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EXPONENTIAL APPROXIMATION ON THE REAL LINE

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1. Introduction

This paper is devoted to exponential approximation, in the norms and seminorms, of some real-valued functions and their derivatives on the real line $R = (-\infty, \infty)$. Theorems of Section 2 concern the functions of Bernoulli's type. Section 3 contains analogous results, but of more general character. They correspond to the Ganelius and author's theorems about one-sided trigonometric approximation, announced in [2] and [7]. We adopt the following convenient notation.

Given a finite $p \geqslant 1$, let $L^p(a, b)$ be the space of all complex-valued functions Lebesgue-integrable with pth power on the interval (a, b). Denote by $L^\infty(a, b)$ the space of all measurable functions essentially bounded on (a, b). As usual, the norm of a function $f \in L^p(a, b)$ is defined by

$$\|f\|_{L^p(a,b)} = \begin{cases} \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{x\in(a,b)} |f(x)| & \text{if } p = \infty. \end{cases}$$

We write $L^p(R)$ or L^p instead of $L^p(-\infty, \infty)$. Moreover, by convention, $L = L^1$.

Let L^p_{loc} be the class of all complex-valued functions belonging to every space $L^p(a, b)$, with finite a, b ($a < b$). Denote by AC^m_{loc} the class of complex-valued functions f having the derivative $f^{(m)}$ absolutely continuous on each finite interval $\langle a, b\rangle$. Write AC_{loc} instead of AC^0_{loc}.

For any function $f \in L^p_{\mathrm{loc}}$, the limit

$$\|f\|_p = \lim_{\substack{a\to-\infty \\ b\to\infty}} \|f\|_{L^p(a,b)}$$

is finite or infinite. In the case of $f \in L^p(R)$, $\|f\|_p = \|f\|_{L^p(R)} < \infty$.

Given a (complex-valued) function f bounded on each finite interval $I = \langle a, b\rangle$, let us define the pth power variation of f on I as

$$V_p(f; I) = \sup_{\Pi} \Big\{\sum_{j=0}^{n-1} |f(t_{j+1})-f(t_j)|^p\Big\}^{1/p} \quad (p>0),$$

where Π denotes the partition $\{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$. Write

$$V_p(f) = V_p(f; R) = \sup_{I} V_p(f; I) \quad (I \subset R).$$

This is our basic seminorm.

Consider, now, a (complex-valued) function f measurable and bounded on finite intervals of R. Introduce its local modulus of continuity

$$\omega[\delta; f](x) = \sup_{u,v \in I_\delta(x)} |f(u)-f(v)|, \quad I_\delta(x) = \langle x-\delta/2, x+\delta/2\rangle,$$

and average modulus of continuity

$$\tau(\delta; f)_p = \|\omega[\delta; f]\|_p \quad (0 \leqslant \delta < \infty, 1 \leqslant p \leqslant \infty).$$

Evidently $\tau(0; f)_p = 0$ always. If δ is a positive number, $\tau(\delta; f)_p$ may be finite or infinite (for existence and basic properties see [4], Sect. 1.3 and [6], Sect. 2).

Let E_σ be the class of all entire functions of exponential type, of order σ at most. Denote by $B_{\sigma,p}$ the class of these functions $F \in E_\sigma$ which belong also to $L^p(R)$.

Consider, next, a function f of class L^p_{loc}. Denote by $H_{\sigma,p}(f)$ the set of all functions $G \in E_\sigma$, such that $f - G \in L^p(R)$. Then, the quantity

$$A_\sigma(f)_p = \begin{cases} \inf\limits_{S \in H_{\sigma,p}(f)} \|f-S\|_p & \text{if } H_{\sigma,p}(f) \text{ is not empty,} \\ \infty & \text{otherwise} \end{cases}$$

is called the *best exponential approximation* of f by entire functions belonging to $H_{\sigma,p}(f)$. In the case of an arbitrary real-valued f, the set of all entire functions $P \in E_\sigma$ [resp. $Q \in E_\sigma$], real-valued on R and such that

1) $P(x) \geqslant f(x)$ $[Q(x) \leqslant f(x)]$ for all real x,
2) $P - f \in L^p(R)$ $[f - Q \in L^p(R)]$

is signified by $H^+_{\sigma,p}(f)$ $[H^-_{\sigma,p}(f)]$. The best one-sided exponential approximations of f are defined by

$$A^\pm_\sigma(f)_p = \begin{cases} \inf\limits_{S \in H^\pm_{\sigma,p}(f)} \|f-S\|_p & \text{if } H^\pm_{\sigma,p}(f) \text{ is not empty,} \\ \infty & \text{otherwise.} \end{cases}$$

The symbols c_k [resp. $c_l(r, \ldots)$] $(k, l \in N)$, occurring in particular formulae, mean some positive absolute constants [positive numbers depending only on the indicated parameters $r, \ldots$].

2. Approximation of Bernoulli's functions

Given a positive number c and a positive integer r, let ϱ be an even real-valued function continuous with its derivatives ϱ', ϱ'', ϱ''' on R, satisfying the conditions

1) $\varrho(0) = \varrho'(0) = \varrho''(0) = 0$,
2) $\varrho''(t) = o(t^{r+1})$ and $\varrho'''(t) = O(t^r)$ as $t \to 0+$,
3) $\varrho(t) = 1$ for all $t \geqslant c$.

