MEAN WITH RESPECT TO A MAP

BY

G. J. MICHAELIDES (TAMPA, FLORIDA)

An \( n \)-mean on a space \( X \) is a continuous function (map) \( m: X^n \to X \), where \( X^n = X \times X \times \cdots \times X \) (the \( n \)-fold Cartesian product of \( X \)), satisfying the following two conditions:

1. \( m(x, x, \ldots, x) = x \) for every \( x \in X \);
2. \( m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \) for each \( n \)-tuple \( (x_1, x_2, \ldots, x_n) \in X^n \), \( \sigma \) being an element of the symmetric group \( S_n \) of \( n \) elements, i.e., \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, n\} \).

A space with a mean (admitting a mean) is called an \( m \)-space (for \( m \)-spaces see [1]-[8]). We now introduce a new idea, i.e. a mean with respect to a map.

Definition 1. Let \( X \) and \( Y \) be topological spaces and let \( f: X \to Y \) be a map. We say that \( m: X^n \to Y \) is an \( n \)-mean with respect to \( f \) if \( m \) is continuous and satisfies the following two conditions:

1. \( m(x, x, \ldots, x) = f(x) \) for every \( x \in X \);
2. \( m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \) for every \( (x_1, x_2, \ldots, x_n) \) in \( X^n \) and for every \( \sigma \in S_n \).

If \( X = Y \) and \( f = id_X \) (identity map on \( X \)), then Definition 1 coincides with the usual definition of a mean on \( X \). For a constant map \( f \), \( m \) always exists, and this is the trivial case where \( m \) is constant. If \( X \) is homeomorphic to \( Y \) and \( f \) is a homeomorphism, then a mean with respect to \( f \) induces a mean on \( X \), namely \( f^{-1}m \), so that both \( X \) and \( Y \) are \( m \)-spaces. For further clarification we give the following two examples:

Example 1. Let \( X = I = [0, 1], \ Y = S^1 \) (the unit circle in the plane) and let \( f: I \to S^1 \) be the map \( f(t) = (\cos 2\pi t, \sin 2\pi t) \). Let \( \mu: I \to I \) be the arithmetic mean. If \( m = f\mu \), then \( m \) is an \( n \)-mean with respect to \( f \) as it can easily be verified.

Example 2. Let \( X \) and \( Y \) be finite polyhedra, \( Y \) convex, and let \( f: X \to Y \) be a piecewise linear map such that \( f(X) \) is not a convex subset of \( Y \). We define \( m(x_1, x_2, \ldots, x_n) \) to be the barycenter of \( f(x_1), f(x_2), \ldots, f(x_n) \). Then \( m \) is an \( n \)-mean with respect to \( f \).
It should be noted that in each of these two examples either $X$ or $Y$ is an $m$-space. In such cases the existence of a mean with respect to a map $f$ is guaranteed by the following

**Proposition 1.** Let $X$ and $Y$ be topological spaces and let $f: X \to Y$ be continuous. If either $X$ or $Y$ admits a mean, then there exists a mean with respect to $f$.

**Proof.** Suppose that $X$ admits a mean $\mu: X^n \to X$ and that $f: X \to Y$ is continuous. The map $m = f\mu$ is obviously a mean with respect to $f$.

If, on the other hand, $Y$ admits a mean $\mu'$, then $m' = \mu'f^n$ is a mean with respect to $f$.

In view of Proposition 1 a question arises as to whether a mean with respect to $f$ exists if neither $X$ nor $Y$ is an $m$-space. The affirmative answer to this question is provided by the following example:

Let $X = Y = S^1$, where $S^1 = \{z \in C : |z| = 1\}$. Let $f: S^1 \to S^1$ be defined by $f(z) = z^2$. Since $S^1$ is an Abelian topological group, the multiplication $m(z_1, z_2) = z_1z_2$ is continuous and $m(z_1, z_2) = m(z_2, z_1)$. Moreover, $m(z, z) = f(z)$ so that $m$ is a mean with respect to $f$.

The converse of Proposition 1 is not true as the above-given example shows, since $S^1$ is not an $m$-space [1]. However, in Proposition 2 we give necessary conditions on $f$ in order that $Y$ is an $m$-space; and in Proposition 3 we examine conditions under which a covering space $X$ is an $m$-space if there is a mean with respect to the projection map $p: X \to Y$.

