*

$$(9) \qquad \left|(|\zeta|^2 - 1)\alpha\left[(1-s)\left(1 - \frac{\zeta g'(\zeta)}{g(\zeta)}\right) - s\frac{\zeta g''(\zeta)}{g'(\zeta)}\right] - \alpha\beta i\right| \leqslant |s|\alpha$$

for $\zeta = E^0$ implies the univalence of g in E^0.

Proof. From (7) we obtain for $a = 1$

$$(10) \quad \left|(|\zeta|^2 - 1)\alpha\left[(1-s)\left(1 - \frac{\zeta g'(\zeta)}{g(\zeta)}\right) - s\frac{\zeta g''(\zeta)}{g'(\zeta)}\right] - \varphi(|\zeta|, 1/\alpha)\beta i\right| \leqslant |s|\,\varphi(|\zeta|, 1/\alpha).$$

Put $S(|\zeta|) = \varphi(|\zeta|;\, 1/\alpha)\beta i$ and $R(|\zeta|) = |s|\,\varphi(|\zeta|;\, 1/\alpha)$. From the property of $\varphi(|\zeta|;\, 1/\alpha)$ stated in Remark 1 and from the inequality $|\beta| \leqslant |s|$ we obtain $|S(|\zeta|) - \alpha\beta i| \leqslant (R(|\zeta|) - |s|\,\alpha)$ and $R(|\zeta|) \geqslant |s|\,\alpha$ with equality for $\alpha = 1$ only. Thus $\bar{K}(\alpha\beta i,\, |s|\,\alpha) \subset \bar{K}(S(|\zeta|),\, R(|\zeta|))$ for each $\zeta \in E^0$. Hence each function g satisfying the hypotheses of Corollary 1 satisfies (10). Thus, by Corollary 1, g is univalent in E^0. This proves Corollary 2.

Earlier, Ruscheweyh [2] obtained the following

THEOREM 3. *Let* $s = \alpha + i\beta$, $\alpha \geqslant 1$ *and let* $g(\zeta) = \zeta + b_1\zeta^{-1} + \ldots \in \Sigma^0$. *If the inequality*

$$\left|(|\zeta|^2 - 1)\alpha\left[(1-s)\left(1 - \frac{\zeta g'(\zeta)}{g(\zeta)}\right) - s\frac{\zeta g''(\zeta)}{g'(\zeta)}\right] - i\beta\right| \leqslant |s|\,\alpha - |\beta|\,(\alpha - 1) \tag{11}$$

holds for $\zeta \in E^0$, *then* g *is univalent in* E^0.

It is easy to see that $\bar{K}(i\beta,\, \alpha|s| - |\beta|(\alpha-1)) \subset \bar{K}(\alpha\beta i,\, |s|\,\alpha)$. It follows that each function g satisfying the hypotheses of Theorem 3 also satisfies all the assumptions of Corollary 2. Thus g is univalent in E^0 by Corollary 2. Hence Corollary 2 and consequently Corollary 1 also extend Theorem 3 in an essential way.

References

[1] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. reine angew. Math. 218 (1965), 159–173.

[2] St. Ruscheweyh, *An extension of Becker's univalence condition*, Math. Ann. 220 (1976), 285–290.

Reçu par la Rédaction le 27.02.1984