Introduce the integral analogues of Bernoulli's functions:

$$\Phi_k(x) = \frac{1}{2\pi} \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} \frac{\varrho(t)}{(it)^k} e^{itx}\, dt \qquad (x \in R,\ k = 1, 2, \ldots, r).$$

It is easy to see that

$$\Phi_1(x) = \frac{1}{\pi} \left\{ \int_0^c \frac{\varrho(t)}{t} \sin tx\, dt + \int_c^{\to\infty} \frac{1}{t} \sin tx\, dt \right\} \qquad (x \in R). \tag{1}$$

Therefore, Φ_1 is a real-valued odd function continuous on $(0, \infty)$, for which

$$\Phi_1(0) = 0, \qquad \Phi_1(0+) = 1/2, \qquad \Phi_1(0-) = -1/2.$$

Under the assumption $x > 0$, the partial integrations lead to

(2) $\Phi_1(x)$

$$= \frac{-1}{\pi x^3} \left\{ \int_0^c \left(\frac{-6}{t^4} \varrho(t) + \frac{6}{t^3} \varrho'(t) - \frac{3}{t^2} \varrho''(t) + \frac{1}{t} \varrho'''(t) \right) \cos tx\, dt - 6 \int_c^{\infty} \frac{\cos tx}{t^4}\, dt \right\};$$

whence

$$\Phi_1(x) = O(x^{-3}) \qquad \text{as } x \to \infty.$$

For $x > 0$, the formula (1) yields

$$\Phi_1'(x) = \frac{1}{\pi} \left\{ \int_0^c \varrho(t) \cos tx\, dt - \frac{\sin cx}{x} \right\} \qquad \text{when } x \neq 0.$$

This implies that Φ_1' is continuous and bounded on $(0, \infty)$. Applying the representation (2), we obtain

$$\Phi_1'(x) = O(x^{-3}) \qquad \text{as } x \to \infty.$$

In the case $2 \leqslant k \leqslant r$, the Bernoulli functions Φ_k are real-valued, even [resp. odd] if k is even [odd]. Moreover, Φ_k are continuous on R and

$$\Phi_k(x) = O(x^{-3}) \qquad \text{as } x \to \infty.$$

Also, it can easily be observed that

$$\Phi_2'(x) = \Phi_1(x) \qquad \text{for all real } x \neq 0,$$

$$\Phi_3'(x) = \Phi_2(x), \ldots, \Phi_r'(x) = \Phi_{r-1}(x) \quad \text{for all } x \in R,$$

whenever $r \geqslant 3$. Consequently, for $m = 1, 2, \ldots, r-1$,

$$\Phi_{m+1}(x) = \int_{-\infty}^{x} \Phi_m(s)\,ds \quad (-\infty < x < \infty).$$

Now, the well-known result ([1], Sect. 101), connected with Bernoulli's functions, to one-sided and ordinary exponential approximations in L^p-metrics will be extended.

THEOREM 1. *Given a finite $\sigma \geqslant c$, there are two entire functions P_σ, Q_σ such that*

(i) $P_\sigma \in H^+_{\sigma,1}(\Phi_1)$, $Q_\sigma \in H^-_{\sigma,1}(\Phi_1)$
and, for every $p \geqslant 1$,
(ii) $\|P_\sigma - \Phi_1\|_p \leqslant c_1 \sigma^{-1/p}$, $\|\Phi_1 - Q_\sigma\|_p \leqslant c_1 \sigma^{-1/p}$,
(iii) $\|P_\sigma' - \Phi_1'\|_p \leqslant c_2 \sigma^{1-1/p}$, $\|\Phi_1' - Q_\sigma'\|_p \leqslant c_2 \sigma^{1-1/p}$
(in the case $p = \infty$, the symbol $1/p$ should be treated as zero).

Proof. Let us consider the entire functions P_σ, Q_σ defined by

$$P_\sigma(z) = G_\sigma(z) + \frac{45\pi^2}{8} D_\sigma^+(z), \qquad Q_\sigma(z) = G_\sigma(z) - \frac{45\pi^2}{8} D_\sigma^-(z),$$

where

$$G_\sigma(z) = \sum_{k=-\infty}^{\infty} \Phi_1(x_k) \frac{\sin\sigma(z-x_k)}{\sigma(z-x_k)}, \qquad x_k = \frac{k\pi}{\sigma}$$

and

$$D_\sigma^\pm(z) = \left\{\sin\frac{\sigma}{2}\left(z \pm \frac{\pi}{2\sigma}\right)\right\}^2 \left\{\frac{\sigma}{2}\left(z \pm \frac{\pi}{2\sigma}\right)\right\}^{-2} \qquad (z = x + iy).$$

As is known ([5], Sect. 2), these P_σ, Q_σ satisfy the relations of (i) and the inequalities of (ii) for $p = 1$. Assuming that $1 \leqslant p < \infty$, we will deduce the first estimates of (ii) and (iii), successively.

Let us start with the identities:

$$P_\sigma(x) - \Phi_1(x) = \frac{45\pi^2}{8} D_\sigma^+(x) - \{\Phi_1(x) - G_\sigma(x)\}, \tag{3}$$

$$\Phi_1(x) - G_\sigma(x) = \frac{1}{\pi} \int_{\sigma}^{\to\infty} \frac{\sin tx}{t}\,dt + \frac{2}{\pi} \int_0^{\pi} \sum_{k=1}^{\infty} \frac{t\sin tx}{(2k\sigma)^2 - t^2}\,dt, \tag{4}$$

$$\Phi_1(x) - G_\sigma(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \int_0^{\to\infty} \frac{\sin\sigma x \cos tx}{t + (2k+1)\sigma}\,dt \tag{5}$$

(see [1], Sects. 87–88 and 100–101).

Clearly,

$$\int_{-\infty}^{\infty} |D_\sigma^+(x)|^p\,dx = \frac{4}{\sigma}\int_0^\infty \left|\frac{\sin u}{u}\right|^{2p} du \leqslant \frac{4}{\sigma}\int_0^\infty \left(\frac{\sin u}{u}\right)^2 du.$$

Consequently,

$$\|D_\sigma^+\|_p \leqslant (2\pi)^{1/p}\sigma^{-1/p} \tag{6}$$

and, by the inequality of Bernstein's type,

$$\|D_\sigma^{+\prime}\|_p \leqslant (2\pi)^{1/p}\sigma^{1-1/p}. \tag{7}$$

Therefore, it is enough to estimate similarly the difference $\Omega(x) = \Phi_1(x) - G_\sigma(x)$ and its derivative. Evidently, putting

$$A(x) = \int_\sigma^{\to\infty} \frac{\sin tx}{t}\,dt, \quad B(x) = \int_0^\sigma \sum_{k=1}^\infty \frac{t\sin tx}{(2k\sigma)^2 - t^2}\,dt$$

and

$$F(x) = \sum_{k=0}^\infty \int_0^{\to\infty} \frac{\cos tx}{t+(2k+1)\sigma}\,dt \qquad (x \neq 0),$$

we have (see (4)–(5))

$$\Omega(x) = \frac{1}{\pi}A(x) + \frac{2}{\pi}B(x) = \frac{2}{\pi}F(x)\sin\sigma x \qquad (x \neq 0). \tag{8}$$

