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On meromorphic solutions of a functional equation, II

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In this paper we consider the problem of the existence and uniqueness of meromorphic solutions of the linear functional equation

$$\varphi[f(z)]-g(z)\varphi(z) = h(z), \tag{1}$$

where φ is unknown and f, g, h are given functions of one complex variable z.

We assume the following hypotheses:

(I) f is analytic in a domain U, $f(U) \subset U$, and the boundary of U contains at least two finite points. Moreover, there exists exactly one point $a \in U$ such that $f(a) = a$ and $|c| < 1$, where $c = f'(a)$.

Without loss of generality we may assume that $a = 0$.

(II) g and h are meromorphic functions in U.

The problem of existence of meromorphic solutions of equation (1) was investigated in [4], where an important role was played by the following theorem of Smajdor [5] (cf. also [1], p. 188), concerning the local analytic solutions of the equation

$$\varphi(z) = h(z, \varphi[f(z)]). \tag{2}$$

THEOREM I. *If $f(z)$ is analytic in a neighbourhood of the point $z = 0$, $f(0) = 0$ and the function $h(z, w)$ is analytic in a neighbourhood of $(z, w) = (0, b)$ such that*

$$h(0, b) = b \tag{3}$$

and $|f'(0)| < 1$, then every formal solution

$$\varphi(z) = b+\sum_{n=1}^{\infty} c_n z^n$$

of equation (2) has a positive radius of convergence. Moreover, we have

$$(1-c^n a_{01})c_n = F_n(c_1, \ldots, c_{n-1}); \qquad n = 1, 2, \ldots, \tag{4}$$

where $c = f'(0)$, $a_{01} = \dfrac{\partial h}{\partial w}(0, b)$ and F_n is a polynomial in the variables $c_1, \ldots, c_{n-1}$.

In [4] the following lemma has been proved:

LEMMA 1. *Suppose that hypothesis* (I) *is fulfilled and* $g_1(z)$, $h_1(z)$ *are meromorphic in* U. *Moreover, let* $g_1(z)$ *and* $h_1(z)$ *be analytic functions at the point* $z = 0$ *and* $g_1(0) = 0$. *If the equation*

$$\varphi(z) = \frac{z^r}{g_1(z)}\varphi[f(z)] - \frac{h_1(z)}{g_1(z)}, \tag{5}$$

where r *is a positive integer or* 0, *has an analytic solution* $\varphi(z)$ *in a neighbourhood* U_0 *of the origin, then there exists a meromorphic solution* $\psi(z)$ *in* U *of equation* (5) *such that* $\psi(z) = \varphi(z)$ *for* $z \in U_0$.

In [4], by the use of Theorem I and of the above lemma, the meromorphic solutions of equation (1) have been determined. However, the case $c = 0$ was omitted in [4], since the problem of the existence of meromorphic solutions of equation (1) in this case often leads to the examination of local analytic solutions of the linear equation

$$\Phi[f(z)] - g^*(z)\Phi(z) = h^*(z), \tag{6}$$

where g^* and h^* are analytic at $z = 0$ and $g^*(0) = 0$. Evidently we cannot apply Theorem I in order to obtain local analytic solutions of this equation, for $h(z, w) = \big(w - h^*(z)\big)/g^*(z)$ is not analytic at any point $(z, w) = (0, b)$.

It is the case $c = 0$ that is the subject of our considerations.

Remark. For the same reasons the case where $g(0) = 0$ and $c \neq 0$ is omitted in [4]. However, in that case there exist formal solutions which are divergent (cf. [3]).

Now suppose that f, g, h are analytic in a neighbourhood of the origin, $f(0) = g'(0) = f'(0) = 0$ and none of these functions is identically zero. Thus we may write

$$f(z) = z^k F(z); \quad F(0) \neq 0; \quad k \geq 2, \tag{7}$$

$$g(z) = z^p G(z); \quad G(0) \neq 0; \quad p \geq 1, \tag{8}$$

$$h(z) = z^q H(z); \quad H(0) \neq 0; \quad q \geq 0, \tag{9}$$

where the functions F, G, H are analytic in a neighbourhood of the origin.

