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PRECOMPLETE CLASSES OF OPERATIONS
ON AN UNCOUNTABLE SET

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Let O be the set of finitary operations on a set A. A precomplete (or maximal) class is a dual atom in the lattice of composition closed subsets of O. We are interested in the precomplete classes containing all unary operations on A. In the late thirties Słupecki found that for A finite the set of all essentially unary or nonsurjective operations is the unique precomplete class containing all unary operations. In the sixties Gavrilov proved that for $|A| = \aleph_0$ there are exactly two such maximal classes. In this paper we look at the problem for $|A| > \aleph_0$. For $|A|$ regular one such maximal class is found by complete analogy with the countable case. This does not work for the other maximal class determined by Gavrilov as for $|A| = \aleph_1$ (assuming CH) the immediate analogue does not have the standard form.

1. Preliminaries. Let $\alpha > 1$ be a cardinal (= initial ordinal) and let A be the set of all ordinals less than α. For $n = 1, 2, \ldots$ let $O^{(n)}$ be the set of all n-ary operations on A (i.e. mappings $A^n \to A$) and let $O = \bigcup_{n=1}^{\infty} O^{(n)}$; for our purposes the zero operations do not fit well and are replaced by unary constant operations. For any $F \subseteq O$ the set $F \cap O^{(n)}$ is denoted by $F^{(n)}$. A set $C \subseteq O$ is called a *closed class* if it contains all compositions (also called compound operations or superpositions) of its elements and any operation obtained by permuting or identifying variables of an operation from C. Let $\mathscr{L} = \mathscr{L}_\alpha$ be the set of closed classes. A. I. Mal'tsev ([17]) pointed out that $\mathscr{L}$ is the set of all subalgebras of a certain algebra $\mathcal{O} = \langle O; *, \zeta, \tau, \Delta \rangle$ on O of type $\langle 2, 1, 1, 1 \rangle$. Hence $(\mathscr{L}, \subseteq)$ is an algebraic lattice ([5], [11]). The

subalgebra of O generated by $M \subseteq O$ (i.e., the least closed class containing M) is denoted by $[M]$. The countable lattice $\mathscr{L}_2$ (known as Post's lattice) is completely described in [20] but little is known for $\alpha > 2$ except that $|\mathscr{L}_\alpha|$ $= 2^{\aleph_0}$ for $2 < \alpha < \aleph_0$ and $|\mathscr{L}_\alpha| = 2^{2^\alpha}$ for $\alpha \geqslant \aleph_0$. The dual atoms of $\mathscr{L}_\alpha$ are called *maximal* or *precomplete classes*. For a finite α the lattice $\mathscr{L}_\alpha$ is dually atomic and the set $\mathscr{M}_\alpha$ of the dual atoms of $\mathscr{L}_\alpha$ is completely described in: [20] for $\alpha = 2$, [12] for $\alpha = 3$, [21], [22] for $\alpha > 3$. For $\alpha \geqslant \aleph_0$ a few classes from $\mathscr{M}_\alpha$ are known ([7], [10] for $\alpha = \aleph_0$, [25] for $\alpha \geqslant \aleph_0$) although it has been proved indirectly that $|\mathscr{M}_\alpha| = |\mathscr{L}_\alpha|$ ([9], [10] for $\alpha = \aleph_0$, [26] for $\alpha \geqslant \aleph_0$).

An operation $f \in O^{(n)}$ ($n > 1$), is said to be *essential* or *Słupecki* if f takes on all values and depends on at least two variables. The following Słupecki Criterion is not only one of the earliest but also one of the most important results in the theory: let $2 < \alpha < \aleph_0$ and let $M \subseteq O$ contain $O^{(1)}$. Then M is complete (i.e. $[M] = O$) iff M contains an essential function ([31], [12], [4]; several generalizations are given in [12], [28], [33], [34], [18]). It can be formulated as follows: For $2 < \alpha < \aleph_0$ the set S of all non-essential operations is a unique maximal class in the interval $\mathscr{K}_\alpha = [O^{(1)}, O]$ (actually $\mathscr{K}_\alpha$ is a chain of length $\alpha + 1$ ([3])).

An analogue of the Słupecki criterion for $\alpha = \aleph_0$ has been found by G. P. Gavrilov ([10]) who proved that $\mathscr{K}_\alpha$ is dually atomic and contains precisely two maximal classes (a description of these classes will be given in Sections 3, 5: classes $\mathrm{Pol}\, G$, $\mathrm{Pol}\, H$). The importance of the Słupecki criterion in the finite case and the fact that $\mathscr{K}_\alpha$ can be expected to be one of the simplest intervals in $\mathscr{L}_\alpha$ lead naturally to the problem of determining the dual atoms in $\mathscr{K}_\alpha$ for $\alpha > \aleph_0$. The following very weak form of the Słupecki criterion is well known and essentially due to Sierpiński ([30]) (see also [15], [5, II. 5.4]).