It can easily be verified that, for $x \neq 0$,

$$A(x) = \int_{\sigma x}^{\to\infty} \frac{\sin u}{u}\,du, \quad A'(x) = -\frac{\sin\sigma x}{x},$$

$$B'(x) = \int_0^\sigma \sum_{k=1}^\infty \frac{t^2\cos tx}{(2k\sigma)^2 - t^2}\,dt,$$

$$F(x) = \sum_{k=0}^\infty \sum_{j=0}^\infty (-1)^j \int_{j\pi}^{(j+1)\pi} \frac{|\sin u|}{\{u+(2k+1)\sigma x\}^2}\,du,$$

and

$$F'(x) = \sum_{k=0}^\infty \sum_{j=0}^\infty (-1)^{j+1} \int_{j\pi}^{(j+1)\pi} \frac{(4k+2)\sigma|\sin u|}{\{u+(2k+1)\sigma x\}^3}\,du;$$

whence (see [5], Sect. 2),

$$0 < F(x) < 5/(\sigma x)^2, \quad 0 < -F'(x) < 10/(\sigma^2 x^3).$$

Write ξ instead of $2^{-1}x_1 = \pi/(2\sigma)$, and observe that

$$A(x) = \frac{\pi}{2} + \int_0^x A'(u)\,du = \frac{\pi}{2} - \int_0^x \frac{\sin\sigma u}{u}\,du \qquad (x \geqslant 0).$$

This identity, together with the obvious estimate

$$\left|\int_0^x \frac{\sin \sigma u}{u}\, du\right| \leqslant \frac{\pi}{2} \qquad (0 \leqslant x \leqslant \xi),$$

gives

$$\left\{\int_0^\xi |A(x)|^p\, dx\right\}^{1/p} \leqslant \pi \left(\frac{\pi}{2\sigma}\right)^{1/p}. \tag{9}$$

By Minkowski's inequalities,

$$\begin{aligned}\left\{\int_0^\xi |B(x)|^p\, dx\right\}^{1/p} &\leqslant \int_0^\sigma \left\{\int_0^\xi \left|\sum_{k=1}^\infty \frac{t \sin tx}{(2k\sigma)^2 - t^2}\right|^p dx\right\}^{1/p} dt \\ &\leqslant \int_0^\sigma \sum_{k=1}^\infty \left\{\int_0^\xi \left|\frac{t \sin tx}{(2k\sigma)^2 - t^2}\right|^p dx\right\}^{1/p} dt \\ &\leqslant \int_0^\sigma \sum_{k=1}^\infty \frac{t}{(2k\sigma)^2 - t^2}\, \xi^{1/p}\, dt \\ &\leqslant \left(\frac{\pi}{2\sigma}\right)^{1/p} \frac{1}{\sigma} \int_0^\sigma \sum_{k=1}^\infty \frac{1}{4k^2 - 1}\, dt \\ &\leqslant \frac{1}{3}\left(\frac{\pi}{2\sigma}\right)^{1/p} \sum_{k=1}^\infty \frac{1}{k^2},\end{aligned}$$

i.e.,

$$\left\{\int_0^\xi |B(x)|^p\, dx\right\}^{1/p} \leqslant \frac{\pi^2}{18}\left(\frac{\pi}{2\sigma}\right)^{1/p}. \tag{10}$$

Further, setting $C(x) = F(x) \sin \sigma x$, we get

$$\begin{aligned}\int_\xi^\infty |C(x)|^p\, dx &\leqslant \int_\xi^\infty \left|\frac{5}{(\sigma x)^2} \sin \sigma x\right|^p dx \\ &= \int_\xi^{x_1} \left|\frac{5}{(\sigma x)^2} \sin \sigma x\right|^p dx + \sum_{\mu=1}^\infty \int_{x_\mu}^{x_{\mu+1}} \left|\frac{5}{(\sigma x)^2} \sin \sigma x\right|^p dx \\ &\leqslant \frac{5^p}{(\sigma\xi)^{2p}} \int_{\pi/2}^{\pi} |\sin u|^p \frac{du}{\sigma} + \sum_{\mu=1}^\infty \frac{5^p}{(\sigma x_\mu)^{2p}} \int_{\mu\pi}^{(\mu+1)\pi} |\sin u|^p \frac{du}{\sigma} \\ &= \frac{5^p}{\pi^{2p}\sigma}(2^{2p} + 2) \sum_{\mu=1}^\infty \frac{1}{\mu^{2p}};\end{aligned}$$

whence

$$\left\{\int_\xi^\infty |C(x)|^p\, dx\right\}^{1/p} \leqslant \frac{40}{(6\sigma)^{1/p}}. \tag{11}$$

From (3), (6), (8)–(11) and the identity

$$\int_{-\infty}^{\infty} |\Omega(x)|^p\, dx = 2\Big(\int_0^{\xi} + \int_{\xi}^{\infty}\Big)|\Omega(x)|^p\, dx$$

the first part of (ii) follows. The second one can be obtained parallelly.