Kuczma [2] has proved the following Lemmas II, III and Theorem II:

LEMMA II. *If either*

(i) $k-1$ *divides* p, $kp/(k-1) = q$; $G(0) \neq [F(0)]^{p/(k-1)}$,

or

(ii) $k-1$ *divides* p, $kp/(k-1) < q$; $G(0) = [F(0)]^{p/(k-1)}$,

or

(iii) $k(q-p) > q$,

then equation (1) *has a local analytic solution*

$$\varphi(z) = z^{\varrho}\Phi(z) \tag{10}$$

in a neighbourhood of the origin. Here $\varrho = p/(k-1)$ *in cases* (i) *and* (ii), $\varrho = q-p$ *in case* (iii) *and* $\Phi(z)$ *is the local analytic solution of the equation*

$$\Phi(z) = \frac{z^{\varrho k-p-\varrho}[F(z)]^{\varrho}}{G(z)}\Phi[f(z)] - \frac{z^{q-p-\varrho}H(z)}{G(z)} \tag{11}$$

in a neighbourhood of the origin. Solution (10) *is unique in cases* (i) *and* (iii) *and depends on the parameter* $b = \Phi(0)$ *in case* (ii).

LEMMA III. *Suppose that* k *divides* q, $q(k-1) < kp$, *and that there exists a polynomial* $P(z) = \sum_{n=r}^{R-1} c_n z^n$, *where* $r = q/k$ *and* R *is the smallest integer such that* $R > r$ *and* $Rk \geqslant R+p$, *with the property that the function* $h^*(z) = h(z) - P[f(z)] - g(z)P(z)$ *has at the origin a zero of an order* $q^* \geqslant R+p$:

$$h^*(z) = z^{q^*}H^*(z), \quad H^*(0) \neq 0.$$

If either

(i) $k-1$ *divides* p, $kp/(k-1) = q^*$ *and* $G(0) \neq [F(0)]^{p/(k-1)}$;

or

(ii) $k-1$ *divides* p, $kp/(k-1) < q^*$ *and* $G(0) = [F(0)]^{p/(k-1)}$;

or

(iii) $k(q^*-p) > q^*$,

then equation (1) *has a local analytic solution*

$$\varphi(z) = P(z) + z^{\varrho}\Phi^*(z) \tag{12}$$

in a neighbourhood of the origin. Here $\varrho = p/(k-1)$ *in cases* (i) *and* (ii), $\varrho = q-q$ *in case* (iii), *and* $\Phi^*(z)$ *is the local analytic solution of the equation*

$$\Phi^*(z) = \frac{z^{\varrho k-p-\varrho}[F(z)]^{\varrho}}{G(z)}\Phi^*[f(z)] - \frac{z^{q^*-p-\varrho}H^*(z)}{G(z)} \tag{13}$$

in a neighbourhood of the origin. Solution (12) *is unique in cases* (i) *and* (iii) *and depends on the parameter* $b = \Phi^*(0)$ *in case* (i).

Remark. Equations (11) and (13) have form (2) and we can apply Theorem I.

THEOREM II. *Suppose that the functions* f, g *and* h *are analytic in a neighbourhood of the origin*, $f(0) = f'(0) = g(0) = 0$, *none of these functions being identically zero. Then equation* (1) *has no local analytic solutions in a neighbourhood of the origin except for the cases covered by Lemmas* II *and* III.

1. In this section we consider meromorphic solutions of equation (1) under the assumption that $c = 0$. The following cases are possible:

(A) $g(z)$ and $h(z)$ are analytic at the origin and $g(0) \neq 0$;

(B) $g(z)$ and $h(z)$ are analytic at the origin and $g(0) = 0$;

(C) $g(z)$ is analytic and $h(z)$ has a pole at the origin;

(D) $g(z)$ has a pole and $h(z)$ is analytic at the origin;

(E) $g(z)$ and $h(z)$ have poles at the origin.

LEMMA 1. *Suppose that hypotheses* (I), (II) *are fulfilled and* $c = 0$. *If* $g(z)$ *and* $h(z)$ *are analytic at the origin, then every meromorphic solution in* U *of equation* (1) *is analytic at the origin.*

Proof. Suppose that $\varphi(z)$ is a meromorphic solution of equation (1) that is not analytic at the origin. Thus we have

$$(14) \qquad \varphi(z) = \frac{\Phi(z)}{z^r}, \quad \Phi(0) \neq 0, \quad r \in N,$$

where $\Phi(z)$ is analytic at the origin and N denotes the set of positive integers. Inserting (14) and (7) into equation (1), we see that the left-hand side of (1) has a pole of the order kr and the right-hand side is analytic at the origin. It is a contradiction and the lemma is proved.