LEMMA 1.1. *Let $\alpha \geqslant \aleph_0$ and let $O^{(1)} \subseteq F \subseteq O$. Then F is complete if and only if $[F]^{(2)}$ contains an injection.*

It seems that very little is known about $\mathscr{K}_\alpha$ for $\alpha > \aleph_0$. In this paper without much trouble for regular α one maximal class in $\mathscr{K}_\alpha$ is found by complete analogy with the countable case. However this does not work for the other maximal class determined by Gavrilov. In Section 2 a subset of $\mathscr{K}_\alpha$ is determined which contains all maximal classes from $\mathscr{K}_\alpha$, then one maximal class in $\mathscr{K}_\alpha$ is found for every $\alpha \geqslant \aleph_0$. In Section 5 it is shown that for $\alpha = \aleph_1$ (assuming CH), the immediate analogue of the second maximal class given by Gavrilov does not have the standard form.

The set $\mathscr{K}_\alpha$ can be conveniently discussed in terms of sets of operations preserving relations. Let I be a non-empty set. An *I-relation* or $|I|$-*ary relation* ϱ on A is a subset of the set A^I of all mappings $I \to A$. If $|I| = k < \aleph_0$ we shall identify A^I and A^k, and the I-relations are simply the k-ary relations.

Throughout ϱ denotes an *I*-relation on *A*. We say that $f \in O^{(n)}$ *preserves* ϱ ([16]) or *f* is a *polymorph* of ϱ if $fg_1 \dots g_n \in \varrho$ whenever all $g_i \in \varrho$. Here and in the sequel $h = fg_1 \dots g_n$ is the mapping $I \to A$ defined by $hi = f(g_1 i) \dots (g_n i)$ for every $i \in I$. (The correspondence "*f* preserves ϱ" induces a Galois connection between the set of all closed classes of operations containing all projections and the subalgebras of a certain naturally defined, but for $\alpha \geqslant \aleph_0$ infinitary, algebra on a set of relations ([2], [24]).) The set of all $f \in O$ preserving ϱ is denoted by Pol ϱ. It has been shown e.g. in [24, Proposition 1] that the relations ϱ for which $O^{(1)} \subseteq \mathrm{Pol}\,\varrho$ can be described as follows:

Denote the set of all (binary) equivalence relations on *I* by $C(I)$. Given $f\colon B \to D$, set $\ker f = \{(x, y) \in B^2\colon fx = fy\}$. For $\gamma \in C(I)$ we set $\Delta_\gamma = \{f \in A^I |\ \ker f \supseteq \gamma\}$. Note that Δ_γ is an *I*-relation on *A* and that $\gamma' \subseteq \gamma'' \Rightarrow \Delta_{\gamma'} \supseteq \Delta_{\gamma''}$. The relation $\bigcup_{\gamma \in G} \Delta_\gamma$ with $G \subseteq C(I)$ will be denoted by ΔG. Let ϱ be an *I*-relation on *A*. Then $\mathrm{Pol}\,\varrho \supseteq O^{(1)}$ if and only if $\varrho = \Delta G$ where $G \subseteq C(I)$. Moreover $\mathrm{Pol}\,\varrho = O$ if and only if $\varrho = \Delta G$ where *G* is closed under intersection ([24, Corollary 1]).

The coordinates (or components) of $a \in A^n$ will be throughout denoted by $(a_1, \dots, a_n)$. The image of *a* under the mapping *f* will be denoted by $fa_1 \dots a_n$ or fa. The image of $B \subseteq A^n$ under *f* is denoted by fB; in particular, for $n = 2$ the image of $C \times D \subseteq A^2$ is denoted by fCD and the images of $\{a\} \times D$ and $C \times \{a\}$ by faD and fCa. The operations $e_i^n \in O^{(n)}$ $(1 \leqslant i \leqslant n)$, defined by $e_i^n a = a_i$ for every $a \in A^n$, are called *projections* ([11]) (other names: trivial, identity, or selective operations). The set of all projections will be denoted by *J*. A closed class containing *J* is called a *polynomial class*.

2. *A*-relations. Let $0 < k < \aleph_0$. By definition any $F \subseteq O^{(k)}$, being a set of mappings $A^k \to A$, is an A^k-relation on *A*. The following statement can be more or less explicitly found in many papers. The proof is routine and therefore omitted.