Passing to Ω', we observe that

$$\int_0^{\xi} |A'(x)|^p\, dx = \sigma^{p-1} \int_0^{\pi/2} \left|\frac{\sin u}{u}\right|^p du \qquad (\xi = \pi/2\sigma).$$

Hence

$$\Big\{\int_0^{\xi} |A'(x)|^p\, dx\Big\}^{1/p} \leqslant \left(\frac{\pi}{2}\right)^{1/p} \sigma^{1-1/p}. \tag{12}$$

In view of Minkowski's inequalities,

$$\begin{aligned}\Big\{\int_0^{\xi} |B'(x)|^p\, dx\Big\}^{1/p} &\leqslant \int_0^{\sigma} \sum_{k=1}^{\infty} \left\{\int_0^{\xi}\left|\frac{t^2 \cos tx}{(2k\sigma)^2 - t^2}\right|^p dx\right\}^{1/p} dt \\ &\leqslant \xi^{1/p} \int_0^{\sigma} \sum_{k=1}^{\infty} \frac{1}{4k^2-1}\, dt \\ &\leqslant \frac{1}{3}\xi^{1/p}\sigma \sum_{k=1}^{\infty}\frac{1}{k^2}.\end{aligned}$$

Consequently,

$$\Big\{\int_0^{\xi} |B'(x)|^p\, dx\Big\}^{1/p} \leqslant \frac{\pi^2}{18}\left(\frac{\pi}{2}\right)^{1/p} \sigma^{1-1/p}. \tag{13}$$

The derivative $C'(x)$ can be estimated (in absolute value) by

$$|F(x)\,\sigma \cos \sigma x| + |F'(x)| \qquad (x > 0).$$

Therefore

$$\begin{aligned}&\int_{\xi}^{\infty} |C'(x)|^p\, dx \\ &\leqslant |F(\xi)|^p \int_{\xi}^{x_1} |\sigma \cos \sigma x|^p\, dx + \sum_{\mu=1}^{\infty} |F(x_\mu)|^p \int_{x_\mu}^{x_{\mu+1}} |\sigma \cos \sigma x|^p\, dx + \int_{\xi}^{\infty} |F'(x)|^p\, dx \\ &\leqslant \sigma^{p-1}\left\{\left(\frac{5}{(\sigma\xi)^2}\right)^p \int_{\pi/2}^{\pi} |\cos u|^p\, du \right. \\ &\qquad\qquad \left. + \sum_{\mu=1}^{\infty}\left|\frac{5}{(\sigma x_\mu)^2}\right|^p \int_{\mu\pi}^{(\mu+1)\pi} |\cos u|^p\, du\right\} + \int_{\xi}^{\infty}\left(\frac{10}{\sigma^2 x^3}\right)^p dx.\end{aligned}$$

Thus,

(14) $$\Big\{\int_{\xi}^{\infty} |C'(x)|^p\,dx\Big\}^{1/p} \leqslant 5\left(\frac{12}{\pi^2}+\frac{1}{3}\right)\sigma^{1-1/p}.$$

The identities (3), (8) together with (7) and (12)–(14) lead to the assertion (iii), immediately.

Remark 1. From (iii) it follows that

$$V_p(P_\sigma-\Phi_1) \leqslant V_1(P_\sigma-\Phi_1) \leqslant 1+c_2.$$

Evidently, the same estimates with Q_σ instead of P_σ are also true.

Remark 2. The assertion (ii) leads to the inequality

(15) $$A_\sigma^{\pm}(f)_p \leqslant \frac{c_3}{\sigma} A_\sigma(f')_p \qquad (0<\sigma<\infty,\ 1\leqslant p<\infty)$$

for each real-valued function $f \in AC_{\text{loc}}$ with $f' \in L^p_{\text{loc}}$ (Th. 3.2 of [5]).

THEOREM 2. *Given a number* $\sigma \geqslant c$, *a positive integer* $n \leqslant r$ *and an arbitrary* $p \geqslant 1$, *there exist entire functions* $P_{\sigma,n}$, $Q_{\sigma,n}$ *such that*

(i) $P_{\sigma,n} \in H^{+}_{\sigma,p}(\Phi_n)$, $Q_{\sigma,n} \in H^{-}_{\sigma,p}(\Phi_n)$

and, for $\nu = 0, 1, \ldots, n$,

(ii) $$\|P^{(\nu)}_{\sigma,n}-\Phi^{(\nu)}_n\|_p \leqslant \frac{c_4(n,\nu)}{\sigma^{n-\nu-1+1/p}},\quad \|\Phi^{(\nu)}_n-Q^{(\nu)}_{\sigma,n}\|_p \leqslant \frac{c_4(n,\nu)}{\sigma^{n-\nu-1+1/p}}.$$

Proof. In view of Theorem 1, we may suppose that $n \geqslant 2$.

Consider functions φ belonging to the space $L^p = L^p(R)$ $(1 \leqslant p \leqslant \infty)$. Introduce the singular integral

(16) $$W[\varphi](z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(t)K_\sigma(z-t)\,dt \qquad (z = x+iy),$$

with

$$K_\sigma(\zeta) = (\cos\sigma\zeta - \cos 2\sigma\zeta)/(\sigma\zeta^2) \qquad (0<\sigma<\infty).$$

Clearly, $K_\sigma \in B_{2\sigma,1}$.

As is well known, $W[\varphi] \in E_{2\sigma}$ and, in the case of $\varphi \in B_{\sigma,p}$,

(17) $$W[\varphi](x) = \varphi(x) \qquad (x \in R).$$

The estimate

$$\|K_\sigma\|_1 \leqslant c_5\pi \qquad (c_5 = (4+2\log 3)/\pi)$$

implies

(18) $$\|W[\varphi]\|_p \leqslant c_5\|\varphi\|_p,$$

i.e., $W[\varphi] \in L^p$ (see [1], Sect. 106).

Write

$$\|\varphi\| = \|\varphi\|_p, \quad A_\sigma(\varphi) = A_\sigma(\varphi)_p, \quad A_\sigma^+(\varphi) = A_\sigma^+(\varphi)_p, \quad \Phi = \Phi_n.$$

Denote by $S[\varphi]$ the entire function of class $B_{\sigma,p}$ corresponding to a function $\varphi \in L^p$ for which

$$\|\varphi - S[\varphi]\| = A_\sigma(\varphi). \tag{19}$$

Choose the entire function $P \in H_{\sigma,p}^+(\Phi)$, such that

$$\|P - \Phi\| \leqslant 2A_\sigma^+(\Phi). \tag{20}$$

Supposing that $0 \leqslant \nu \leqslant n$ and using the operator (16), we have

$$\begin{aligned}\|\Phi^{(\nu)} - P^{(\nu)}\| &\leqslant \|\Phi^{(\nu)} - W[\Phi^{(\nu)}]\| + \|S^{(\nu)}[W[\Phi]] - P^{(\nu)}\| \\ &\quad + \|W[\Phi^{(\nu)}] - S^{(\nu)}[W[\Phi]]\| \\ &= M_1 + M_2 + M_3, \text{ say.}\end{aligned}$$