**Proposition 2.** Let $\mu$ be an $n$-mean with respect to a map $f: X \to Y$, where $f$ is open and onto. If for any two $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$ in $X^n$ such that $f(x_i) = f(x'_i)$ for some permutation $\sigma \in S_n$ we have $\mu(x_1, x_2, \ldots, x_n) = \mu(x'_1, x'_2, \ldots, x'_n)$, then $Y$ admits an $n$-mean.

**Proof.** Since $f$ is onto, for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Let $(y_1, y_2, \ldots, y_n) \in Y^n$ and define $m: Y^n \to Y$ by

$$m(y_1, y_2, \ldots, y_n) = \mu(x_1, x_2, \ldots, x_n), \quad \text{where} \quad y_i = f(x_i).$$

We shall show that $m$ is an $n$-mean on $Y$.

First we show $m$ is well defined. Let $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$ be two $n$-tuples in $X^n$ such that $f(x_i) = f(x'_i)$. Then $\mu(x_1, x_2, \ldots, x_n) = \mu(x'_1, x'_2, \ldots, x'_n)$ since $f(x_i) = f(x'_i)$. Here we take $\sigma$ to be the identity permutation in $S_n$. Hence $m(y_1, y_2, \ldots, y_n) = \mu(x_1, x_2, \ldots, x_n)$ is a unique point in $Y$ and $m$ is well defined.

To show continuity we observe that, by the definition of $m$, we have $\mu = mf^n$. If $U$ is an open subset of $Y$, then $\mu^{-1}(U) = (f^n)^{-1}m^{-1}(U)$ is open in $X^n$ by the continuity of $\mu$. Since $f$ is open, $f^n$ is also open, and since $f$ is onto, $(f^n)^{-1}m^{-1}(U) = m^{-1}(U)$ is open. Therefore, $f$ is continuous.

We now show that $m$ satisfies the conditions for a mean.
1. If \( y \in Y \) and \( x \in X \) are such that \( f(x) = y \), then

\[
m(y, y, \ldots, y) = \mu(x, x, \ldots, x) = f(x) = y.\]

2. Let \((y_1, y_2, \ldots, y_n) \in Y^n\) and let \((y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})\) be a permutation of \((y_1, y_2, \ldots, y_n)\). Let \((x_1, x_2, \ldots, x_n)\) and \((x'_1, x'_2, \ldots, x'_n)\) be two \(n\)-tuples in \(X^n\) such that \(f(x_i) = y_i\) and \(f(x'_i) = y_{\sigma(i)}\). From the latter equality we obtain \(f(x'_i) = y_i\), where \(\tau = \sigma^{-1}\), and therefore \(f(x_i) = f(x'_i)\). By hypothesis, \(\mu(x_1, x_2, \ldots, x_n) = \mu(x'_1, x'_2, \ldots, x'_n)\). Thus

\[
m(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = \mu(x_1, x_2, \ldots, x_n) = m(y_1, y_2, \ldots, y_n).\]

Since this is true for every permutation \(\sigma\), \(m\) satisfies the symmetric property of a mean, therefore \(m\) is an \(n\)-mean on \(Y\).

**Definition 2.** Let \(X\) and \(Y\) be pathwise connected and locally arcwise connected spaces and let \(p: X \to Y\) be continuous. The pair \((X, p)\) is called a covering space of \(Y\) if

1. \(p\) is onto,
2. for each \(x \in X\), there exists an open set \(U\) in \(X\) containing \(x\) such that \(p^{-1}(U)\) is a disjoint union of open sets, each of which maps homeomorphically onto \(U\) by \(p\).

**Proposition 3.** Let \(X\) be a covering space of \(Y\) and let \(p: X \to Y\) be the projection map. If \(\mu\) is a mean with respect to \(p\) and \(\mu_\pi(X^n) \subset p_\pi(X)\), then \(X\) admits a mean \((\mu_\pi \text{ and } p_\pi \text{ are the homomorphisms induced by } \mu \text{ and } p, \text{ respectively})\).