PROPOSITION 2.1. *Let* *C* *be a polynomial class and let* $D_k = \mathrm{Pol}\,C^{(k)}$ $(0 < k < \aleph_0)$. *Then*

$$D_1 \supseteq D_2 \supseteq \dots, \qquad D_k^{(k)} = C^{(k)}, \qquad C = \bigcap_{k=1}^{\infty} D_k.$$

We need also the following equivalence relations $\underset{k}{\sim}$ on $\mathscr{L}_\alpha$: we set

$$M \underset{k}{\sim} N \quad \Leftrightarrow \quad M^{(k)} = N^{(k)}.$$

The relation $\underset{1}{\sim}$ was implicitly used by Post ([20]) in his classification of $\mathscr{L}_2$. The relations $\underset{1}{\sim}$ and $\underset{2}{\sim}$ on $\mathscr{L}_\alpha$ $(\alpha < \aleph_0)$ have been used in [16], [12] in proving that $\mathscr{L}_\alpha$ is dually atomic.

PROPOSITION 2.2. *The relation* $\underset{k}{\sim}$ *properly contains* $\underset{k+1}{\sim}$ *for every* *k*.

Proof. This was shown in [20] for $\alpha = 2$, and we can imbed $\mathscr{L}_2$ into any $\mathscr{L}_\alpha$, $\alpha > 2$. $\square$

Post ([20, p. 46]) observed that each equivalence class of $\underset{1}{\tau}$ in $\mathscr{L}_2$ has a greatest element and asked whether this holds in general. The answer is affirmative for every k. Indeed from 2.1 we get:

PROPOSITION 2.3. *Let M and N be polynomial classes. Then*

$$M \underset{\tau}{\tau} N \quad \Leftrightarrow \quad [M^{(k)}] \subseteq N \subseteq \operatorname{Pol} M^{(k)}.$$

Thus $\operatorname{Pol} M^{(k)}$ is the greatest element in the equivalence class of $\underset{\tau}{\tau}$ containing M. The dual atoms of intervals in $\mathscr{L}$ satisfy:

PROPOSITION 2.4. *Let H be a polynomial class and let M be a dual atom of the interval $[J, H]$. Then $M = H \cap \operatorname{Pol} M^{(k)}$ where k is the least integer such that $M^{(k)} \subset H^{(k)}$.*

(We use $\subset$ for strict inclusion.)

Proof. Since $M \subset H$ there exists the least $k > 0$ such that $M^{(k)} \subset H^{(k)}$. Let $T = H \cap \operatorname{Pol} M^{(k)}$. Then $T \subset H$ because by Proposition 2.1 we have $T^{(k)} = H^{(k)} \cap M^{(k)} \subset H^{(k)}$. From M being a dual atom, $M \subseteq T \subset H$ and $T \in \mathscr{L}$ we obtain $M = T$. $\square$

PROPOSITION 2.5. *Each maximal class is a polynomial class.*

Proof. Let M be maximal. Suppose that $J \nsubseteq M$. Then $M \subset [M \cup J] \subseteq O$ implies $[M \cup J] = O$. Choose $a, b \in A$, $a \neq b$, and define $fab = b$, $fbb = a$ and $fx_1 x_2 = x_1$ otherwise. Now $f \in O = [M \cup J]$ has no fictitious variables and therefore belongs to M (since $[M \cup J]$ consists of M, J, and the functions obtained from those of M by adding fictitious variables). However from $f(fx_1 x_2) x_2 = x_1$ we see that $e_1^2 \in M$ and therefore $J \subseteq M$, a contradiction. $\square$

COROLLARY 2.6. *Let M be a maximal class. If $M^{(1)} \subset O^{(1)}$, then $M = \operatorname{Pol} M^{(1)}$. If $M^{(1)} = O^{(1)}$, then $M = \operatorname{Pol} M^{(2)}$ and $M^{(2)}$ is of the form ΔG.*

Proof. This follows from 2.5, 2.4 (with $H = O$), the fact that $[O^{(2)}] = O$ (Słupecki criterion and Lemma 1.1) and the remark towards the end of Section 1. $\square$

From now on we assume that A is infinite. We are interested in maximal classes containing $O^{(1)}$. Let $Q = \{ f \in O^{(2)} : f$ depends on at most one variable$\}$. For $h_i \in O^{(2)}$ ($i = 0, 1, 2$) let $h = h_0 h_1 h_2$ be the function from $O^{(2)}$ satisfying $hx = h_0(h_1 x)(h_2 x)$ for every $x \in A^2$. Setting $N = M^{(2)}$ in Corollary 2.6 we obtain that each maximal class containing $O^{(1)}$ has the form $\operatorname{Pol} N$ where the A^2-relation N on A is a subset of $O^{(2)}$ such that $J \subseteq N \subseteq M = \operatorname{Pol} N$ and $N = \Delta G$ for some $G \subseteq C(I)$. Thus N has the following properties:

(a) $e_1^2, e_2^2 \in N$;

(b) $h \in N$, $g \in O^{(2)}$, $\ker g \supseteq \ker h$ $\Rightarrow$ $g \in N$,

(c) $h_i \in N$ ($i = 0, 1, 2$) $\Rightarrow$ $h_0 h_1 h_2 \in N$

(the last one follows since $N = (\text{Pol} N)^{(2)}$) and therefore is a special kind of simple Menger algebra ([32]). The sets $N \subseteq O^{(2)}$ satisfying (a)–(c) will be called *basic Menger algebras*. Note that every basic Menger algebra N contains Q and that $h \in N$ $\Rightarrow$ $h^* \in N$ where $h^* = h e_2^2 e_1^2$ satisfies $h^* x = hx_2 x_1$ for every $x \in A^2$. We say that a basic Menger algebra $N \subset O^{(2)}$ is 2-*precomplete* if for each $h \in O^{(2)} \setminus N$ the set $O^{(2)}$ is the least basic Menger algebra containing $N \cup \{h\}$ (i.e., $[N \cup \{h\}] \supseteq O^{(2)}$ for each $h \in O^{(2)} \setminus N$).

LEMMA 2.7. *A basic Menger algebra N is 2-precomplete if and only if $[N \cup \{h\}]^{(2)}$ contains an injection for every $h \in O^{(2)} \setminus N$.*

Proof. Necessity is obvious since $\alpha \geqslant \aleph_0$. Sufficiency: if $[\{h\} \cup N]^{(2)}$ contains an injection, then since $O^{(1)} \subseteq N$, it follows from Lemma 1.1 that $[\{h\} \cup N] = O$, whence $[\{h\} \cup N]^{(2)} = O^{(2)}$. □

THEOREM 2.8. *If a basic Menger algebra N is 2-precomplete, then $\text{Pol} N$ is maximal.*

Proof. Let $g \in O^{(n)} \setminus \text{Pol} N$. Then there exist $h_1, \ldots, h_n \in N$ such that $h \in O^{(2)}$ defined by $hx = g(h_1 x) \ldots (h_n x)$ for every $x \in A^2$ does not belong to N. By virtue of (c) in the definition of basic Menger algebras $h_i \in N \subseteq \text{Pol} N$. Thus $h \in B = [\{g\} \cup \text{Pol} N]$, hence $B \supseteq [N \cup \{h\}] \supseteq O^{(2)}$ proving $B = O$ (by Lemma 1.1).

3. A maximal class for regular α. For every $f \in O^{(2)}$ let $f^* \in O^{(2)}$ be defined by $f^* x = fx_2 x_1$ for every $x \in A^2$. Let G denote the set of all $f \in O^{(2)}$ with the property that either f or f^* maps each $B \times A$ with $B \subset A$, $|B| < \alpha$ onto a set of cardinality less than α. Note that for regular α it suffices to consider only singletons for B.

LEMMA 3.1. *The set G is a basic Menger algebra.*

Proof. Clearly it suffices to verify the condition (c) only. Let $h_i \in G$ ($i = 0, 1, 2$), $h = h_0 h_1 h_2$ and suppose, for example, that h_0 and h_1^* have the above property. Then for every $B \subset A$, $|B| < \alpha$ we have

$$|h^* BA| \leqslant |h_0 h_1^* BAh_2^* BA| \leqslant |h_0 CA| < \alpha$$

because $|C| = |h_1^* BA| < \alpha$. The other cases are similar. Hence $h \in G$. □

THEOREM 3.2. *The class $\text{Pol} G$ is maximal for each regular α.*

Proof. We assume that A is the set of all ordinals less than ω_μ (where $\alpha = \omega_\mu$). By 3.1 and 2.8 it suffices to show that G is 2-precomplete. Let $h \in O^{(2)} \setminus G$. Because α is regular, there exists $a_1, a_2 \in A$ such that $|ha_1 A| = |hAa_2| = \alpha$. Using functions from $O \subseteq G$ we can achieve that $a_1 = a_2 = 0$. Choose sets $A_i \subseteq A$ such that $0 \in A_i$, $|A_i| = \alpha$ ($i = 1, 2$), and h is injective on $(A_1 \times \{0\}) \cup (\{0\} \times A_2)$. Let $T_1 = \{(x, y) \in A^2: x \geqslant y\}$, $T_2 = A^2 \setminus T_1$, and let

$g_i \in O^{(2)}$ be such that (i) $g_i|T_i$ is an injection into A_i and (ii) $g_i(A^2 \setminus T_i)$ = $\{0\}$. It is easy to see that $g_1 \in G$ (because $|g_1 aA| \leqslant 1 + |\{x: x \leqslant a\}| < \alpha$) and similarly $g_2 \in G$. Moreover $hg_1 g_2 \in [\{h\} \cup G]^{(2)}$ is injective and the required 2-precompleteness now follows from 2.7. $\square$