By (17), (19) and (18),

$$M_1 \leqslant \|\Phi^{(\nu)} - S[\Phi^{(\nu)}]\| + \|W[S[\Phi^{(\nu)}] - \Phi^{(\nu)}]\| \leqslant A_\sigma(\Phi^{(\nu)}) + c_5 A_\sigma(\Phi^{(\nu)}).$$

The Bernstein type inequality and (19), (17), (20) give

$$\begin{aligned}M_2 &\leqslant \sigma^\nu \|S[W[\Phi]] - P\| \\ &\leqslant \sigma^\nu \{\|S[W[\Phi]] - W[\Phi]\| + \|W[\Phi] - S[\Phi]\| + \|S[\Phi] - \Phi\| + \|\Phi - P\|\} \\ &\leqslant \sigma^\nu \{A_\sigma(W[\Phi]) + (c_5+1)A_\sigma(\Phi) + 2A_\sigma^+(\Phi)\}.\end{aligned}$$

Under the assumption $\nu \leqslant n-1$,

$$W[\Phi^{(\nu)}](x) = W^{(\nu)}[\Phi](x) \quad \text{for all real } x.$$

Applying, in this case, the Bernstein type inequality and (19), we get

$$M_3 \leqslant (2\sigma)^\nu \|W[\Phi] - S[W[\Phi]]\| = (2\sigma)^\nu A_\sigma(W[\Phi]).$$

But

$$A_\sigma(W[\Phi]) \leqslant \|W[\Phi] - S[\Phi]\| = \|W[\Phi - S[\Phi]]\| \leqslant c_5 A_\sigma(\Phi),$$

by (17)–(19). Consequently,

$$M_3 \leqslant c_5 (2\sigma)^\nu A_\sigma(\Phi).$$

Therefore, if $0 \leqslant \nu \leqslant n-1$, the estimate (15) yields

$$\begin{aligned}M_1 &\leqslant (1+c_5)A_\sigma(\Phi^{(\nu)}) \leqslant (1+c_5)c_3^{n-1-\nu}\sigma^{\nu-n+1}A_\sigma(\Phi^{(n-1)}), \\ M_2 &\leqslant \sigma^\nu(2c_5+3)A_\sigma^+(\Phi) \leqslant 3(1+c_5)c_3^{n-1}\sigma^{\nu-n+1}A_\sigma(\Phi^{(n-1)}), \\ M_3 &\leqslant (2\sigma)^\nu c_5 c_3^{n-1}\sigma^{1-n}A_\sigma(\Phi^{(n-1)}) = 2^\nu c_5 c_3^{n-1}\sigma^{\nu-n+1}A_\sigma(\Phi^{(n-1)}).\end{aligned}$$

Since (see Theorem 1) $A_\sigma(\Phi^{(n-1)}) = A_\sigma(\Phi_1) \leq c_1 \sigma^{-1/p}$ and

$$\|P^{(n)} - \Phi^{(n)}\| \leq (c_1 + c_2)\sigma^{1-1/p} + \sigma \|P^{(n-1)} - \Phi^{(n-1)}\|,$$

the first parts of the assertions (i) and (ii), with $P_{\sigma,n} = P$, are established. The second parts can be obtained parallelly.

3. General theorems

Let us start with the following

LEMMA. *Let $\psi \in AC_{\text{loc}} \cap L$ and let $\psi' \in L^p$ for some finite or infinite $p \geq 1$. Then, for the operator W defined by* (16),

$$W'[\psi](x) = W[\psi'](x) \quad (x \in R).$$

Proof. The Lebesgue dominated convergence theorem ensures that

$$\pi W'[\psi](x) = \int_{-\infty}^{\infty} \psi(t) K'_\sigma(x-t)\,dt = \int_{-\infty}^{\infty} \psi(x-u) K'_\sigma(u)\,du \quad (x \in R).$$

Since

$$|\psi(u) - \psi(0)| = \Big|\int_0^u \psi'(t)\,dt\Big| \leq \|\psi'\|_p |u|^{1/q} \quad (p+q = pq),$$

we have

$$\psi(u) = O(u) \quad \text{as } u \to \pm\infty.$$

Therefore, by partial integration,

$$\pi W'[\psi](x) = \int_{-\infty}^{\infty} \psi(x-u) K'_\sigma(u)\,du = \int_{-\infty}^{\infty} K_\sigma(u)\psi'(x-u)\,du = \pi W[\psi'](x).$$

PROPOSITION. *Let ψ be as in the Lemma. Suppose that, for some entire function G of class E_σ $(\sigma > 0)$, the estimate*

$$\|G - \psi\|_p \leq c_6 \sigma^{-1} A_\sigma(\psi')_p \quad (1 \leq p \leq \infty)$$

holds. Then

$$\|G' - \psi'\|_p \leq c_7 A_\sigma(\psi')_p.$$

Proof. Consider functions $g \in L^p_{\text{loc}}$ and the operator W defined by (16). Denote by $S[g]$ the entire function of class E_σ, such that

$$\|g - S[g]\|_p \leq 2A_\sigma(g)_p. \tag{21}$$

Writing $\|\ \|$ instead of $\|\ \|_p$, we have

$$\|\psi'-G'\| \leqslant \|\psi'-W[\psi']\|+\|S'[W[\psi]]-G'\|+\|W[\psi']-S'[W[\psi]]\|$$
$$= N_1+N_2+N_3, \text{ say.}$$

The identity (17), together with estimates (21) and (18), gives

$$N_1 \leqslant \|\psi'-S[\psi']\|+\|W[S[\psi']-\psi']\| \leqslant 2(1+c_5)A_\sigma(\psi')_p.$$

By the Bernstein inequality and our Lemma (see also (21) and (18)),

$$N_2 \leqslant \sigma\|S[W[\psi]]-G\|$$
$$\leqslant \sigma\{\|S[W[\psi]]-W[\psi]\|+\|W[\psi]-S[\psi]\|+\|S[\psi]-\psi\|+\|\psi-G\|\}$$
$$\leqslant \sigma\{2A_\sigma(W[\psi])_p+(2c_5+2)A_\sigma(\psi)_p+c_6\sigma^{-1}A_\sigma(\psi')_p\},$$

$$N_3 \leqslant 2\sigma\|W[\psi]-S[W[\psi]]\| \leqslant 4\sigma A_\sigma(W[\psi])_p.$$

Observing that $A_\sigma(W[\psi])_p \leqslant c_5 A_\sigma(\psi)_p$ (see the proof of Theorem 2) and applying the estimate like (15), we get at once the desired assertion.

