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THE GELFAND TRANSFORMS OF A CONVOLUTION MEASURE ALGEBRA

BY

COLIN C. GRAHAM (EVANSTON, ILLINOIS)

1. INTRODUCTION

This paper* discusses certain "thinness" properties of commutative convolution measure algebras. Most of the paper is devoted to the "thinness" of the set $\hat{B}$ of Gelfand transforms of the commutative convolution measure algebra B in the set of all continuous functions on the maximal ideal space $\hat{S}$ of B. Obviously, if $F: \boldsymbol{C} \to \boldsymbol{C}$ is continuous and $\hat{f} \in \hat{B}$, then $F \circ \hat{f} \in C(\hat{S})$. However, if one demands that $F \circ \hat{f} \in \hat{B}$, then (in general) restrictions must be placed on F. We say that F *operates in* $\hat{B}$ if $U \subseteq \boldsymbol{C}$, $F: U \to \boldsymbol{C}$ and $F \circ \hat{f} \in \hat{B}$ whenever $\hat{f} \in \hat{B}$ and $\hat{f}(\hat{S}) \subseteq U$. Throughout the paper the reader is expected to be familiar with the elementary theory of convolution measure algebras as developed in [13] or the first four chapters of [9]; in particular, the reader should be aware that B may be assumed to be an L-subalgebra of the regular Borel measure algebra $M(S)$ on a commutative bicontinuous compact semigroup S and that $\hat{S}$ is the set of continuous semicharacters on S (so $\hat{S}$ is a semigroup).

A *group* G in $\hat{S}$ is a subset G which is a group under the (pointwise) multiplication of $\hat{S}$; a *chain of idempotents* C in S is a non-empty subset $C \subseteq S$ such that $x, y \in C$ implies $x^2 = x$ and is totally ordered under $x \leqslant y$ if $xy = y$; a *bar* in S is a non-empty set $D \subseteq S$ with a distinguished element $d \in D$ such that $x, y \in D$, $x \neq y$ imply $xy = d$ and $x^2 = x$.

THEOREM 1. *Let B be a commutative convolution measure algebra with maximal ideal space $\hat{S}$. Suppose $F: \boldsymbol{C} \to \boldsymbol{C}$ operates in $\hat{B}$ and that $\hat{S}$ contains groups of arbitrarily large finite cardinality or an infinite group. Then F is real-analytic in a neighborhood of* 0.

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THEOREM 2. *Let B be a commutative convolution measure algebra with maximal ideal space $\hat{S}$. Suppose $F\colon \boldsymbol{C} \to \boldsymbol{C}$ operates in $\hat{B}$. Then, for some constants $C > 0$ and $\delta > 0$,*

$$|F(z)| \leqslant C|z| \quad \textit{if } |z| \leqslant \delta. \tag{1.1}$$

THEOREM 3. *Suppose $\hat{S}$ contains an infinite chain of idempotents and B contains an identity. If $F\colon \boldsymbol{C} \to \boldsymbol{C}$ operates in B, then, for each $R > 0$, there exists $C > 0$ such that*

$$|F(z)-F(w)| \leqslant C|z-w| \quad \textit{if } |z| \leqslant R,\ |w| \leqslant R.$$

Remark. Katznelson (see [5], Theorem 2) has proved a similar theorem.

THEOREM 4. *Let B be a semisimple commutative convolution measure algebra whose maximal ideal space $\hat{S}$ is a chain. Suppose $F\colon \boldsymbol{R} \to \boldsymbol{C}$ is absolutely continuous and that, on each compact interval, F' is essentially bounded and of bounded variation. Then $F\circ\hat{\mu} \in \hat{B}$ for all real $\hat{\mu}$ in $\hat{B}$.*

In combination with Theorem 3, we see that, if B obeys the hypotheses of Theorem 3 and has an identity, then exactly the absolutely continuous functions with bounded derivatives operate in $\hat{B}$.

The preceding three results describe the sparseness of $\hat{B}$ as a subset of $C(\hat{S})$. The next result describes the sparseness of the image of B (by the natural injection $\mu \mapsto \mu_S$, see [13]) in $M(S)$.

THEOREM 5. *Let B be a commutative convolution measure algebra with structure semigroup S. Suppose that S contains a perfect (non-empty) subset. Then the natural embedding of B into $M(S)$ does not cover the set $M_d(S)$ of discrete measures on S.*

Examples are given in Section 6 which show that each of the preceding results is sharp. Theorem 1 is proved in Section 2, Theorems 2 and 3, which generalize a result of Ross [7], are proved in Section 3; Theorems 4 and 5 are proved in Sections 4 and 5, respectively, and Section 7 contains some questions.

Notation otherwise unexplained is that of Rudin's monograph [8], or of [13].

2. PROOF OF THEOREM 1

The proof is somewhat involved. We first give some preliminary lemmas and results (Section 2.1), and then prove the theorem when $\hat{S}$ contains an infinite group (Section 2.2). This result is used to extend the theorem to the case of finite groups $G_j \subseteq \hat{S}$ of increasing cardinality whose identities χ_j obey $\chi_{j+1}\chi_j = \chi_j$ $(\chi_{j+1} \geqslant \chi_j)$ (Section 2.3). We then prove the theorem when $\hat{S}$ is discrete (Section 2.4). When $\hat{S}$ is not discrete, F is shown to be continuous, and this assumption is combined with the remainder of Section 2.5 to prove the theorem in that case. Thus Theorem 1 is proved for $\hat{S}$ discrete and $\hat{S}$ non-discrete, and so it is proved in general.

2.1. Some preliminaries.

LEMMA 1. *Let $f: \hat{S} \to \mathbf{C}$. Then there exists a $\nu \in M(S)$ with $f(\gamma) = \int \gamma \, d\nu$ for all $\gamma \in \hat{S}$ iff there is a constant $C \geqslant 0$ such that $c_1, \ldots, c_n \in \mathbf{C}$ and $\gamma_1, \ldots, \gamma_n \in \hat{S}$ imply*

$$\left|\sum_j c_j f(\gamma_j)\right| \leqslant C \sup_{x \in S} \left|\sum_j c_j \gamma_j(x)\right|, \tag{2.1}$$

and the total variation of ν is the infimum of such constants C.

This is the Riesz representation theorem (for the proof see [8], 1.9).

LEMMA 2. *Let $G \subseteq \hat{S}$ be a group $c_1, \ldots, c_n \in \mathbf{C}$, and $\gamma_1, \ldots, \gamma_n \in G$. Then*

$$\sup_{x \in S} \left|\sum_j c_j \gamma_j(x)\right| = \sup_{x \in \hat{G}} \left|\sum_j c_j \gamma_j(x)\right|. \tag{2.2}$$

Proof. Since the elements of S separate G, if $x \in S$ and $\gamma(x) \neq 0$ for some (and hence all) $\gamma \in G$, then $\gamma \to \gamma(x)$ is a character on G_d, G with the discrete topology. It is easy to see that there is a continuous multiplicative map from $\{x \in S: \gamma(x) \neq 0, \text{ all } \gamma \in G\} = S'$ to $(G_d)^\wedge$. Of course, the image of S' is a compact separating subsemigroup of $(G_d)^\wedge$, and hence equals $(G_d)^\wedge$. Since $\hat{G}$ is dense in $(G_d)^\wedge$, the lemma is proved.

A third technical lemma we shall need is the following. A proof can be extracted from [8], 6.3.

LEMMA 3. *If $F: \mathbf{C} \to \mathbf{C}$ is not real-analytic in a neighborhood of zero, then, for each $\varepsilon > 0$ and $C > 0$, there exists an integer $N = N(\varepsilon, C)$ such that if E is any abelian group of cardinality at least N, or an arithmetic progression (containing 0) of length at least N in an abelian group, then there exists an $f \in A(E)$ such that $f(0) = 0$, $\|f\| < \varepsilon$ and $\|F \circ f\| \geqslant C$.*

Finally, we make some simple observations about finite groups in $\hat{S}$. Each finite group $G \subseteq \hat{S}$ partitions S into $1 + \operatorname{card} G$ maximal open-closed sets $X_0, \ldots, X_m$ ($m = \operatorname{card} G$) on each of which every $\gamma \in G$ is constant. Let X_0 be the common zero set. Thus, if μ is any measure on S, the numbers

$$a_1 = \mu(X_1), \quad \ldots, \quad a_m = \mu(X_m) \tag{2.3}$$

completely determine $\hat{\mu}|_G$.

Let $X_1 = \{x \in S: \gamma(x) = 1, \text{ all } \gamma \in G\}$. Then $\nu \in M(X_j)$ for $1 \leqslant j \leqslant m$ implies $\nu^m \in M(X_1)$, and $\nu \in M(X_j)$ and $\omega \in M(X_1)$ imply $\omega * \nu \in M(X_j)$. (These follow from consideration of elements of the supports of the measures concerned, and from the particular choice of the exponent $m = \operatorname{card} G$.) We shall use this notation in Sections 2.3 and 2.5.