4. A maximal class for singular α. In what follows we suppose that α is a singular infinite cardinal, and that A is a set with $|A| = \alpha$. A subset $C \times D$ of A^2 with $|C| = |D| = \gamma$ will be called a γ-*square*. Let F denote the set of all functions $f \in O^{(2)}$ with the property: (I)   *There exists a selfmap γ (depending on f) of the set C_α of cardinals less than α such that for every $\beta \in C_\alpha$, each $\gamma(\beta)$-square contains a β-square on which f depends at most on one variable.*

THEOREM 4.1.   *The set F is a basic Menger algebra.*

P r o o f.   Clearly conditions (a) and (b) in the definition of basic Menger algebra are satisfied.

Now let $h_1, h_2, h_3 \in F$, let $h = h_3 h_1 h_2$, and denote the associated functions, whose existence is given by (I), by $\gamma_1, \gamma_2, \gamma_3$. We shall show that for the associated function γ for h we may take the function $\gamma(\beta)$ = $\gamma_1(\gamma_2(\gamma_3(\beta)^+))$; here we suppose, as we may, that $\beta \geqslant \aleph_0$.

Suppose that $C \times D \subseteq A^2$ and $|C| = |D| = \gamma_1(\gamma_2(\gamma_3(\beta)^+))$. We can find a set $C_1 \times D_1 \subseteq C \times D$ with $|C_1| = |D_1| = \gamma_2(\gamma_3(\beta)^+)$ on which h_1 depends on at most one variable, and then a set $C_2 \times D_2 \subseteq C_1 \times D_1$ with $|C_2| = |D_2|$ = $\gamma_3(\beta)^+$ on which h_2 depends on at most one variable. If it is the same variable in both cases, then also h depends only on this variable on $C_2 \times D_2$, and we are done. Otherwise, we may suppose that on $C_2 \times D_2$ the operation h_1 depends only on x, say $h_1(x, y) = g_1(x)$, and h_2 depends only on y, say $h_2(x, y) = g_2(y)$. Since $\gamma_3(\beta)^+$ is a regular cardinal, we can find a set $C_3 \subseteq C_2$ with $|C_3| = \gamma_3(\beta)^+$ on which either g_1 is constant, in which case h depends at most on y on $C_3 \times D_2$ and we are done, or g_1 is injective, as we may therefore suppose. Similarly we can find a set $D_3 \subseteq D_2$ with $|D_3| = \gamma_3(\beta)^+$ on which we may suppose g_2 injective. But now $|g_1 C_3| = |g_2 D_3|$ = $\gamma_3(\beta)^+ > \gamma_3(\beta)$, so we can find a set $C_4 \times D_4 \subseteq g_1 C_3 \times g_2 D_3$ with $|C_4|$ = $|D_4| = \beta$ on which h_3 depends on at most one variable. Upon taking C_0 = $C_3 \cap g_1^{-1}(C_4)$, $D_0 = D_3 \cap g_2^{-1}(D_4)$, we have $C_0 \times D_0 \subseteq C \times D$, $|C_0| = |D_0|$ = β, and h depends on at most one variable on $C_0 \times D_0$. Thus F satisfies condition (c), and is therefore a basic Menger algebra.

In order to prove that F is also 2-precomplete, it is convenient to express the defining property in a somewhat different form; to do this we appear to need to assume the generalized continuum hypothesis (GCH).

THEOREM 4.2 (GCH).   *The following three properties are equivalent for $f \in O^{(2)}$.*

(I)   *For every cardinal $\beta < \alpha$ there exists a cardinal $\gamma = \gamma(\beta) < \alpha$ such that each γ-square contains a β-square on which f depends on at most one variable.*

(II)   *There exists a cardinal $\beta < \alpha$ such that f is injective on no β-square.*

(III)   *f is not injective on any disjoint union of β_i-squares $(i \in I)$ such that* $|I| < \alpha$ *and* $\sum_{i \in I} \beta_i = \alpha$.

Remark 4.3. It would be easy to show that assuming GCH the following property is also equivalent to (I); however, we shall not need this fact.

(IV)   *f is not injective on any disjoint union* $\bigcup_{i \in I, \, j \in J} (C_{ij} \times D_{ij}) \subseteq A^2$ *where*

$|C_{ij}| = \beta_i < \alpha$, $|D_{ij}| = \beta_j < \alpha$, $|I| < \alpha$, *and* $\sum_{i \in I} \beta_i = \alpha$.