Now, the main results will be presented.

THEOREM 3. *Let f be a real-valued function of class AC_{loc}^{r-1} ($r\in N$) having (a.e.) the derivative $f^{(r)}\in L_{\text{loc}}^p$ ($1\leqslant p<\infty$), and let $A_c(f^{(r)})_p<\infty$ for some positive number c. Then, for every finite $\sigma\geqslant c$, there exist entire functions T_σ, U_σ such that*

(i) $T_\sigma\in H_{\sigma,p}^+(f)$, $U_\sigma\in H_{\sigma,p}^-(f)$

and, for $\nu=0, 1, \ldots, r-1$,

(ii) $\|T_\sigma^{(\nu)}-f^{(\nu)}\|_p\leqslant \dfrac{c_8(r,\nu)}{\sigma^{r-\nu}}A_\sigma(f^{(r)})_p$, $\|f^{(\nu)}-U_\sigma^{(\nu)}\|_p\leqslant \dfrac{c_8(r,\nu)}{\sigma^{r-\nu}}A_\sigma(f^{(r)})_p$.

If, in addition, $f^{(r-1)}\in L$ and $f^{(r)}\in L^p$, then

(iii) $\|T_\sigma^{(r)}-f^{(r)}\|_p\leqslant c_9(r)A_\sigma(f^{(r)})_p$, $\|f^{(r)}-U_\sigma^{(r)}\|_p\leqslant c_9(r)A_\sigma(f^{(r)})_p$.

Proof. Let g_σ be a function of class E_σ, real-valued on R, satisfying the inequality

$$\|f^{(r)}-g_\sigma^{(r)}\|_p\leqslant 2A_\sigma(f^{(r)})_p.$$

Retain the symbols Φ_n, $P_{\sigma,n}$, $Q_{\sigma,n}$ used in Theorem 2 with $p=1$.

By the well-known theorem ([1], Sects. 101–102, [3], pp. 113–116), for every $x\in R$,

$$f(x)-g_\sigma(x)=A_c(x)+\int_{-\infty}^{\infty}\{f^{(r)}(t)-g_\sigma^{(r)}(t)\}\Phi_r(x-t)\,dt \qquad (\sigma\geqslant c),$$

where Λ_c denotes some entire function of class E_c. From this identity it follows that

$$f^{(\nu)}(x)-g_\sigma^{(\nu)}(x) = \Lambda_c^{(\nu)}(x)+\int_{-\infty}^{\infty}\{f^{(r)}(t)-g_\sigma^{(r)}(t)\}\,\Phi_r^{(\nu)}(x-t)\,dt$$

for $\nu = 0, 1, \ldots, r-1$ and all real x. Therefore, putting

$$h_r^+(t) = \tfrac{1}{2}\{|f^{(r)}(t)-g_\sigma^{(r)}(t)|+f^{(r)}(t)-g_\sigma^{(r)}(t)\},$$

$$h_r^-(t) = \tfrac{1}{2}\{|f^{(r)}(t)-g_\sigma^{(r)}(t)|-f^{(r)}(t)+g_\sigma^{(r)}(t)\},$$

we can write

$$f^{(\nu)}(x) = g_\sigma^{(\nu)}(x)+\Lambda_c^{(\nu)}(x)+\int_{-\infty}^{\infty} h_r^+(t)\,\Phi_r^{(\nu)}(x-t)\,dt-\int_{-\infty}^{\infty} h_r^-(t)\,\Phi_r^{(\nu)}(x-t)\,dt.$$

Introduce the function of a complex variable z:

$$T_\sigma(z) = g_\sigma(z)+\Lambda_c(z)+\int_{-\infty}^{\infty} h_r^+(t)\,P_{\sigma,r}(z-t)\,dt-\int_{-\infty}^{\infty} h_r^-(t)\,Q_{\sigma,r}(z-t)\,dt.$$

It is easy to show (see [3], pp. 137, 308–309) that $T_\sigma\in E_\sigma$ and that, for $\nu\in\langle 0, r-1\rangle$,

$$(22)\qquad T_\sigma^{(\nu)}(x)-f^{(\nu)}(x) = \int_{-\infty}^{\infty} h_r^+(t)\{P_{\sigma,r}^{(\nu)}(x-t)-\Phi_r^{(\nu)}(x-t)\}\,dt$$

$$+\int_{-\infty}^{\infty} h_r^-(t)\{\Phi_r^{(\nu)}(x-t)-Q_{\sigma,r}^{(\nu)}(x-t)\}\,dt \qquad (x\in R).$$

In particular, by Theorem 2 (i),

$$T_\sigma(x)\geqslant f(x)\quad \text{for all real } x.$$

Applying to (22) the Minkowski inequalities and Theorem 2 (ii), we obtain

$$\begin{aligned}\|T_\sigma^{(\nu)}-f^{(\nu)}\|_p &\leqslant \|h_r^+\|_p\|P_{\sigma,r}^{(\nu)}-\Phi_r^{(\nu)}\|_1+\|h_r^-\|_p\|\Phi_r^{(\nu)}-Q_{\sigma,r}^{(\nu)}\|_1\\ &\leqslant \|f^{(r)}-g_\sigma^{(r)}\|_p\cdot 2c_4(r,\nu)\,\sigma^{\nu-r}\\ &\leqslant 4c_4(r,\nu)\,\sigma^{\nu-r}A_\sigma(f^{(r)})_p,\end{aligned}$$

whenever $\nu\leqslant r-1$. Thus, the first parts of (i) and (ii) are proved. The second parts can be deduced parallelly.