2.2. *$\hat{S}$ contains an infinite group.* Suppose $\hat{S}$ contains an infinite group G. Then Lemmas 1 and 2 show immediately that $\hat{B}|_G$ is a translation invariant subalgebra of the algebra $B(G_d)$ of Fourier-Stieltjes transforms on G_d. If χ is any multiplicative linear functional on $B(G_d)$, then $\mu \to \hat{\mu}|_G \to \chi(\hat{\mu}|_G)$ is a multiplicative linear functional on B. Thus, if F operates in $\hat{B}$, then F operates in $\hat{B}|_G$ (on its maximal ideal space).

We may assume that G is discrete, even though $\hat{B}|_G$ may not induce the discrete topology on G. Then $\hat{B}|_G = \hat{A}|_G$, where A is a (necessarily) weak*-dense L-subalgebra of $M\big((G_d)^\wedge\big)$. A straightforward application of the methods of [2] completes the proof; where Lemma 1.6 of [2] is used, one should apply [3] instead.

2.3. *$\hat{S}$ contains G_j with increasing supports.* We now assume $\hat{S}$ contains no infinite group, but does contain finite maximal groups $G_1, G_2, \ldots$ of increasing cardinality and with identities $\chi_1 \in G_1$, $\chi_2 \in G_2, \ldots$ such that $\chi_j\chi_{j+1} = \chi_j$ for $j = 1, 2, \ldots$ Note that if $k \geqslant j$, then $\chi_j G_k \subseteq G_j$, since G^j is maximal. Thus, for fixed j and $k \geqslant j$, the kernels H_{kj} of the maps $G_k \to G^j$ given by $\gamma \to \gamma\chi_j$ have increasing cardinality.

We proceed by induction. Choose $j(1)$ so large that (by Lemma 3) there exists an $f \in A(G_{j(1)})$ such that

$$\|f\|_{A(G_{j(1)})} < \tfrac{1}{2}f(0) = 0 \qquad \text{and} \qquad \|F\circ f\|_{A(G_{j(1)})} \geqslant 1.$$

Now apply the remarks at the end of Section 2.1, the definition of the norm on $A(G_{j(1)})$ and the weak*-density of B in $M(S)$ to find $\mu_1 \in B$ such that μ_1 is concentrated on $\{x\colon\ \gamma(x) \neq 0$ for $\gamma \in G_{j(1)}\}$, $\|\mu_1\| \leqslant \frac{1}{2}$, $\int d\mu_1 = 0$, and $\hat{\mu}_1|_{G_{j(1)}} = f$. Thus if $\nu \in M(S)$ and $\hat{\nu} = F\circ\hat{\mu}_1$ on $G_{j(1)}$, then, by Lemmas 1 and 2, $\|\nu\| \geqslant 1$.

We now assume we have found measures $\mu_1, \ldots, \mu_n \in B$ $(n \geqslant 1)$ and $j(1) < \ldots < j(n)$ such that, for $1 \leqslant k \leqslant m \leqslant n$,

$$\text{(2.4)} \qquad \int d\mu_k = 0, \quad \|\mu_k\| \leqslant 2^{-k}, \quad \sum_1^m \hat{\mu}_i|_{G_{j(k)}} = \hat{\mu}_k|_{G_{j(k)}} \quad (1 \leqslant k \leqslant m),$$

$$\text{(2.5)} \qquad \|F\circ\hat{\mu}_k|_{G_{j(k)}}\|_{A(G_{j(k)})} \geqslant k,$$

(2.6) μ_k is concentrated on $\{x\colon\ \gamma(x) \neq 0,\ \gamma \in G_{j(k)}\}$.

It follows immediately from (2.4)-(2.6) and Lemmas 1 and 2 that if $\nu \in M(S)$ has

$$\hat{\nu}|_{G_{j(k)}} = F\circ\Big(\sum_1^n \hat{\mu}_k|_{G_{j(k)}}\Big),$$

then $\|\nu\| \geqslant k$ for $1 \leqslant k \leqslant n$, and so $\|\nu\| \geqslant n$.

Choose $j(n+1) \geqslant j(n)$ so large that the kernel H of the map $G_{j(n+1)} \to G_{j(n)}$ given by $\gamma \to \chi_{j(n)}\gamma$ has cardinality at least $N = N(2^{-n-1}, n+1)$, where N is the integer produced by Lemma 3. Then there exists an $f \in A(H)$ such that $\|f\|_{A(H)} < 2^{-n-1}$, $f(0) = 0$, and $\|F \circ f\|_{A(H)} \geqslant n+1$. Note that $\gamma \in H$ implies $\gamma(x) = 1$ whenever $\chi_{j(n)} \neq 0$ Thus, those level sets (see the end of Section 2.1) $X_1, \ldots, X_k$ ($k \geqslant \operatorname{card} H$) for $G_{j(n+1)}$ which are not contained in (the prime subsemigroup) $S_n = \{x: \chi_{j(n)}(x) \neq 0\}$ separate H. Of course, $X_1 \nsubseteq S_n$. (If $X_1 \subseteq S_n$, and $x \in X_j \setminus S_n$, then $x^m \notin S_n$, but $x^m \in X_1$ if $m = \operatorname{card} G_{j(n+1)}$.) Thus, as for $n = 1$, we can find a measure $\mu_{n+1} \in B$ such that

$$\|\mu_{n+1}\| < 2^{-n-1}, \qquad \int d\mu_{n+1} = 0, \qquad \hat{\mu}_{n+1}|_{G_{j(n+1)}} = f,$$

and μ_{n+1} is concentrated on $\{x: \chi_{j(n)}(x) = 0,\ \chi_{j(n+1)}(x) \neq 0\}$. It is then straightforward to check that (2.4)-(2.6) now hold for $1 \leqslant k \leqslant m \leqslant n+1$.

We set

$$\mu = \sum_1^\infty \mu_j \in B$$

and see that if $\nu \in M(S)$ has $\hat{\nu}(\gamma) = F \circ \hat{\mu}(\gamma)$ for all $\gamma \in G_{j(n)}$, then $\|\nu\| \geqslant n$. Thus $F \circ \hat{\mu} \in \hat{B}$ would imply (by Lemma 1) $\|F \circ \mu\| = \infty$. This completes the proof in this case.

2.4. *$\hat{S}$ is discrete.* We assume again that $\hat{S}$ contains no infinite groups, and that $\hat{S}$ is discrete. The discreteness of $\hat{S}$ implies (by a result of Baker in [1]) that S is a union of compact groups H_a, and B is contained in the algebra generated by the radicals of the $L^1(H_a)$ and contains the algebra generated by the algebras $L^1(H_a)$.

LEMMA. *Under these hypotheses, if $H \subseteq S$ is a group with $L^1(H) \subseteq B$, then H is finite.*

Proof. Let $\nu \in B$ be Haar measure on H restricted to a compact subset of H with $\nu \neq 0$. Then $\hat{S}$ discrete implies $\hat{\nu} \in C_0(\hat{S})$. Thus there are only a finite number of $\gamma \in \hat{S}$, say $\gamma_1, \ldots, \gamma_n$, such that $\hat{\nu}(\gamma) = \|\hat{\nu}\|$.

Let $\chi = |\gamma_1| \ldots |\gamma_n|$. If $\Lambda = \{\gamma \in \hat{S}: \chi\gamma = \gamma\}$, then Λ is a group, and $\gamma \in \hat{S}$, $\hat{\nu}(|\gamma|) \neq 0$, implies $\chi\gamma \in \Lambda$. It is clear that Λ separates H. Thus, if the support of ν generates an infinite group (it must if ν is not discrete), then Λ is infinite which is a contradiction. Thus, if H is a group with $L^1(H) \subseteq B$, then H is discrete and of torsion.

Let x be the identity of H, and let $\nu = \delta_x \in B$. Then $\hat{\nu} \in C_0(\hat{S})$, so there are only a finite number of $\nu \in \hat{S}$ such that $\hat{\nu}(\gamma) = \hat{\nu}(\gamma)^2 = 1$. Thus H must be finite. This completes the proof of the Lemma.

Remark. Note that the proof of the Lemma implies that there exists a group $\Lambda \subseteq \hat{S}$ having minimal supports such that $\Lambda|_H = \hat{H}$.

Let $H_j \subseteq S$ be a sequence of maximal finite groups of increasing cardinality with $L^1(H_j) \subseteq B$. Let x_j be the identity of H_j. As in Section 2.2, we choose $j(1)$ so large that there exists an $f \in A(\hat{H}_{j(1)})$ such that

$$\|f\|_{A(\hat{H}_{j(1)})} < \tfrac{1}{2}, \quad f(0) = 0 \quad \text{and} \quad \|F \circ f\|_{A(\hat{H}_{j(1)})} \geqslant 1.$$

Choose $\gamma_1, \ldots, \gamma_n \in \hat{S}$ such that $\{\gamma_i|_{H_{j(1)}}: 1 \leqslant i \leqslant n\} = \hat{H}_{j(1)}$. We may assume that the γ's form a finite group Λ_1 which is separated by $H_{j(1)}$, and so is (isomorphic to) $\hat{H}_{j(1)}$. Then there exists a $\mu_1 \in L^1(H_{j(1)}) \subseteq B$ such that $\|\mu_1\| \leqslant \frac{1}{2}$ and $\hat{\mu}_1|_\Lambda = f$. This starts the induction.