Proof of theorem. Since obviously (I) $\Rightarrow$ (II) $\Rightarrow$ (III), it would be enough to prove (III) $\Rightarrow$ (I); but it is convenient to prove (III) $\Rightarrow$ (II) and (II) $\Rightarrow$ (I).

(III) $\Rightarrow$ (II).   Write $\alpha = \sum_{\theta < \omega_\varphi} \beta_\theta$ where $\aleph_\varphi < \beta_0 < \beta_1 < \ldots < \alpha$ and each β_θ is regular. If (II) is false then for each θ, f is injective on a β_θ-square $C_\theta^* \times D_\theta^*$. Define sets $S_\theta = \bigcup_{\psi < \theta} (C_\psi^* \times D_\psi^*)$. Because β_θ is regular we have $|S_\theta| \leqslant \sum_{\psi < \theta} \beta_\psi < \beta_\theta$, hence there exist β_θ-squares $C_\theta \times D_\theta \subseteq (C_\theta^* \times D_\theta^*) \backslash S_\theta$. Now f is injective on the disjoint union of the squares $C_\theta \times D_\theta$, so (III) is false.

(II) $\Rightarrow$ (I) (GCH).   Let (II) hold and let $\beta_0 < \alpha$ be an infinite cardinal such that f is not injective on any β_0-square $C \times D$. We show that (I) holds with $\gamma(\beta) = \beta^{+++++}$ (this could be decreased by a more careful argument) supposing, as we may, that $\beta \geqslant \beta_0$. Given a β^{+++++}-square $C \times D$ let $D_1 \subseteq D$ satisfy $|D_1| = \beta^{+++}$. For every fixed $a \in A$ define $f_a, f^a \colon A \to A$ by setting $f_a(x) = fax$ and $f^a(x) = fxa$ for every $x \in A$. Let

$$C^* = \{x \in C \colon f_x \text{ takes some value } \beta^{+++} \text{ times on } D_1\},$$

and $C^{**} = C \backslash C^*$. If $x \in C^*$, choose a set $D_2(x) \subseteq D_1$ with cardinal β^{+++} on which f_x is constant; if $x \in C^{**}$, then f_x takes β^{+++} values on D_1, and we choose a set $D_2(x) \subseteq D_1$ with cardinal β^{+++} on which f_x is injective. Choose $C_1 \subseteq C$ with $|C_1| = \beta^{+++++}$ and either $C_1 \subseteq C^*$ or $C_1 \subseteq C^{**}$. Assuming GCH, there are only β^{++++} subsets of D_1, so we can choose $C_2 \times D_2 \subseteq C_1 \times D_1$ with $|C_2| = |D_2| = \beta^{+++}$ and $D_2(x) = D_2$ for all $x \in C_2$. If $C_2 \subseteq C^*$ we are done, since f depends at most on x on $C_2 \times D_2$, so we may suppose that $C_2 \subseteq C^{**}$ and $f_x | D_2$ is injective for $x \in C_2$.

Arguing the same way with x, y interchanged, we get a set $C_3 \times D_3 \subseteq C_2 \times D_2$ with $|C_3| = |D_3| = \beta^+$ on which either f depends only on y, and we are done, or (as we may suppose) f^y is injective for each y.

Let $D_4 \subseteq D_3$, $|D_4| = \beta$. No value can be in $f_x D_4$ for more than β values of $x \in C_3$, because otherwise $f(x_1, y) = f(x_2, y)$ for two different values $x_1, x_2 \in C_3$, and some $y \in D_4$, making f^y non-injective on C_3. Let $\beta = \aleph_\mu$, let

$<$ be a well-ordering of C_3 of type $\omega_{\mu+1}$, and for $0 \leqslant \varrho < \omega_\mu$ let $x(\varrho)$ be the least (with respect to $<$) element of C_3 such that

$$f_{x(\varrho)} D_4 \cap \bigcup \{f_{x(\sigma)} D_4 : 0 \leqslant \sigma < \varrho\} = \emptyset.$$

Let $C_4 = \{x(\varrho): 0 \leqslant \varrho < \omega_\mu\}$; then f is injective on $C_4 \times D_4$, contrary to hypothesis. The proof is complete. □

THEOREM 4.3 (GCH). *The set F is a 2-precomplete basic Menger algebra; consequently* $\mathrm{Pol}\, F$ *is a maximal class in O containing* $O^{(1)}$.

Proof. The set F being a basic Menger algebra by Theorem 4.1, in view of Lemma 2.7 it will be enough to show that if $h \in O^{(2)} \backslash F$, then $[\{h\} \cup F]^{(2)}$ includes an injection.