Hence according to Theorem I and Lemma I we obtain the following

THEOREM 1. *Suppose that hypotheses* (I), (II), (A) *are fulfilled and* $c = 0$. *Then equation* (1) *has a meromorphic solution* $\varphi(z)$ *in* U *if and only if either*

(i) $g(0) \neq 1$,

or

(ii) $g(0) = 1$ *and* $h(0) = 0$.

In case (i) *the solution* $\varphi(z)$ *is unique and in case* (ii) *it depends on the parameter* $b = \varphi(0)$. *This solution is analytic at the origin.*

THEOREM 2. *Suppose that hypotheses* (I), (II), (A) *are fulfilled and* $c = 0$. *Then equation* (1) *has no meromorphic solutions in* U *except for the cases covered by Lemmas* (II) *and* (III).

Proof. If $\varphi(z)$ is a meromorphic solution of equation (1), then by Lemma 1 and Theorem II it must have form (10) or (12). Since equations (11) and (13) have form (2), we can extend Φ and Φ^* by Lemma I to meromorphic solution of equation (1). This completes the proof.

The problem of finding the meromorphic solutions of equation (1) in cases (C), (D), (E) can be reduced to case (A) or (B).

For example consider case (C). We may write

$$(15) \qquad h(z) = \frac{H(z)}{z^q}; \quad H(0) \neq 0; \quad q \in N,$$

where $H(z)$ is analytic at the origin. In this case every meromorphic solution $\varphi(z)$ of equation (1) (if it exists) must have form (14) (r is to be determined). Inserting (7), (14) and (15) into equation (1), we obtain the equation

$$\frac{\Phi[f(z)]}{z^{kr}[F(z)]^r} - g(z)\frac{\Phi(z)}{z^r} = \frac{H(z)}{z^q}.$$

Hence we get $kr = q$. Thus q must be divisible by k. If this condition is fulfilled, then we obtain for $\Phi(z)$ equation (6), where

$$g^*(z) = z^{q-q/k}[F(z)]^{q/k}g(z); \qquad h^*(z) = [F(z)]^{q/k}H(z).$$

Since $g^*(0) = 0$ and $h^*(z)$ is analytic at the origin, we can apply Theorem 2. Thus case (C) is reduced to case (B).

We proceed analogously in cases (D) and (E).

2. In this section we prove the following

THEOREM 3. *Suppose that $g(z)$ and $h(z)$ are analytic functions in $\overline{U}$, where $U = \{z : |z| < 1\}$ and $g(1) = 1$, $h(1) \neq 0$. If $\varphi(z)$ is a meromorphic solution of equation*

$$\varphi(z^k) - g(z)\varphi(z) = h(z) \tag{16}$$

in U, then the boundary ∂U of the disc U is the natural boundary of the function $\varphi(z)$.

Proof. Let us note that $\varphi(z)$ cannot be continued analitically through the point $z = 1$. Indeed, if $\varphi(z)$ is analytic at the point $z = 1$, then by (16) we obtain $\varphi(1) - g(1)\varphi(1) = h(1)$, i.e. $h(1) = 0$. This is a contradiction. Hence and by (16) it follows that $\varphi(z)$ cannot be continued through any point ϑ such that $\vartheta^{k^n} = 1$ for some $n \in N$. Since the set of such numbers ϑ is dense in ∂U, Theorem 3 is proved.

EXAMPLE. The function $\varphi(z) = \sum_{n=0}^{\infty} z^{k^n}$ satisfies the equation

$$\varphi(z^k) - \varphi(z) = -z, \qquad k \geqslant 2.$$

By Theorem 3 the set $\{z : |z| = 1\}$ is the natural boundary of the function $\varphi(z)$.

THEOREM 4. *Suppose that $h(z)$ is an analytic function in $\overline{U}$, where $U = \{z : |z| < 1\}$, and $h'(1) \neq 0$. If $\varphi(z)$ is an analytic solution of equation*

$$\varphi(z^k) - k\varphi(z) = h(z), \qquad k \in N \tag{17}$$

in U, then the boundary ∂U of the disc U is the natural boundary of the function $\varphi(z)$.

Proof. Differentiating (17) we obtain the following functional equation for $\varphi'(z)$:

$$k\varphi'(z^k)z^{k-1} - k\varphi'(z) = h'(z).$$

It is easy to see that $\varphi'(z)$ cannot be continued through any point ϑ such that $\vartheta^{k^n} = 1$ for some $n \in N$.

References

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[4] J. Matkowski, *On meromorphic solutions of a functional equation*, ibidem 22 (1969), p. 303-311.

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