The assertion (iii) follows at once from (ii) and Proposition given above.

THEOREM 4. *Let f be a real-valued function measurable and bounded on each finite interval. Suppose that, for some finite $p\geqslant 1$ and $\delta>0$, $\tau(\delta;f)_p<\infty$.*

Then, for every finite $\sigma > 0$, there are entire functions T_σ^*, U_σ^* such that

(i) $T_\sigma^* \in H_{\sigma,p}^+(f)$, $U_\sigma^* \in H_{\sigma,p}^-(f)$

and

(ii) $\|T_\sigma^* - f\|_p \leqslant c_{10}\tau(\sigma^{-1}; f)_p$, $\|f - U_\sigma^*\|_p \leqslant c_{10}\tau(\sigma^{-1}; f)_p$.

If, in addition, f is of bounded p-th power variation on R, then

(iii) $\|T_\sigma^* - f\|_p \leqslant c_{11}\sigma^{-1/p} V_p(f)$, $\|f - U_\sigma^*\|_p \leqslant c_{11}\sigma^{-1/p} V_p(f)$

and

(iv) $V_p(T_\sigma^* - f) \leqslant c_{12} V_p(f)$, $V_p(f - U_\sigma^*) \leqslant c_{12} V_p(f)$.

The proof of (i) and (ii) is given in Section 3 of [5]. The assertions (iii) and (iv) can be stated similarly to their trigonometric analogues (3.2), (3.3) in Theorem 3 of [7].

Remark 3. In the case of real-valued $f \in AC_{\mathrm{loc}}^{r-1}$ ($r \in N$) such that $f^{(r)} \in L_{\mathrm{loc}}^p \cap BV_p$ ($1 \leqslant p < \infty$), the entire functions T_σ, U_σ ($\sigma > 0$) determined in Theorem 3 satisfy the inequalities

$$\|T_\sigma^{(\nu)} - f^{(\nu)}\|_p + \|f^{(\nu)} - U_\sigma^{(\nu)}\|_p \leqslant 2c_8(r, \nu)c_{11}\sigma^{\nu - r - 1/p} V_p(f^{(r)}) \qquad (\nu = 0, 1, \ldots, r-1).$$

This is an immediate consequence of the last two theorems.

THEOREM 5. *Suppose that f is a real-valued function of class AC_{loc}^{r-1} ($r \in N$), with $f^{(r)} \in L^p \cap BV_p$ for some finite $p \geqslant 1$. Then, for every finite $\sigma > 0$, there exist entire functions $\tilde{T}_\sigma$, $\tilde{U}_\sigma$ such that*

(i) $\tilde{T}_\sigma \in H_{\sigma,p}^+(f)$, $\tilde{U}_\sigma \in H_{\sigma,p}^-(f)$

and, for $\nu = 0, 1, \ldots, r-1$,

(ii) $\|\tilde{T}_\sigma^{(\nu)} - f^{(\nu)}\|_p + \|f^{(\nu)} - \tilde{U}_\sigma^{(\nu)}\|_p \leqslant c_{13}(r, \nu)\sigma^{\nu - r - 1/p} V_p(f^{(r)})$,

(iii) $V_p(\tilde{T}_\sigma^{(\nu)} - f^{(\nu)}) + V_p(f^{(\nu)} - \tilde{U}_\sigma^{(\nu)}) \leqslant c_{14}(r, \nu)\sigma^{\nu - r} V_p(f^{(r)})$.

Proof. In view of Theorem 4, there is an entire function $T_{\sigma,r}^* \in B_{\sigma,p}$ ($\sigma > 0$), real-valued on R, satisfying the inequalities

$$(23) \qquad \|T_{\sigma,r}^* - f^{(r)}\|_p \leqslant c_{11}\sigma^{-1/p} V_p(f^{(r)}), \qquad V_p(T_{\sigma,r}^* - f^{(r)}) \leqslant c_{12} V_p(f^{(r)}).$$

Suppose further that $\sigma \geqslant c > 0$. Retain the symbols Φ_r, $P_{\sigma,r}$, $Q_{\sigma,r}$ used in the proof of Theorem 3 and start with the identities

$$f(x) = F_c(x) + \int_{-\infty}^{\infty} f^{(r)}(t)\Phi_r(x-t)\,dt$$

$$= F_c(x) + J_\sigma(x) + \int_{-\infty}^{\infty} \{f^{(r)}(t) - T_{\sigma,r}^*(t)\}\Phi_r(x-t)\,dt \qquad (x \in R),$$

where F_c means some entire function of class E_c, and

$$J_\sigma(z) = \int_{-\infty}^{\infty} \Phi_r(u)\, T^*_{\sigma,r}(z-u)\, du \qquad (z = x+iy,\ x,\ y \in R).$$

It is easily seen that $J_\sigma \in B_{\sigma,p}$.

Introduce the auxiliary function

$$g(x) = f(x) - F_c(x) - J_\sigma(x) = \int_{-\infty}^{\infty} \{f^{(r)}(t) - T^*_{\sigma,r}(t)\}\, \Phi_r(x-t)\, dt;$$

write

$$h_r^+(t) = \tfrac{1}{2}\{|f^{(r)}(t) - T^*_{\sigma,r}(t)| + f^{(r)}(t) - T^*_{\sigma,r}(t)\},$$
$$h_r^-(t) = \tfrac{1}{2}\{|f^{(r)}(t) - T^*_{\sigma,r}(t)| - f^{(r)}(t) + T^*_{\sigma,r}(t)\}.$$

Then

$$g(x) = \int_{-\infty}^{\infty} h_r^+(t)\, \Phi_r(x-t)\, dt - \int_{-\infty}^{\infty} h_r^-(t)\, \Phi_r(x-t)\, dt \qquad (x \in R).$$