Let h be injective on the disjoint union of β_i-squares $C_i \times D_i$ $(i \in I, |I| < \alpha$, and $\sum_{i \in I} \beta_i = \alpha$; see Theorem 4.2, property (III)). It will be enough to construct two functions $h_1, h_2 \in F$ such that the mapping $(x, y) \to (h_1 xy, h_2 xy)$ is an injection from A^2 to $\bigcup_{i \in I} (C_i \times D_i)$, since then clearly hh_1h_2 is an injection belonging to $[\{h\} \cup F]^{(2)}$.

We may suppose that $|I| = cf\alpha = \aleph_\psi$, say, that I consists of the ordinals less than ω_ψ, and that $\omega_\psi < \beta_0 < \beta_1 < \dots$ We can write $A = \bigcup_{i \in I} A_i$ where $A_i \subset A_j$ for $i < j$. Let u_i: $A_i \to C_i$ and v_i: $A_i \to D_i$ be injective $(i \in I)$. We can now define h_1 and h_2 as follows. For any $(x, y) \in A^2$ let i be the least ordinal such that $(x, y) \in A_i^2$, and then let $h_1(x, y) = u_i(x)$, $h_2(x, y) = v_i(y)$. It is easy to see that these functions have the desired properties; in particular, the fact that h_2 takes only $\aleph_\psi$ values in each row and h_1 only $\aleph_\psi$ values in each column obviously implies that they belong to F. □

5. A counter-example. Two strictly increasing sequences $\langle b_\lambda \rangle_{\lambda < \omega_v}$ and $\langle c_\lambda \rangle_{\lambda < \omega_v}$ in A determine a *lower triangle* $\Delta = \{(b_\lambda, c_\mu): \mu \leqslant \lambda < \omega_v\}$ and an *upper triangle* $\{(b_\lambda, c_\mu): \lambda \leqslant \mu < \omega_v\}$ in A^2. Let T denote the set of all lower and upper triangles and set

$$H = \{f \in O^{(2)}: f \mid \Delta \text{ is injective on no } \Delta \in T\}.$$

For $v = 0$ Gavrilov showed that $\mathrm{Pol}\, H$ is a maximal class in $K_{\aleph_v}$. The proof is based on a lemma asserting that for any $f \in H$ and $\Delta \in T$ there exists $\Delta' \in T$, $\Delta' \subseteq \Delta$, such that $f \mid \Delta'$ depends on at most one variable. Unfortunately the proof fails for $v > 0$; and in fact we prove the following result, assuming the continuum hypothesis (CH).

PROPOSITION 5.1 (CH). *For $v = 1$, H is not a basic Menger algebra.*

Proof. In view of Corollary 2.6 and Lemma 2.7, it will be sufficient to establish the existence of $h_0, h_1, h_2 \in H$ such that $h_0 h_1 h_2 \notin H$, and this is an

immediate consequence of the lemmas below: with the notation of both lemmas, take $h_1 = e_1^2 k$, $h_2 = e_2^2 k$, and take as h_0: $A^2 \to A$ any function injective on the set S and constant on $A^2 \backslash S$. $\square$

LEMMA 5.2. *There exists a subset S of A^2 that has $\aleph_1$ elements in each row and each column of A^2, but contains no set $B_1 \times B_2$ with $|B_1| = |B_2| = 2$.*

Proof. Let $\langle \lambda_\mu \rangle_{\mu < \omega_1}$ be a sequence in which each element of A appears $\aleph_1$ times, and let S consist of all elements (θ, λ_θ) and (λ_θ, θ) with $\lambda_\theta < \theta$; evidently S has $\aleph_1$ elements in each row and each column of A^2. Now we show that if $\alpha_1 < \alpha_2 < \omega_1$ and $\beta_1 < \beta_2 < \omega_1$ then not all of the elements (α_i, β_j) can belong to S. If $\alpha_2 \leqslant \beta_2$, then neither of the elements (α_1, β_2), (α_2, β_2) can be of the form (θ, λ_θ) with $\lambda_\theta < \theta$, and of course at most one of them is the element $(\lambda_{\beta_2}, \beta_2)$, so they do not both belong to S. The case $\alpha_2 > \beta_2$ can be treated in a similar way and the lemma is proved. $\square$

LEMMA 5.3 (CH). *If $S \subseteq A^2$ has $\aleph_1$ elements in each row and each column, then there exists an injection k: $A^2 \to S$ such that $e_1^2 k$, $e_2^2 k \in H$.*

Proof. With each (lower or upper) triangle $\Delta \in T$ associate the set δ consisting of the elements in the first ω_0 rows and first ω_0 columns of Δ, and call it the (lower or upper) *starter* of Δ; let T^* denote the set of all starters. Each starter being countable, we have $|T^*| \leqslant \aleph_1^{\aleph_0} = \aleph_1$, assuming CH, and thus we can list the elements of T^* as $\langle \delta_\varrho \rangle_{\varrho < \omega_1}$.