We now suppose we have found $j(1) < \ldots < j(n)$ $(n \geqslant 1)$, $\mu_1, \ldots, \mu_n \in B$ and $\Lambda_1, \ldots, \Lambda_n \subseteq \hat{S}$ such that, for $1 \leqslant k$ and $m \leqslant n$,

$$\|\mu_k\| \leqslant 2^{-k}, \quad \int d\mu_k = 0, \tag{2.7}$$

$$\sum_1^m \hat{\mu}_j|_{\Lambda_k} = \hat{\mu}_k \quad (1 \leqslant k \leqslant m) \tag{2.8}$$

and

$$\|F \circ \hat{\mu}_k\|_{A(\Lambda_k)} \geqslant k+1. \tag{2.9}$$

Since

$$\Big(\sum_1^n |\mu_j|\Big)^\wedge \in C_0(\hat{S}),$$

there exists a J so large that if $j \geqslant J$, if $\chi \in S$ and $|\chi| \equiv 1$ on H_j, then

$$\chi \equiv 0 \text{ a.e. } d\Big(\sum_1^n |\mu_j|\Big),$$

that is, $\chi \equiv 0$ on $H_{j(1)} \cup \ldots \cup H_{j(n)}$. (This is immediate from the fact that $\sum_1^n |\mu_j|$ has a finite (discrete) support.)

Let Λ_j'' be the group of elements in $\hat{S}$ which have minimal supports and are non-zero on H_j. Then the set of non-zero restrictions of elements of $\Lambda_1 \cup \ldots \cup \Lambda_n$ to H_j generates a finite subgroup Λ_j' (taken to be the trivial subgroup if all elements of all Λ_k are zero on H_j) of Λ_j'' of order bounded independently of j. Thus, there exists a $j \geqslant J$ such that $\operatorname{card} \Lambda_j''/\Lambda_j' \geqslant N$, where $N = N(2^{-n-1}, n+1)$ is provided by Lemma 3.

We now choose $\mu_{n+1} \in L^1(H_j) \subseteq B$, so that $\hat{\mu}_{n+1}(\gamma) = 0$ if $\gamma \in \Lambda_j'$, $\|\mu_{n+1}\| < 2^{-n-1}$, $\hat{\mu}_{n+1}$ is constant on cosets of Λ_j', and $\|F \circ \hat{\mu}\|_{A(\Lambda_j'')} \geqslant n+1$. ($\mu_{n+1}$ can be found by standard harmonic analysis arguments; cf. [8], Chapter 2.) Set $j(n+1) = j$, and $\Lambda_{n+1} = \Lambda_j''$. It is straightforward to see that (2.7)-(2.9) hold for $1 \leqslant k$ and $m \leqslant n$. The proof of Theorem 1 now follows in this case exactly as at the end of Section 2.2.

2.5. *F is continuous.* We assume as before that each group $G \subseteq \hat{S}$ is finite and that $\hat{S}$ is not discrete. A simple but tedious topological argument shows that $\hat{S}$ not discrete implies F is continuous. (This requires showing first that $I = \{\chi \in \hat{S}: \chi \geqslant 0\}$ is not discrete and then using the methods of Section 3 below.)

We let $G_j \subseteq \hat{S}$ be maximal groups of increasing cardinality with identity χ_j, and let $j(1)$ be so large that there is an $f \in A(G_{j(1)})$ with $\|f\| < \frac{1}{2}$, $f(0) = 0$, and $\|F \circ f\| \geqslant 1$. As at the end of Section 2.1, we let $X_0, X_1, \ldots, X_m$ ($m = \operatorname{card} G_{j(1)}$) be the maximal level sets of $G_{j(1)}$, where X_0 (X_1) is the common zero (one) set.

Let $\nu_1, \ldots, \nu_m \in B$ be probability measures such that $\nu_1 \in M(X_1), \ldots$ $\ldots, \nu_m \in M(X_m)$.

Let $\chi \in \hat{S}$ be an accumulation point of $\{\chi_j\}_{j=1}^{\infty}$. Then $\chi \geqslant 0$, since each $\chi_j \geqslant 0$. Therefore, $\chi = \chi^2$ (otherwise $\{\chi^{ia}: a \in \boldsymbol{R}\}$ is an infinite group). It is easy to see (cf. [13], 5.1.5) that if $\chi_{j(a)}$ converges to χ, then

$$\int |\chi_{j(a)} - \chi| \, d\mu \to 0 \quad \text{for each } \mu \in B.$$

Indeed, a straightforward computation (using $\chi_{j(a)} = 0, 1$, and $\chi = 0, 1$) shows that

$$|\chi_{j(a)} - \chi| = (\chi - \chi_{j(a)})\chi + (1-\chi)\chi_{j(a)},$$

and so

$$\int |\chi_{j(a)} - \chi| \, d\mu = \int \chi \, d\mu - \int \chi_{j(a)} d(\chi\mu) + \int \chi_{j(a)} d(1-\chi)\mu$$

converges.

Let

$$\mu = \sum_{n=1}^{\infty} 2^{-n} m^{-n} (\nu_1 + \ldots + \nu_m)^n.$$

Then, by the above, we may assume χ_j converges a.e. $d\mu$ to an idempotent function $g \in L^{\infty}(\mu)$.

We have two cases: first $g = 0$ a.e. $d(\nu_1 * \ldots * \nu_m)$ and, second, $g = 1$ for a set of non-zero $(\nu_1 * \ldots * \nu_m)$-measure, in which case

$$\int g \, d\nu_1 \ldots \int g \, d\nu_m = \int g \, d(\nu_1 * \ldots * \nu_m) \neq 0.$$

(This follows from the a.e. convergence of the multiplicative functions χ_j.) In this second case, set $d\nu_j' = g d(\nu_j * \nu_1 * \ldots * \nu_m)$ for $1 \leqslant j \leqslant m$. In the first case, set $\nu_j' = \nu_j * \nu_1 * \ldots * \nu_m$. Note that no ν_j' is zero.

There exist numbers $a_1, \ldots, a_m$ such that if $\mu(X_1) = a_1, \ldots, \mu(X_m) = a_m$, then $\hat{\mu}|_{G_1} = f$, and $\sum_j |a_j| < \frac{1}{2}$.

Set

$$\mu_1 = \sum_{1}^{m} a_j \|\nu_j'\|^{-1} \nu_j'.$$

Then $\|F\circ\hat{\mu}_1|_{G_1}\|_{A(G_1)} > 1$, $\|\mu_1\| < \frac{1}{2}$, $\hat{\mu}_1(0) = 0$, and either $\chi_j \to 0$ a.e. $d\mu_1$ or $\chi_j \to 1$ a.e. $d\mu_1$.

Since F is continuous, there is a number $\varepsilon_1 > 0$ such that $\nu \in B$ and $\|\nu\| < \varepsilon_1$ imply that

$$\|F\circ(\mu_1+\nu)\|_{A(G_1)} > 1 \quad \text{and} \quad \|\mu_1+\nu\| < \tfrac{1}{2}.$$

We claim that, by modifying the G_j and μ_1, we may assume that there exists an $M_1 \geqslant 0$ such that either

$$\operatorname{card}\chi_{j(1)}G_k \leqslant M_1 < \infty \quad (1 \leqslant k < \infty) \tag{2.10}$$

or there exists a J such that $j \geqslant J$ implies $\chi_j \equiv 0$ a.e. $d\mu_1$. We now carry out such a modification.

Suppose $\lim \chi_k = 0$ a.e. $d\mu_1$. Then choose a J so large that $|\mu_1(X_i)| \neq 0$ implies

$$\int_{X_i} (1-\chi_J)\,d|\mu_1| \neq 0 \quad (1 \leqslant i \leqslant m).$$

Let $d\nu_i'' = (1-\chi_J)f_i\,d|\mu_1|$, where f_i is the characteristic function of X_i for $1 \leqslant i \leqslant m$. We may renormalize the ν_i'' so that each has norm one. Then set

$$\mu_1' = \sum_i a_i\nu_i''.$$

It is clear that μ_1' has the required properties: since $\chi_j \to 0$ a.e. $d\mu_1'$, the χ_j are converging monotonically to zero. Thus $j \geqslant J$ implies $\chi_j \leqslant \chi_J$ a.e. $d\mu_1$, and so $\chi_j \equiv 0$ a.e. $d\mu_1'$.

Thus, we may assume $\lim \chi_k \equiv 1$ a.e. $d\mu_1$. Let $\chi \in \hat{S}$ be an accumulation point of the χ_k, so $\chi \equiv 1$ a.e. $d\mu_1$. We may replace $G_{j(1)}$ by $\chi G_{j(1)}$, and may assume that $\operatorname{card}\chi_{j(1)}G_k$ is still increasing. Thus, we may assume that $\chi_{j(1)}G_k = G_k$ for all $k = 1, 2, \ldots$ Since the evaluation at points of the set $X_1 \cup \ldots \cup X_m$ will separate points of G_k, we may assume that the elements of G_k for each $k \geqslant j(1)$ restrict to distinct elements of $L^\infty(\mu_1)$. (Since we may perturb μ_1 by small amounts, we may assume $L^1(\mu_1)$ separates $\bigcup_{k \geqslant j(1)} \chi_{j(1)}G_k$.)