**Proof.** Let \(Y_0\) be a point in \(Y\) and let \(* \in p^{-1}(Y_0)\) be a base point of \(X\). By the definition of a covering space, \(X\) is pathwise connected and locally arcwise connected, therefore \(X^n\) is pathwise connected and locally arcwise connected with \((*, *, \ldots, *)\) as a base point. Since \(\mu\) is a mean with respect to \(p\), we have \(\mu(*, *, \ldots, *) = p(*, \ldots, *)\) and because of the condition \(\mu_\pi(X^n) \subset p_\pi(X)\) there is a unique lifting \(m: X^n \to X\) of \(\mu\) such that \(m(*, *, \ldots, *) = *\) and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{m} & X^n \\
p \downarrow & & \downarrow \mu \\
Y & \xrightarrow{p} & Y
\end{array}
\]

commutes. The map \(m\) is a mean on \(X\). To show this we first prove that \(m(x, x, \ldots, x) = x\) for every \(x \in X\). Let \(x \in X\) and let \(w\) be a path from \(*\) to \(x\). We have \(w(0) = *\) and \(w(1) = x\). If \(i: I \to I^n\) is the imbedding \(i(t) = (t, t, \ldots, t)\), we let

\[
\varphi = mw^n i = m(w \times w \times \ldots \times w) i: [0, 1] \to X.
\]
We have
\[ \varphi(0) = m(w \times w \times \ldots \times w) \text{i}(0) = m(w(0), w(0), \ldots, w(0)) = m(*, *, \ldots, *) = * \]
and
\[ p\varphi(t) = pm(w(t), w(t), \ldots, w(t)) = \mu(w(t), w(t), \ldots, w(t)) = pw(t) \]
by the commutativity of the diagram above and the fact that \( \mu \) is a mean with respect to \( p \). Since \( p\varphi \) and \( pw \) agree at one point, namely 0, we have
\[ \varphi = w \]
and
\[ x = w(1) = \varphi(1) = m(w(1), w(1), \ldots, w(1)) = m(x, x, \ldots, x). \]

To show that
\[ m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \]
we let \((x_1, x_2, \ldots, x_n), (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \in X^n \) and \( \sigma \in S_n \). Let us define \( m: X^n \to X \) by
\[ \tilde{m}(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}). \]

We observe that
\[ \tilde{m}(*, *, \ldots, *) = * \]
and
\[ p\tilde{m}(x_1, x_2, \ldots, x_n) = pm(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \]
\[ = \mu(x_1, x_2, \ldots, x_n). \]

But this means that \( \tilde{m} \) is a lifting of \( \mu \). Since
\[ m(*, *, \ldots, *) = \tilde{m}(*, *, \ldots, *) \]
by the uniqueness of the lifting \( m = \tilde{m} \), we have
\[ m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}). \]

Thus \( m \) is a mean on \( X \).

**Corollary 1.** Let \( X \) and \( X \) be as in Proposition 3. Let \( \bar{X} \) be an \( m \)-space with a mean \( \bar{m} \) such that \((mp^n)_{*} x_1(X^n) \subseteq p_{*} x_1(X)\). Then \( X \) also admits a mean.

**Proof.** If we let \( mp^n = \mu \), then \( \mu \) is obviously continuous. Moreover,
\[ \mu(x, x, \ldots, x) = m(p(x), p(x), \ldots, p(x)) = p(x) \]
and
\[ \mu(x_1, x_2, \ldots, x_n) = m(p(x_1), p(x_2), \ldots, p(x_n)) = m(p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \quad \text{for all } \sigma \in S_n. \]

Thus \( \mu \) is a mean with respect to \( p \), and all assumptions of Proposition 3 are satisfied. Therefore, \( X \) admits a mean.
COROLLARY 2. If \( Y \) in Corollary 1 is an \( m \)-space and \( X \) is its universal covering space, then \( X \) admits a mean.

Proof. Since \( X \) is the universal covering of \( Y \), \( \pi_1(X) \) and \( \pi_1(X^n) \) are both zero. Also \( \mu = mp^\pi \) is easily shown to be a mean with respect to \( p \), and \( \mu \cdot \pi_1(X^n) < p \cdot \pi_1(X) \) since \( \pi_1(X) = \pi_1(X^n) = 0 \). By Proposition 3, \( X \) admits a mean.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA

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