For any $\theta < \omega_1$, let

$$C_\theta = \{(x, y): \max(x, y) = \theta\} \quad \text{and} \quad D_\theta = \bigcup \{C_\psi: \psi < \theta\}.$$

Let θ_0 be the least ordinal such that there are infinitely many lower and infinitely many upper starters $\delta_\varrho \subseteq D_{\theta_0}$ with $\varrho < \theta_0$. First, define $k \mid D_{\theta_0}$ in an arbitrary way as an injection into S. Since the sets C_θ, $\theta_0 \leqslant \theta < \omega_1$, are disjoint and their union is $A^2 \backslash D_{\theta_0}$, it will now be sufficient to define $k \mid C_\theta$ by transfinite induction on θ for $\theta_0 \leqslant \theta < \omega_1$.

Suppose, then, that $\theta_0 \leqslant \theta < \omega_1$ and that k: $D_\theta \to S$ has already been defined and is injective. There are $\aleph_0$ lower respectively upper starters $\delta_\varrho \subseteq D_\theta$ with $\varrho < \theta$; list them as

$$\langle \delta_\theta(2m) \rangle_{m < \omega_0} \quad \text{respectively} \quad \langle \delta_\theta(2m+1) \rangle_{m < \omega_0}.$$

Define elements $(x_\theta(n), y_\theta(n)) \in C_\theta$, together with the corresponding values of k, by induction on n as follows.

If $n = 4m$ or $4m+1$, we set $x_\theta(n) = \theta$, define $y_\theta(n)$ to be the least ordinal in the set $e_2^2 [\delta_\theta(2m)] \backslash \{y_\theta(\mu): \mu < n\}$, and choose $\xi_\theta(n)$ so that $(\xi_\theta(n), y_\theta(n)) \in \delta_\theta(2m)$. For $n = 4m$ respectively $4m+1$, we select as $kx_\theta(n) y_\theta(n)$ any value of the form

$$(e_1^2 k\xi_\theta(n) y_\theta(n), q) \quad \text{respectively} \quad (p, e_2^2 k\xi_\theta(n) y_\theta(n))$$

in the set

$$(3) \qquad (S \setminus k[D_\theta]) \setminus \{kx_\theta(\mu) \, y_\theta(\mu): \ \mu < n\};$$

such an element exists because S has $\aleph_1$ elements in each row and each column.

Similarly, if $n = 4m + 2$ or $4m + 3$, we set $y_\theta(n) = \theta$, define $x_\theta(n)$ to be the least ordinal in the set $e_1^2[\delta_\theta(2m+1)] \setminus \{x_\theta(\mu); \ \mu < n\}$, and choose $\eta_\theta(n)$ so that $(x_\theta(n), \eta_\theta(n)) \in \delta_\theta(2m+1)$. For $n = 4m+2$ respectively $4m+3$, we select as $kx_\theta(n) \, y_\theta(n)$ any value of the form

$$(e_1^2 \, kx_\theta(n) \, \eta_\theta(n), \ q) \quad \text{respectively} \quad (p, \ e_2^2 \, kx_\theta(n) \, \eta_\theta(n))$$

in the set (3).

Clearly, as now defined on $D_\theta \cup \{(x_\theta(n), \ y_\theta(n)): \ n < \omega_0\}$, k is injective. Finally, on $C_\theta \setminus \{(x_\theta(n), \ y_\theta(n)): \ n < \omega_0\}$ we assign to k arbitrary values in S subject to the requirement that $k|(D_\theta \cup C_\theta)$ be injective; this is possible because $|S| = \aleph_1$.

The inductive definition of k on A^2 is now complete, and it remains only to show that $e_1^2 k$ and $e_2^2 k$ belong to H. Consider for example the former, and let T be (for example) a lower triangle $\{(b_\lambda, c_\mu): \ \mu \leqslant \lambda < \omega_1\}$, with starter δ_ϱ. Choose $\lambda_0 \geqslant \omega_0$ so large that $b_{\lambda_0} > \varrho$ and $b_{\lambda_0} \geqslant \theta_0$. It will be sufficient to show that $e_1^2 k$ is not injective on the subset $U = \delta_\varrho \cup \{(b_{\lambda_0}, c_\mu'): \ \mu < \omega_0\}$ of T. Write θ for b_{λ_0}, and let δ_ϱ be listed as $\delta_\theta(2m)$. Then by the construction of k, we see that $e_1^2 k\theta y_\theta(n) = e_1^2 k\xi_\theta(n) \, y_\theta(n)$; since $(\theta, \ y_\theta(n))$ and $(\xi_\theta(n), \ y_\theta(n))$ are distinct elements of U, our assertion is proved. $\square$

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