Since $\lim \chi_k = \chi_{j(1)}$ a.e. $d\mu_1$, and since $\chi_k \leqslant \chi_{j(1)}$ and $\chi_k = 0, 1$, $\chi_{j(1)} = 0, 1$, the sequence $\{\chi_k\}$ must be increasing a.e. $d\mu_1$. Thus, we may replace χ_k by

$$\chi_k' = \lim_{K\to\infty} \chi_k\chi_{k+1}\cdots\chi_{k+K} \text{ (limit in } \hat{S}),$$

and G_k by $\chi_k'G_k = G_k'$. Note that $\chi_k' = \chi_k$ a.e. $d\mu_1$, and so $\operatorname{card}G_k'$ is increasing.

Now, we may assume

$$G_k' = \bigcup_{j \geqslant k} \chi_k'G_j'.$$

If some G'_k is infinite, then, by Section 2.1, the proof is complete. If all G'_k are finite, we are in the situation of Section 2.3, and in that case Theorem 1 is proved. Thus we see that we may assume $\operatorname{card}\chi_{j(1)}G_k \leqslant M_1 < \infty$ for all $k \geqslant 1$. This completes the modification of μ_1 and $G_{j(1)}$.

Now suppose that groups $G_{j(1)}, \ldots, G_{j(n)} \subseteq \hat{S}$, measures $\mu_1, \ldots, \mu_n \in B$, and numbers $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > 0$ and $M_1, \ldots, M_N > 0$ have been found such that, for $1 \leqslant i \leqslant n$ and $2 \leqslant k \leqslant n$,

$$\varepsilon_i < 2^{-i} \quad \text{and} \quad \|\mu_i\| < 2^{-i}, \quad \|\mu_k\| < \varepsilon_{k-1}/2, \quad \hat{\mu}_i(1) = 0, \tag{2.11}$$

$$\text{either } \lim \chi_j \equiv 1 \text{ a.e. } d\mu_i \text{ or } \lim \chi_j \equiv 0 \text{ a.e. } d\mu_i, \tag{2.12}$$

$$\nu \in B \text{ and } \|\nu\| < \varepsilon_i \text{ imply } \|F\circ(\hat{\mu}_i + \hat{\nu})|_{G_{j(i)}}\|_{A(G_{j(i)})} > i, \tag{2.13}$$

$$\sup_{1\leqslant k<\infty} \operatorname{card}\chi_{j(i)}G_k \leqslant M_i. \tag{2.14}$$

We now show how to find μ_{n+1}, $G_{j(n+1)}$, ε_{n+1} and M_{n+1}.

First choose a J so large that $j \geqslant J$ implies that

$$\operatorname{card}G_j \geqslant N \operatorname{card}G_{j(1)} \ldots \operatorname{card}G_{j(n)} M_1 \ldots M_n,$$

where $N = N(\varepsilon_n 2^{-n-2}, n+1)$, the integer provided by Lemma 3.

For $j \geqslant J$, let L_j denote the intersection of the kernels of the maps $G_j \to \chi_{j(i)}G_j$ given by $\gamma \to \chi_{j(i)}\gamma$ for $1 \leqslant i \leqslant n$. (If $\chi_{j(i)}G_j = \{0\}$, the kernel is defined to be G_j.)

There are two possibilities: either there is a constant $M > 1$ such that $\gamma \in L_j$ for all $j \geqslant J$ implies order $\gamma \leqslant M$ or there is no such constant. In the first case, it is easy to see that L_j contains an increasing sequence of subgroups

$$\{\chi_j\} = H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_N = L_j \quad \text{with } \operatorname{card}H_i/H_{i-1} \leqslant M.$$

We choose

$$\delta = [3\cdot 2^{n+2} MNn]^{-1}\varepsilon_n. \tag{2.15}$$

Now choose a $j \geqslant J$ so large that

$$\|\chi_j - \lim_k \chi_k\|_{L^1(\mu_i)} < \delta \quad (1 \leqslant i \leqslant n). \tag{2.16}$$

In the second case, where no such M exists, set $M = 1$ and define δ by (2.15), choosing $j \geqslant J$ so large that L_j contains an arithmetic progression of length N and so that (2.16) holds. Of course, in either case, we may assume (passing to a subset L'_j of L_j) that $L'_j \subseteq L_j$ is a subgroup or progression of cardinality at least N and at most MN which contains the identity of the group L_j.

We claim $\gamma \in L_j$ implies $|\hat{\mu}_i(\gamma)| < \delta$, for

$$\begin{aligned}|\hat{\mu}_i(\gamma)| &= \left|\int \chi_{j(i)}\gamma\, d\mu_i\right| = \left|\int \chi_{j(i)}\chi_j\, d\mu_i\right| \\ &\leqslant \left|\int \lim_k \chi_k\, d\mu_i\right| + \left|\int (\chi_j - \lim_k \chi_k)\, d\mu_i\right| \\ &\leqslant |\hat{\mu}_i(1)| + \delta = \delta.\end{aligned}$$

We now construct μ_{n+1}. First, there exists an $f \in A(G_j)$ such that

$$\|f\|_{A(G_j)} < \min(2^{-n-2},\ \varepsilon_n/3) \quad \text{and} \quad \|F \circ f_{L_j'}\| > n+1.$$

Since F is continuous, there exists an $\varepsilon_n > \varepsilon_{n+1} > 0$ such that $g \in A(G_j)$ and $\|g\| < 2\varepsilon_{n+1}$ imply

$$\|F \circ (f+g)_{L_j'}\|_{A(L_j')} > n+1 \quad \text{and} \quad \|f+g\| < 2^{-n-2}.$$

Let $X_0, X_1, \ldots, X_m$ be the $m+1 = \operatorname{card} G_j + 1$ level sets for G_j with X_0 (X_1) the common zero (one) set. Let

$$\nu_1 \in B \cap M(X_1), \quad \ldots, \quad \nu_m \in B \cap M(X_m)$$

be positive measures of norm one. Set

$$\mu = \sum_{k=1}^{\infty} 2^{-k} m^{-k} (\nu_1 + \ldots + \nu_m)^k.$$

We may apply the arguments we used in the case $n = 1$ to conclude that there exists

$$\lim_k \chi_k = g \text{ a.e. } d\mu$$

and that

$$\int g\, d\nu_k * \nu_1 * \ldots * \nu_m = \int g\, d\nu_k \int g\, d\nu_1 \ldots \int g\, d\nu_m \quad (1 \leqslant k \leqslant m).$$

(We pass to a subsequence of $\{\chi_k\}$.)

If $\int g\, d\nu_1 * \ldots * d\nu_m = 0$, set $\nu_j' = \nu_j * \nu_1 * \ldots * \nu_m$. Otherwise, set $d\nu_j' = g\, d(\nu_j * \nu_1 * \ldots * \nu_m)$. As before, no $\nu_j' = 0$.

There exist numbers $a_1, \ldots, a_m$ such that if $\mu' \in B$, and $\mu'(X_1) = a_1, \ldots, \mu'(X_m) = a_m$, then

$$\hat{\mu}'|_{G_j} = f \quad \text{and} \quad \sum_k |a_k| < \min(2^{-n-2},\ \varepsilon_n/3).$$

Also, there exist numbers $b_1, \ldots, b_m$ such that if $\nu(X_1) = b_1, \ldots, \nu(X_m) = b_m$, then

$$\hat{\nu}\,|_{L_j'} = \sum_1^n \hat{\mu}_i\Big|_{L_j'} \quad \text{and} \quad \sum_k |b_k| \leqslant \Big\| \sum_1^n \hat{\mu}_i \Big|_{G_j} \Big\|_{A(L_j')}.$$

The fact that $|\hat{\mu}_i(\gamma)| < \delta$ on L_j for $1 \leqslant i \leqslant n$ implies that the restriction of $\sum_1^n \hat{\mu}_i$ to L_j' has norm at most

$$(\operatorname{card} L_j')\,\delta = MN\delta < \min(\varepsilon_n/3,\ 2^{-n-2}).$$

Thus we can find $b_1, \ldots, b_m$ with $\sum_k |b_k| < \min(\varepsilon_n/3,\ 2^{-n-2})$ so that $\nu(X_k) = b_k$ implies

$$\hat{\nu}\,|_{L_j'} = \sum_1^n \hat{\mu}_i \Big|_{L_j'}.$$

Set $\hat{\mu}_{n+1} = \sum_k (a_k - b_k)\|\nu_j'\|^{-1}\nu_j$. Then

$$\sum_1^{n+1} \hat{\mu}_i \Big|_{L_j'} = f\,|_{L_j'},$$

and so

$$\Big\| F \circ \Big(\sum_1^{n+1} \hat{\mu}_i\Big)\Big|_{G_j} \Big\|_{A(G_j)} \geqslant \Big\| F \circ \Big(\sum_1^{n+1} \hat{\mu}\,\Big|_{L_j'}\Big) \Big\|_{A(L_j')} \geqslant n+1,$$

while $\|\mu_{n+1}\| < \min(\varepsilon_n/2,\ 2^{-n-1})$. Thus, setting $G_{j(n+1)} = L_j'$, we see that we can find ε_{n+1} so that (2.11)-(2.13) hold for $1 \leqslant i \leqslant n+1$ and $2 \leqslant k \leqslant n+1$.

The proof that we may assume (formula (2.14)) that $\operatorname{card} \chi_{j(n+1)} G_k \leqslant M_{n+1}$ for some $M_{n+1} < \infty$ is exactly the same as that given for $\operatorname{card} \chi_{j(1)} G_k \leqslant M_1$.