Putting

$$Y_\sigma(z) = \int_{-\infty}^{\infty} h_r^+(t)\, P_{\sigma,r}(z-t)\, dt - \int_{-\infty}^{\infty} h_r^-(t)\, Q_{\sigma,r}(z-t)\, dt \qquad (z = x+iy),$$

we obtain

$$Y_\sigma(x) - g(x) = \int_{-\infty}^{\infty} h_r^+(t)\, \{P_{\sigma,r}(x-t) - \Phi_r(x-t)\}\, dt + \int_{-\infty}^{\infty} h_r^-(t)\, \{\Phi_r(x-t) - Q_{\sigma,r}(x-t)\}\, dt. \tag{24}$$

Evidently, $Y_\sigma \in B_{\sigma,p}$ and $Y_\sigma(x) \geqslant g(x)$ for all real x. Moreover, by Theorem 2,

$$\|Y_\sigma - g\|_p \leqslant \|h_r^+\|_p \|P_{\sigma,r} - \Phi_r\|_1 + \|h_r^-\|_p \|\Phi_r - Q_{\sigma,r}\|_1$$
$$\leqslant 2c_4(r,0)\, \sigma^{-r} \|f^{(r)} - T^*_{\sigma,r}\|_p,$$

i.e. (see (23)),

$$\|Y_\sigma - g\|_p \leqslant 2c_4(r,0)\, c_{11}\, \sigma^{-r-1/p}\, V_p(f^{(r)}). \tag{25}$$

Taking the entire function $\tilde{T}_\sigma$ defined by

$$\tilde{T}_\sigma(z) = F_c(z) + J_\sigma(z) + Y_\sigma(z),$$

we observe that

$$\tilde{T}_\sigma(x) - f(x) = Y_\sigma(x) - g(x).$$

Therefore $\tilde{T}_\sigma \in H^+_{\sigma,p}(f)$ and, in view of (25),

$$\|\tilde{T}_\sigma - f\|_p \leqslant 2c_4(r, 0)\, c_{11}\, \sigma^{-r-1/p}\, V_p(f^{(r)}). \tag{26}$$

Further, by Minkowski's inequalities and Theorem 2 (ii),

$$\begin{aligned} V_p(\tilde{T}_\sigma - f) &= V_p(Y_\sigma - g) \\ &\leqslant V_p(h_r^+)\, \|P_{\sigma,r} - \Phi_r\|_1 + V_p(h_r^-)\, \|\Phi_r - Q_{\sigma,r}\|_1 \\ &\leqslant V_p(f^{(r)} - T^*_{\sigma,r})\, 2c_4(r, 0)\, \sigma^{-r}; \end{aligned}$$

whence, in view of (23),

$$V_p(\tilde{T}_\sigma - f) \leqslant 2c_4(r, 0)\, c_{12}\, \sigma^{-r}\, V_p(f^{(r)}). \tag{27}$$

Analogously, we can construct the entire function $\tilde{U}_\sigma \in H^-_{\sigma,p}(f)$ having the same approximation properties as $\tilde{T}_\sigma$, determined by (26) and (27). Thus, for $\nu = 0$, the desired estimates of (ii) and (iii) are proved.

From the identity (24) it follows that

$$\begin{aligned} Y_\sigma^{(\nu)}(x) - g^{(\nu)}(x) = &\int_{-\infty}^{\infty} h_r^+(t)\, \{P_{\sigma,r}^{(\nu)}(x-t) - \Phi_r^{(\nu)}(x-t)\}\, dt \\ &+ \int_{-\infty}^{\infty} h_r^-(t)\, \{\Phi_r^{(\nu)}(x-t) - Q_{\sigma,r}^{(\nu)}(x-t)\}\, dt \qquad (1 \leqslant \nu \leqslant r-1) \end{aligned}$$

for all real x. Hence, by Minkowski's inequalities and Theorem 2,

$$\begin{aligned} \|Y_\sigma^{(\nu)} - g^{(\nu)}\|_p &\leqslant \|h_r^+\|_p\, \|P_{\sigma,r}^{(\nu)} - \Phi_r^{(\nu)}\|_1 + \|h_r^-\|_p\, \|\Phi_r^{(\nu)} - Q_{\sigma,r}^{(\nu)}\|_1 \\ &\leqslant 2c_4(r, \nu)\, \sigma^{\nu-r}\, \|f^{(r)} - T^*_{\sigma,r}\|_p. \end{aligned}$$

Consequently (see (23)),

$$\|\tilde{T}_\sigma^{(\nu)} - f^{(\nu)}\|_p = \|Y_\sigma^{(\nu)} - g^{(\nu)}\|_p \leqslant 2c_4(r, \nu)\, c_{11}\, \sigma^{\nu-r-1/p}\, V_p(f^{(r)}). \tag{28}$$

Since

$$\begin{aligned} V_p(Y_\sigma^{(\nu)} - g^{(\nu)}) &\leqslant V_p(h_r^+)\, \|P_{\sigma,r}^{(\nu)} - \Phi_r^{(\nu)}\|_1 + V_p(h_r^-)\, \|\Phi_r^{(\nu)} - Q_{\sigma,r}^{(\nu)}\|_1 \\ &\leqslant 2c_4(r, \nu)\, \sigma^{\nu-r}\, V_p(f^{(r)} - T^*_{\sigma,r}), \end{aligned}$$

we have

$$V_p(\tilde{T}_\sigma^{(\nu)} - f^{(\nu)}) = V_p(Y_\sigma^{(\nu)} - g^{(\nu)}) \leqslant 2c_4(r, \nu)\, c_{12}\, \sigma^{\nu-r}\, V_p(f^{(r)}). \tag{29}$$

Obviously, the estimate (28) [resp. (29)] in which $\|\tilde{T}_\sigma^{(\nu)} - f^{(\nu)}\|_p$ [$V_p(\tilde{T}_\sigma^{(\nu)} - f^{(\nu)})$] is replaced by $\|\tilde{U}_\sigma^{(\nu)} - f^{(\nu)}\|_p$ [$V_p(\tilde{U}_\sigma^{(\nu)} - f^{(\nu)})$] remains valid. Thus, in the case $1 \leqslant \nu \leqslant r-1$, the assertions (ii) and (iii) are established.

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