The induction is now complete, and the proof runs along the same lines as at the end of Section 2.2.

3. PROOF OF THEOREMS 2 AND 3

The proof of Theorem 2 appears in Section 3.1; two lemmas used for this proof appear in 3.2 and 3.3. The proof of Theorem 3 is a combination of the methods in 3.1 and 3.2, and appears in 3.4.

3.1. Proof of Theorem 2. By applying Theorem 1 we may assume that there is an integer $n \geqslant 1$ such that if $G \subseteq \hat{S}$ is any group, then $\operatorname{card} G \leqslant n$. Let π be the map $\pi\colon \hat{S} \to \hat{S}$ given by $\pi(\chi) = \chi^{n!}$. Then $\pi(\hat{S})$ is the set of idempotents in $\hat{S}$, which must be infinite if $\hat{S}$ is infinite, since $\pi^{-1}(\chi)$ has cardinality at most $n!$ for all $\chi \in \pi(\hat{S})$.

We prove in Section 3.3 that any infinite idempotent semigroup contains either an infinite chain or an infinite bar.

Suppose $C \subseteq \hat{S}$ is an infinite chain of idempotents. We may assume C is countable, say $C = \{\chi_j\}_{j=1}^{\infty}$.

We have two possibilities: $\chi_j\chi_{j+1} = \chi_{j+1}$ for all j or $\chi_j\chi_{j+1} = \chi_j$ for all j.

Suppose first that $\chi_{j+1}\chi_j = \chi_{j+1}$ for all j. Choose measures $\mu_j \in B$ such that μ_j is positive, of norm one and supported on $\{x\colon \chi_{j+1}(x) = 0, \chi_j(x) = 1\}$. If $c_1, c_2, \ldots$ are numbers with $\sum_j |c_j| < \infty$, then

$$F\circ\Big(\sum_{k=1}^{\infty} c_k\mu_k\Big)^{\wedge}(\chi_j) = F\Big(\sum_{k=j}^{\infty} c_k\Big). \tag{3.1}$$

Then

$$\Big\|F\circ\Big(\sum_{k=1}^{\infty} c_k\mu_k\Big)\Big\| \geqslant \sum_{j=1}^{\infty}\Big|F\Big(\sum_{k=j}^{\infty} c_k\Big) - F\Big(\sum_{k=j+1}^{\infty} c_k\Big)\Big|,$$

since C a chain and $\omega \in M(S)$ imply

$$\|\omega\| \geqslant \sum_j |\hat{\omega}(\chi_j) - \hat{\omega}(\chi_{j+1})|.$$

(This follows from the location of mass on the pairwise disjoint sets $T_j = \{x\colon \chi_j(x) = 1, \chi_{j+1}(x) = 0\}$; cf. [7].)

Suppose

$$\sum_1^{\infty} |d_k| < \infty, \qquad d_k \in C.$$

Set $c_{k+1} \doteq d_k - d_{k+1}$, $1 \leqslant k < \infty$. Then

$$\sum_{j=1}^{\infty} |c_j| < \infty \quad \text{and} \quad \sum_{j=k}^{\infty} c_j = d_k.$$

Thus

$$F\circ\Big(\sum_j c_j\hat{\mu}_j(\chi_k)\Big) = F(d_k), \tag{3.2}$$

so

$$\Big\|F\circ\Big(\sum_j c_j\mu_j\Big)\Big\| \geqslant \sum_k |F(d_k) - F(d_{k+1})|. \tag{3.3}$$

Since we may assume $F(0) = 0$ without loss of generality, the Lemma in Section 3.2 below completes the proof.

We now suppose the chain $C = \{\chi_j\}$ obeys $\chi_{j+1} \geqslant \chi_j$ $(\chi_{j+1}\chi_j = \chi_j)$ for all $j = 1, 2, \ldots$ Choose measures $\mu_j \in B$ such that each μ_j is positive, of norm one, and concentrated on $\{x\colon \chi_{j+1}(x) = 1, \chi_j(x) = 0\}$. If $c_1, c_2, \ldots$ are numbers with $\sum_j |c_j| < \infty$, then (3.1) is replaced by

$$F\circ\Big(\sum_{k=1}^{\infty} c_k\hat{\mu}_k(\chi_j)\Big) = F\Big(\sum_{k=1}^{j} c_k\Big). \tag{3.4}$$

Then

$$\Big\|F\circ\Big(\sum_{k=1}^{\infty} c_k\hat{\mu}_k\Big)\Big\| \geqslant \sum_{j=1}^{\infty}\Big|F\Big(\sum_{k=1}^{j} c_k\Big) - F\Big(\sum_{k=1}^{j+1} c_k\Big)\Big|. \tag{3.5}$$

Let $\sum_{k=1}^{\infty}|d_k| < \infty$. Let $d_0 = 0$ and set $c_k = d_k - d_{k-1}$, $1 \leqslant k < \infty$. Then

$$\sum_{k=1}^{j} c_k = d_j.$$

Thus

$$\left\|F\circ\Big(\sum_{k=1}^{\infty} c_k\mu_k\Big)\right\| \geqslant \sum_{j=1}^{\infty} |F(d_j)-F(d_{j+1})|. \tag{3.6}$$

Again, the Lemma in Section 3.2 completes the proof, since we may assume $F(0) = 0$.

We now suppose $\hat{S}$ contains an infinite bar $B = \{\chi_j\}_{j=1}^{\infty}$ with $\chi_i\chi_j = \chi_0$ if $0 \leqslant i \neq j < \infty$ and, without loss of generality, that $F(0) = 0$. Let μ_k $(1 \leqslant k < \infty)$ be positive measures in B of norm one concentrated on $\{x\colon \chi_k(x) = 1,\ \chi_0(x) = 0\}$. Then $c_1, c_2, \ldots \in C$ and $\sum_k |c_k| < \infty$ imply

$$\begin{aligned}\left\|F\circ\Big(\sum_{1}^{\infty} c_k\mu_k\Big)\right\| &\geqslant \sum_j \left|F\circ\Big(\sum_{1}^{\infty} c_k\hat{\mu}_k(\chi_j)\Big) - F\circ\Big(\sum_{1}^{\infty} c_k\hat{\mu}_k(\chi_0)\Big)\right| \\ &= \sum_j |F(c_k)-F(0)|.\end{aligned} \tag{3.7}$$

Again the Lemma in Section 3.2 completes the proof.

3.2. Lemma. *Let $F\colon C \to C$ be any function with $F(0) = 0$. Then the following are equivalent:*

(i) *there exist $C > 0$ and $\delta > 0$ such that $|z| < \delta$ implies $|F(z)| \leqslant C|z|$;*

(ii) *$c_1, c_2, \ldots \in C$ and $\sum_{j=1}^{\infty}|c_j| < \infty$ imply $\sum_{j=1}^{\infty}|F(c_j)| < \infty$.*

Proof. That (i) implies (ii) is obvious. So suppose that (i) failed. Then there would exist $z_k \in C$ such that $|z_k| \leqslant 2^{-2k}$ and $|F(z_k)| > 2^{2k}|z_k|$. Assume (without loss of generality) that $z_k \neq 0$ and choose a map $n \mapsto z_n'$ from the integers to $\{z_k\}_{k=1}^{\infty}$ such that

$$2^{-2k} < |z_k|\operatorname{card}\{n\colon z_n' = z_k\} \leqslant 2^{-k}.$$

Then

$$\sum_{n=1}^{\infty} |z_n'| = \sum_{k=1}^{\infty} |z_k| \operatorname{card}\{n\colon z_n' = z_k\} \leqslant \sum_{k=1}^{\infty} 2^{-k} < \infty$$

while

$$\sum_{n=1}^{\infty} |F(z_n')| = \sum_{k=1}^{\infty} |F(z_k)| \operatorname{card}\{n\colon z_n' = z_k\} = \infty.$$

Thus, if (i) fails, then (ii) fails, so (i) and (ii) are equivalent.

3.3. Lemma. *Let S be an infinite idempotent commutative semigroup with 0. Then S contains either an infinite chain or an infinite bar.*

COROLLARY. *If S is the structure semigroup of the convolution measure algebra B, then at least one of the following holds*:

$$\sup\{\operatorname{card} G\colon G \subseteq \hat{S} \text{ is a group}\} = \infty, \tag{3.8}$$

(3.9) *S contains an infinite chain,*

(3.10) *S contains an infinite bar.*

Proof of the Corollary. Suppose that (3.8) fails; then

$$\sup\{\operatorname{card} G\colon G \subsetneq S,\ G \text{ a group}\} \neq \infty.$$

Indeed, if S contained an infinite compact group, then the fact that $\hat{S}|_G = \hat{G} \cup \{0\}$ implies (3.8). Thus every group in S is finite. If $G \subseteq S$ is a finite group, pick $\chi_1, \ldots, \chi_n$ such that $\{\chi_j|_G\} = \hat{G}$. Then

$$\Big\{\Big(\prod_{j=1}^{n} |\chi_j|\Big)\chi_k\colon 1 \leqslant k \leqslant n\Big\}$$

is a group in $\hat{S}$, so

$$\sup\{\operatorname{card} G\colon G \subseteq S;\ G \text{ a group}\} = \sup\{\operatorname{card} G\colon G \subseteq \hat{S},\ G \text{ a group}\}.$$

Thus, for some integer $n \geqslant 1$, $\pi\colon x \to x^n$ maps S onto the set S' of idempotents in S (continuously). Since S is compact for each $y \in S'$ and $\pi^{-1}(y)$ has finite cardinality, the Lemma implies that $S' \subseteq S$ contains an infinite bar or an infinite chain. (Since S' is compact, $0 \in S'$.)

Proof of the Lemma. We argue by contradiction. Without loss of generality, we may assume that S has an identity 1. We shall say that a chain $C \subseteq S$ *begins at* x and *ends at* y if $z \in C$ implies $xz = x$, $yz = z$ and $x, y \in C$.

We first show that if $x, y \in S$ and $x = xy \neq y$, then

(3.11) any maximal chain beginning at x and ending at y is finite;

(3.12) there are only a finite number of such maximal chains.

Taking $x = 0$ and $y = 1$, one sees that each $z \in S$ belongs to at least one maximal chain beginning at 0 and ending at 1, so S is finite, thus contradicting the hypotheses.

To prove (3.11) suppose $\{x, y\}$ is not maximal among chains beginning at x and ending at y. Then there exists a $z \in S \setminus \{x, y\}$ such that $zx = x$ and $zy = z$. So $\{x, y, z\} = C_2$ is a longer chain. If C_2 were not maximal, there would exist a $v \in S \setminus C_3$ such that $\{x, y, z, w\}$ was a chain with $xw = x$ and $yw = y$. Since S contains no infinite chains, this process stops after a finite number of steps. This proves (3.11).

To prove (3.12), let $\{C_j\}$ be any infinite set of distinct maximal chains beginning at x and ending at y. Let x_j be the least element of $C_j \setminus \{x\}$.

Since the C_j are maximal, $x \neq x_i x_j \neq x_i$ cannot hold, so $\{x_j\} \cup \{x\}$ is a (finite) bar.

Let $\{C_j(1)\}$ be an infinite subset of distinct C_j's such that the least elements of $C_j(1) \setminus \{x\}$ are the same. Then $C_j(1) \setminus \{x\}$ are chains between $x_1 = \min C_j(1) \setminus \{x\}$ and y, so the same argument shows there exists an infinite subset $\{C_j(2)\}_{j=1}^{\infty}$ of $\{C_j(1)\}_{j=1}^{\infty}$ such that the lowest two elements of $C_j(2)$ are the same. Proceeding in this "diagonal" way, we obtain an infinite chain. This contradicts the hypotheses, and so (3.12) is proved.

3.4. Proof of Theorem 3. We argue by contradiction, and suppose $\hat{S}$ contains an infinite chain $C = \{\chi_j\}_{j=1}^{\infty}$ and that, for some $R > 0$,

$$\sup_{\substack{|z| \leqslant R, |w| \leqslant R \\ z \neq w}} \big(|F(z) - F(w)|/|z-w|\big) = \infty. \tag{3.13}$$

Since F is continuous, there exists a z_0 with $|z_0| \leqslant R$ and $z_j \neq z$ such that $|z_j| \leqslant R$, $|z_j'| \leqslant R$, $\lim z_j = \lim z_j' = z$ and

$$\sup_j \big(|F(z_j) - F(z_j')|/|z_j - z_j'|\big) = \infty.$$

By replacing $F(z)$ by $G(z) = F(z-z_0) - F(z_0)$ (this uses the identity of B), we see that we may assume $z_0 = 0 = F(z_0)$.

By induction, we may assume that

$$|F(z_j) - F(z_j')| \geqslant 2^{2j} |z_j - z_j'|, \quad 1 \leqslant j < \infty, \tag{3.14}$$

and that

$$|z_j| \leqslant 2^{-2j}, \quad |z_j'| \leqslant 2^{-2j}, \quad 1 \leqslant j < \infty. \tag{3.15}$$

Set $n_0 = 0$ and choose integers $n_1 \geqslant 1$, $n_2 \geqslant 1, \ldots$ such that

$$2^{-k-1} < n_k |z_k - z_k'| < 2^{-k}, \quad 1 \leqslant k < \infty. \tag{3.16}$$

Suppose the chain $\{\chi_j\}$ obeys $\chi_{j+1}\chi_j = \chi_j$. Choose positive measures $\mu_1, \mu_2, \ldots \in B$ of norm one with μ_j concentrated on $\{x\colon \chi_{j+1}(x) = 1, \chi_j(x) = 0\}$. Then $c_1, c_2, \ldots \in C$ and $\sum_j |c_j| < \infty$ imply (cf. (3.4))

$$\sum_{j=1}^{\infty} c_j \hat{\mu}_j(\chi_k) = \sum_{j=1}^{k} c_j. \tag{3.17}$$

Define $c_1, c_2, \ldots$ by

$$c_m = \begin{cases} z_k, & m = 2(n_0 + \ldots + n_{k-1} + k - 1) + 1, \\ z_k' - z_k, & m = 2(n_0 + \ldots + n_{k-1} + k - 1) + 2j, \quad 1 \leqslant j \leqslant n_k, \\ z_k - z_k', & m = 2(n_0 + \ldots + n_{k-1} + k - 1) + 2j + 1, \quad 1 \leqslant j \leqslant n_k, \\ -z_k', & m = 2(n_0 + \ldots + n_{k-1} + k - 1) + 2n_k + 1, \\ 0, & m = 2(n_0 + \ldots + n_k + k). \end{cases} \tag{3.18}$$

It is a straightforward verification that, for $1 \leqslant k < \infty$,

$$(3.19)\qquad \sum \{c_m\colon\ 2(n_0+\ldots+n_k+k-1) < m \leqslant 2(n_0+\ldots+n_k+k)\} = 0$$

and that

$$(3.20)\qquad \sum \{c_m\colon\ m \leqslant 2(n_0+\ldots+n_{k-1}+k-1)+2j+1\} = z_k, \quad 0 \leqslant j < n_k,$$

while

$$(3.21)\qquad \sum \{c_m\colon\ m \leqslant 2(n_0+\ldots+n_{k-1}+k-1)+2j\} = z_k', \quad 1 \leqslant j \leqslant n_k.$$

Also

$$\sum_{m=1}^{\infty} |c_m| \leqslant \sum_{k=1}^{\infty} (|z_k|+|z_k'|+2n_k|z_k-z_k'|) < \infty.$$

Applying (3.17), (3.19) and (3.20) we see that

$$\Big|F\Big(\sum c_j\hat{\mu}_j(\chi_m)\Big)-F\Big(\sum c_j\hat{\mu}_j(\chi_{m+1})\Big)\Big| = |F(z_k)-F(z_k')|$$

if $2(n_0+\ldots+n_{k-1}+k-1)+1 \leqslant m \leqslant 2(n_0+\ldots+n_k+k-1)$.

Therefore, since $C = \{\chi_j\}$ is a chain (cf. [4]),

$$\begin{aligned}\Big\|F\circ\Big(\sum_j c_j\mu_j\Big)\Big\| &\geqslant \sum_m \Big|F\Big(\sum_j c_j\hat{\mu}_j(\chi_m)\Big)-F\Big(\sum_j c_j\hat{\mu}_j(\chi_{m+1})\Big)\Big| \\ &\geqslant \sum_k n_k|F(z_k)-F(z_k')| \geqslant \sum_k n_k 2^{2k}|z_k-z_k'| \\ &\geqslant \sum_k 2^{2k}2^{-k-1} = \infty,\end{aligned}$$

which is absurd. Thus Theorem 3 is proved in the case $\chi_{j+1}\chi_j = \chi_j$ for all j.

Suppose the chain $\{\chi_j\}$ obeys $\chi_{j+1}\chi_j = \chi_{j+1}$ for all j. Choose positive measures $\mu_1, \mu_2, \ldots \in B$ of norm one with μ_j concentrated on $\{x\colon \chi_{j+1}(x) = 0,\ \chi_j(x) = 1\}$. Then $c_1, c_2, \ldots \in C$ and $\sum_j |c_j| < \infty$ imply

$$(3.22)\qquad \sum_{j=k}^{\infty} c_j\hat{\mu}_j(\chi_k) = \sum_{j=k}^{\infty} c_j.$$

Define $c_1, c_2, \ldots$ by

$$(3.23)\qquad c_m = \begin{cases} -z_k', & m = 2(n_0+\ldots+n_{k-1}+k)-1, \\ z_k-z_k', & m = 2(n_0+\ldots+n_k+k)-2j-1,\ 1 \leqslant j < n_k, \\ z_k'-z_k, & m = 2(n_0+\ldots+n_k+k)-2j, \quad 1 \leqslant j \leqslant n_k, \\ z_k, & m = 2(n_0+\ldots+n_k+k)-1, \\ 0, & m = 2(n_0+\ldots+n_k+k). \end{cases}$$

(Formula (3.23) was constructed from the bottom line up, using (3.22) as a guide.) It is a straightforward verification that, for $1 \leqslant k < \infty$,

$$(3.24) \quad \sum \{c_m\colon\ 2(n_0+\ldots+n_k+k-1) \leqslant m \leqslant 2(n_0+\ldots+n_k+k)\} = 0,$$

and that

$$(3.25) \quad \sum \{c_m\colon\ m \geqslant 2(n_0+\ldots+n_k+k)-2j-1\} = z_k, \quad 0 \leqslant j < n_k,$$

while

$$(3.26) \quad \sum \{c_m\colon\ m \geqslant 2(n_0+\ldots+n_k+k)-2j\} = z'_k, \quad 1 \leqslant j \leqslant n_k.$$

Also

$$\sum_{m=1}^{\infty} |c_m| \leqslant \sum_{k=1}^{\infty} (|z'_k|+|z_k|+2n_k|z'_k-z_k|) < \infty.$$

Applying (3.22) and (3.24)-(3.26), we see that

$$\Big|F\Big(\sum c_j\hat{\mu}_j(\chi_m)\Big)-F\Big(\sum c_j\hat{\mu}_j(\chi_{m+1})\Big)\Big| = |F(z_k)-F(z'_k)|$$

if $2(n_0+\ldots+n_{k-1}+k) < m < 2(n_0+\ldots+n_k+k)-1$.

Therefore

$$\begin{aligned}\Big\|F\circ\Big(\sum_j c_j\mu_j\Big)\Big\| &\geqslant \sum_m \Big|F\Big(\sum_j c_j\hat{\mu}_j(\chi_m)\Big)-F\Big(\sum c_j\hat{\mu}_j(\chi_{m+1})\Big)\Big| \\ &\geqslant \sum_k n_k|F(z_k)-F(z'_k)| = \sum_k n_k 2^{2k}|z_k-z'_k| \\ &\geqslant \sum_k 2^{2k}2^{-k-1} = \infty,\end{aligned}$$

which is absurd, and so Theorem 3 is proved in the case $\chi_{j+1}\chi_j = \chi_{j+1}$ for all j. This completes the proof of Theorem 3.

4. PROOF OF THEOREM 4

This is motivated by the example of $B = L^1(0,1)$, where $\hat{B}$ is the set of absolutely continuous functions of bounded variation on $(0,1)$ (where $(0,1)$ is given the multiplication $xy = \max(x,y)$). In that case the proof of the theorem is easy: the measure associated with $F\circ\hat{\mu}$ is $(F'\circ\mu)\hat{\mu}'$. That is, $F\circ\hat{\mu}$ can be expressed by

$$F\circ\hat{\mu}(\chi) = \int\chi\,d\nu,$$

where ν is absolutely continuous with respect to μ.

We show that $F\circ\hat{\mu}$ is given by $\int\chi d\nu$, where $\nu\in M(S)$, and, furthermore, that ν is absolutely continuous with respect to μ. Since B_S is an L-subspace of $M(S)$, this is enough to prove Theorem 4.

4.1. LEMMA. *Under the hypotheses of Theorem 4, for each $\mu \in B$ there exists a $\nu \in M(S)$ such that*

$$\int \chi d\nu = F \circ \hat{\mu}(\chi) \quad \textit{for all } \chi \in \hat{S}.$$

Proof. Since $\hat{S}$ is totally ordered (see [5]), a necessary and sufficient condition for a function G on $\hat{S}$ to be expressible in the form $G(\chi) = \int \chi d\nu$ for all $\chi \in \hat{S}$ and (fixed) $\nu \in M(S)$ is that

$$\sup \sum_{j=1}^{n} |G(\chi_j) - G(\chi_{j-1})| < \infty, \tag{4.1}$$

where the supremum is taken over all finite sets $\chi_0, \ldots, \chi_n$ of elements of $\hat{S}$ such that

$$\chi_j \chi_{j-1} = \chi_j \quad \text{for } j = 1, \ldots, n.$$

Note that

$$\sum_{j=1}^{n} |F(\hat{\mu}(\chi_j)) - F(\hat{\mu}(\chi_{j-1}))| \leqslant \sup_{|x| \leqslant \|\mu\|} |F'(x)| \sum_{j=1}^{n} |\hat{\mu}(\chi_j) - \hat{\mu}(\chi_{j-1})|,$$

so the fact that (4.1) holds for $G = \hat{\mu}$ implies that (4.1) holds for $G = F \circ \hat{\mu}$. This completes the proof of the Lemma.

4.2. LEMMA. *Let F and B satisfy the hypotheses of Theorem 4 and let ν be the measure given by Lemma 4.1 for (a fixed) $\mu \in B$. Then ν is absolutely continuous with respect to μ.*

Proof. Let E be any set (Borel) on which μ has no mass. Let $\varepsilon > 0$. Then there exist χ_n and ϱ_n in $\hat{S}$ ($n = 1, \ldots, N$) such that $\chi_n \leqslant \varrho_n \leqslant \chi_{n+1}$ and such that μ gives mass at most ε to

$$\bigcup_n \{x\colon\ \varrho_n(x) = 1,\ \chi_n(x) = 0\} = U$$

and U contains E. This is an application of the regularity of μ and the fact that the intervals $\{x\colon \chi(x) = 1,\ \chi'(x) = 0\}$ $(\chi'\chi = \chi')$ form a basis for the topology of S.

The mass ν gives to U is given by

$$\sup \sum_{n,j} |\hat{\nu}(\chi_{n,j}) - \hat{\nu}(\chi_{n,j-1})|, \tag{4.2}$$

where the supremum is taken over finite subsets $\{\chi_{n,j}\}$ with $\chi_n \leqslant \chi_{n,j} < \chi_{n,j-1} \leqslant \varrho$ for all n, j. Of course, (4.2) is majorized by

$$\sup_{|x| \leqslant \|\mu\|} |F'(x)| \sup \sum_{n,j} |\hat{\mu}(\chi_{n,j}) - \hat{\mu}(\chi_{n,j-1})|, \tag{4.3}$$

where the inner supremum is taken over the same sets of $\{\chi_{n,j}\}$ as the supremum in (4.2). Since μ has mass at most ε on U, we infer that ν has mass at most $\varepsilon \sup |F'|$ on U, so ν is absolutely continuous with respect to μ. This completes the proof of Theorem 4.

5. PROOF OF THEOREM 5

The proof will be carried out in several steps. But before beginning, note that Theorem 5 implies the following:

If S is an infinite compact semigroup such that the semicharacters on S separate the points of S and S contains a perfect subset, then there exists a discontinuous semicharacter on S.

Indeed, let $B = M_d(S)$. If every semicharacter on S were continuous, then S would be the structure semigroup of B, so Theorem 5 is contradicted.

Also note that if $M(S) = B_S$, then $B_S = C(S)^*$. Of course, if this happens, then, by Theorem 5, S contains no perfect subset, so $B_S = M_d(S) = M(S)$. This suggests the following question:

For which commutative Banach algebras B is B the dual space of the norm closed subspace of B^* generated by the multiplicative linear functionals on B? (See Problem 7.5 in Section 7.)

It is easy to see that an infinite-dimensional uniform algebra does not have this property.

5.1. *We may assume that B has an identity.*

Indeed, adjoining an identity to B if B lacks an identity consists in enlarging S by adding an isolated point e to S and defining $se = es = s$ for all $s \in S \cup \{e\}$. Thus, no perfect set has been added to or subtracted from S.

5.2. *Every group $G \subseteq S$ is finite.*

Indeed, if G were infinite, then the Bohr compactification $\bar{G}$ would be an infinite compact subgroup and $M_d(\bar{G})$ would be a closed subalgebra of B. Of course, $(\bar{G}_d)\hat{}$ is the Šilov boundary of $M_d(\bar{G})$, and so $\Delta B = \hat{S}|_{M_d(\bar{G})} = \hat{S}|_{(\bar{G})}$ equals $(\bar{G}_d)\hat{}$ (cf. [2]). Therefore, $\bar{G}$ has the discrete topology in S, which implies $\bar{G}$ is finite.

5.3. *Any chain C of idempotents in S is discrete.*

Indeed, the Šilov boundary of $M_d(C)$ is $\Delta M_d(C)$, so $\hat{S}|_C = \Delta M_d(C)$. But, if $c_0 \in C$, then the map

$$\mu \mapsto \mu(\{c\colon cc_0 = c_0\})$$

is a multiplicative linear functional on $M_d(C)$. Therefore, $\{c\colon cc_0 = c_0\}$ is a component of C. Similarly, $\{c\colon c \neq c_0 \text{ and } cc_0 = c_0\}$ is a component of C. Therefore, $\{c_0\}$ is a component, so C is discrete.

5.4. *Any chain C of idempotents is finite.*

Otherwise, C contains an infinite descending (or ascending) subchain C'. By 5.3, C' is discrete, but, by the compactness of S, C' must have an accumulation point c_0. Then $C' \cup \{c_0\}$ is an infinite compact chain which contradicts C.

5.5. Let I denote the (closed) subsemigroup of idempotent elements of S. For $x \in I$, put

$$x^+ = \{y \in I\colon xy = x;\ u \in I,\ uy = u,\ ux = x \Rightarrow u = x \text{ or } u = y\}.$$

LEMMA. *Under the hypotheses of Theorem 5, x^+ is finite if and only if x is isolated in I.*

Proof. Suppose x^+ is infinite. Then $u, v \in x^+$ imply $uv = x$, so if $\chi \in \hat{S}$, $\chi(x) = 1$ implies $\chi(u) = 1$ for all $u \in x^+$. If $\chi(x) = 0$, and $\chi(u_1) = 1$ for $u_1 \in x^+$, then $\chi(v) = 0$ for all $v \in x^+ \setminus \{u_1\}$ Thus $(x^+)^-$ contains x, by the definition of the (subspace) topology on I.

Suppose x^+ is finite. Set

$$\mathscr{S} = \{y \in I\colon xy = y,\ y \neq x\}.$$

Then $\varrho\colon \mathscr{S} \cup \{x\} \to \{0, 1\}$ defined by $\varrho \equiv 0$ on $\mathscr{S}$, $\varrho(x) = 1$ is the restriction of a continuous semicharacter on S to $\mathscr{S} \cup \{x\}$. Therefore, $x \notin \mathscr{S}^-$. Set

$$\mathscr{L} = \{y \in I\colon xy = x,\ y \neq x\},$$

and let $y_1, \ldots, y_n$ be an enumeration of the elements of x^+. For each $j = 1, \ldots, n$, there must exist $\chi_j \in \hat{S}$ such that $\chi_j(y_i) = \delta_{ij}$, $\chi_j(x) = 0$. Clearly,

$$\mathscr{L} \subseteq \Big\{y\colon \sum_j |\chi_j(y)| \geqslant 1\Big\},$$

so $x \notin \mathscr{L}^-$.

If $\{y_\alpha\}_{\alpha \in A} \subseteq I$ is a net which converges to x, then either $\{\alpha\colon xy_\alpha \neq x\} = A_1$ or $\{\beta\colon xy_\beta = x\} = A_2$ is cofinal. If A_1 were cofinal, then $\{xy_\alpha\colon \alpha \in A_1\} \subseteq \mathscr{S}$, so $x \notin \{xy_\alpha\colon \alpha \in A_1\}^-$. Therefore, A_2 is cofinal, so $\{xy_\beta\colon \beta \in A_2\}^-$ contains x. But $\{xy_\beta\colon \beta \in A_2\} \subseteq \mathscr{L}$. Therefore, x is isolated.

Here is a proof that for each j there exists a $\chi_j \in \hat{S}$ such that $\chi_j(y_i) = \delta_{ij}$. Let $T = \{x\} \cup x^+$, and $\mathscr{B} = M(T)$. Then $\mathscr{B}$ is a closed L-subalgebra of $\mathscr{A}$ and $\hat{\mathscr{B}}$ is symmetric, so the Šilov boundary of $\mathscr{B}$ is all of the maximum ideal space $\Delta_{\mathscr{B}}$. Therefore, the restriction of $\hat{S}$ to T maps onto $\Delta_{\mathscr{B}} = \hat{T}$.

5.6. *The set I of idempotents in S contains no perfect subsets.*

Indeed, let $P \subseteq I$ be a perfect non-empty subset, and let $x_1 \in P$. Then, by the Lemma in 5.5, x_1^+ is infinite, and $x_1^+ \cap P \neq \emptyset$, since P is perfect. The argument used in 5.5 to show $x \notin \mathscr{S}^-$ shows $x_\alpha \to x$ implies $xx_\alpha = x$. Therefore, there exists an $x_2 \in P$ such that $x_1 x_2 = x_1$. By induction, we thereby construct a set of elements $x_1, x_2, \ldots \in P$ such that $x_{j+1} x_j = x_j$. Therefore, P contains an infinite chain, which contradicts step 5.4.

5.7. *S contains no perfect subsets P.*

Indeed, let $\pi: S \to S$ be the map which sends each element $s \in S$ to

$$\lim_{n\to\infty} n!s = \pi(s).$$

Of course, $\pi(s)$ is the idempotent element which is the identity of the maximal group containing s. We claim that π is continuous. If $s_a \to s$ in S, then $\chi(s_a) \to \chi(s)$ for all $\chi \in \hat{S}$, so $\chi(s_a) \to \chi(s)$ for all idempotent $\chi \in \hat{S}$. But if $\chi = \chi^2$, then $\chi(s) = \chi(\pi s)$, so $\chi(\pi s_a) \to \chi(\pi s)$ for all idempotent $\chi \in \hat{S}$. But the idempotent elements of $\hat{S}$ induce the topology on I. Indeed, if $s \in I$, then $\chi(s)^2 = \chi(s^2) = \chi(s)$, so by the compactness (use 5.1) of $\hat{S}$, there exists an idempotent

$$\varrho = \lim_a \chi^{n_a}$$

which agrees with χ on I. Therefore, π is continuous.

We claim that $\pi(P)$ is perfect if P is. For if $x \in \pi(P)$ were isolated in $\pi(P)$, then $\pi^{-1}(x) \cap P$ would be a component of P. But $\pi^{-1}(x)$ is finite for each $x \in I$, by 5.2. Therefore, $\pi(P)$ is perfect. But by the preceding 5.6, I has no perfect subsets.

The author is grateful to D. E. Ramirez for pointing out an error in the original formulation of Theorem 5 and suggesting Example 6.5 which shows that one cannot conclude that $B \supseteq M_d(S)$ implies S has discontinuous semicharacters.

6. EXAMPLES

6.1. Theorem 1 is sharp. Take $B = L^1(T)$ and apply Chapter 6 of [8].

6.2. Theorem 2 is sharp. Let $S_1 = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with multiplication $xy = 0$ if $x \neq y$ and $xx = x$. Set $B_1 = M(S_1)$; then S_1 is the structure semigroup of B, and if $F: C \to C$ satisfies (1.1), then F operates in $\hat{B}$. Let $S_2 = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with multiplication $xy = \min(x, y)$; if $F: C \to C$ satisfies (1.1), then F operates in $\hat{B}_2$ for $B_2 = M(S_2)$. These assertions follow from straightforward computation.

6.3. From Taylor [13], p. 162, it is easy to see that if $\hat{S}$ contains χ such that $|\chi|^{-1}(0, 1) \neq \emptyset$, then only analytic functions operate.

6.4. It is not true that non-symmetry of B implies only analytic functions operate. Here is an example B for which $\hat{S}$ is a product of an idempotent group with the group of three elements and such that all C^2-functions $F: C \to C$ give $F \circ (\hat{\mu}^2) \in \hat{B}$ whenever $\hat{\mu} \in \hat{B}$.

Let $B_1 = L^1(0, 1)$, where $(0, 1)$ is given the multiplication $xy = \min(x, y)$. Let

$$B_2 = \{\mu \in B_1 \hat{\otimes} B_2 \colon \operatorname{supp} \mu \subseteq \{(r, s) \colon r+s \leqslant 1,\ 0 \leqslant r,\ s \leqslant 1\}\}.$$

Let ν_0 be the usual length measure on $\{(r,s): r+s = 1,\ 0 \leqslant r,\ s \leqslant 1\}$; then $\nu_0 * \nu_0$ is a constant multiple of area measure on

$$\{(r,s):\ r+s \leqslant 1,\ 0 \leqslant r,\ s \leqslant 1\}.$$

$((0,1)\times(0,1)$ is given the product semigroup multiplication.) If $\mu \in B_2$, then $\mu * \nu_0 \in B_2$. Let Z_3 be the cyclic group of order three, and let δ denote the unit point mass at some (fixed) element of Z_3 not the identity of Z_3. Put

$$B = \{(M(Z_3)\hat{\otimes} B_2)\} \cup L^1(\delta \times \nu_0).$$

A series of straightforward computations shows that B has the required properties (and that the above-given assertions are correct).

6.5. If B_1 is the algebra of 6.2, then $M(S_1) = M_d(S_1) = B_1$, and *every* semicharacter on S_1 is continuous. Thus Theorem 5 is sharp, for one cannot conclude $B \supseteq M_d(S)$ implies S is finite.

7. SOME QUESTIONS

7.1. Let T be the structure semigroup of $M(S)$, where S is the structure semigroup of $\mathscr{A}$. Can there exist $F\colon C \to C$ such that $F\circ\hat{\mu} \in M(S)^\wedge$ for all $\hat{\mu} \in M(S)^\wedge$, and such that, for some $\hat{\mu}_0 \in \hat{\mathscr{A}}$, $F\circ\hat{\mu}_0 \notin \hat{\mathscr{A}}$? (Assume $\mathscr{A}$ has an identity.) (**P 950**)

7.2. If only real-analytic functions operate in $\mathscr{A}$, must $\hat{S}$ contain arbitrarily large groups? (**P 951**)

7.3. Find those algebras $\mathscr{B}$ of functions $f\colon C \to C$ such that there is a commutative convolution measure algebra $\mathscr{A}$ such that all (and only) functions $F \in \mathscr{B}$ operate in $\hat{\mathscr{A}}$. Does there exist a commutative convolution measure algebra $\mathscr{A}$ on an idempotent semigroup such that only real-analytic functions operate in $\hat{\mathscr{A}}$? (**P 952**)

7.4. If I is a closed ideal of a convolution measure algebra $\mathscr{A}$, and $\mathscr{A}/I$ is a uniform algebra on its maximal ideal space X, does $\mathscr{A}/I = C(X)$? (**P 953**)

7.5. Which commutative Banach algebras B have the property that B is the dual space of the norm-closure (in B^*) of the linear span of the multiplicative linear functionals on B? (**P 954**)

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