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ON INFINITE COMPLETE ALGEBRAS

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1. In this paper the following problem is considered. Find a family $\boldsymbol{F}$ of operations on a set A, as small as possible with regard to the contents and arity of operations, and such that each operation on A could be presented as a composition of some operations from $\boldsymbol{F}$. Our considerations are concerned only with infinite sets. Many results for the finite case are collected in [5].

It is convenient here to use the terminology of abstract algebras (cf. [7]). We denote by $\boldsymbol{O}^{(n)}$ the set of all n-ary operations on A. Let $\boldsymbol{O}^{(1)} \cup \boldsymbol{O}^{(2)} \cup \ldots = \boldsymbol{O}$. A system $\mathfrak{A} = (A, \boldsymbol{F})$ with $\boldsymbol{F} \subset \boldsymbol{O}$ is said to be an *algebra*. We denote by $\boldsymbol{A}(\mathfrak{A})$ the family of all operations which can be obtained as compositions of operations from $\boldsymbol{F}$ and *trivial operations*

$$e_k^{(n)}(x_1, \ldots, x_n) = x_k, \quad 1 \leqslant k \leqslant n.$$

$\boldsymbol{A}(\mathfrak{A})$ is called the *family of algebraic operations of* $\mathfrak{A}$.

Definition 1. An algebra $\mathfrak{A} = (A, \boldsymbol{F})$ is *complete* if $\boldsymbol{A}(\mathfrak{A}) = \boldsymbol{O}$.

Finite complete algebras are also called *primal*. They were recently examined in numerous papers (see, e.g., [8]).

2. From this moment on we assume that the set A is infinite. First, we recall one of Sierpiński's [11] results: *for each infinite set A, the algebra $\mathfrak{A} = (A, \boldsymbol{O}^{(2)})$ is complete.* According to this we can restrict our considerations to operations of arity not greater than 2.

THEOREM 1. *If A is infinite and $g: A^2 \to A$ is one-to-one on a rectangle $A_1 \times A_2$, where $|A_1| = |A_2| = |A|$, then the algebra $\mathfrak{A} = (A, \boldsymbol{O}^{(1)} \cup \{g\})$ is complete.*

Proof. If $f_i: A \to A_i$, $i = 1, 2$, are one-to-one mappings, then also $g_1(x, y) = g(f_1(x), f_2(y))$ is one-to-one. Any binary operation f can be obtained as a composition of g_1 and of some unary operation. In fact, it suffices to define a unary operation h on the range of g_1 by $h(x) = fg_1^{-1}(x)$. Now, $f(x, y) = h(g_1(x, y))$ which completes the proof.

Using Theorem 1, we can easily prove Theorem 2 which confirms the Bieberbach [2] hypothesis that each operation of the set of real numbers can be obtained as a composition of some unary operations and the binary operation $+$. It was proved (in a different way) by Sierpiński [9] for the case $i = 2$.

THEOREM 2. *Let A_i for $i = 1, 2, 3, 4, 5$ denote the set of complex, real, rational, integer, and natural numbers, respectively. Then the algebras*

(a) $\mathfrak{A}_i = (A_i, \boldsymbol{O}^{(1)} \cup \{+\})$ *for* $i = 1, 2, 3, 4, 5$,

(b) $\mathfrak{B}_i = (A_i, \boldsymbol{O}^{(1)} \cup \{\cdot\})$ *for* $i = 1, 2$

are complete.

Proof. 1° To prove (a) for $i = 1, 2$ it suffices to show that there are two subsets B_1 and B_2 of real numbers, $|B_1| = |B_2| = \mathfrak{c}$, such that $+|_{B_1 \times B_2}$ is one-to-one. In fact, let B_1 be the set of numbers which have binary expansion of the form $0, a_1 0 0 a_2 0 0 \ldots$ and B_2 the set of numbers of the form $0, 0 b_1 0 0 b_2 0 0 \ldots$ It is easy to see that we can reconstruct each component from the sum of any two numbers of B_1 and B_2.

2° In the case of $i = 3, 4, 5$, let B_1 be the set of all natural numbers of the form 2^{2n} and B_2 the set of all numbers of the form 2^{2n-1}, $n \geqslant 1$. Use the argument similar to that of 1°.

3° To prove (b) it is sufficient to observe that $+$ can be obtained as a composition of $\cdot$ and some unary operations, viz. $x + y = \log(e^x \cdot e^y)$.

Theorem 1 helped to solve some particular problems, but it does not answer the following more general

PROBLEM 1. For which operations g the algebra $\mathfrak{A} = (A, \boldsymbol{O}^{(1)} \cup \{g\})$ is complete? (**P 885**)

In the finite case the solution is given by a theorem of Słupecki [12]: *if A is a finite set, then the algebra $\mathfrak{A} = (A, \boldsymbol{O}^{(1)} \cup \{g\})$ is complete if and only if g is onto and depends essentially on at least two variables.* It is easy to see that in the infinite case this assumption is neither sufficient nor necessary for the completeness.

3. Theorem 1 gives a possibility to replace all operations on A by all unary operations and one binary operation. It is obvious that some unary operations can be obtained by compositions of the others, whence we do not need all of them. We shall find a possibly small family of unary operations $\boldsymbol{F}$ for which there exists an operation g such that the algebra $\mathfrak{A} = (A, \boldsymbol{F} \cup \{g\})$ is complete.

Let $\boldsymbol{F}$ be a family of unary operations on an infinite set A. We consider the following properties of $\boldsymbol{F}$:

(o) *There is a binary operation g such that the algebra $(A, \boldsymbol{F} \cup \{g\})$ is complete.*

(p) *There are countably many operations* $g_1, g_2, \ldots$ *such that the algebra* $(A, F \cup \{g_1, g_2, \ldots\})$ *is complete.*

(o_1) *There are two unary operations* f_1, f_2 *such that each unary operation can be obtained as a composition of some operations from* $F \cup \{f_1, f_2\}$.

(p_1) *There are countably many unary operations* $f_1, f_2, \ldots$ *such that each unary operation can be obtained as a composition of some operations from* $F \cup \{f_1, f_2, \ldots\}$.

Obviously, (o) $\Rightarrow$ (p) and (o_1) $\Rightarrow$ (p_1) $\Rightarrow$ (p).

We obtain (p_1) $\Leftrightarrow$ (o_1) from the following result of Sierpiński [10] (see [1] for a simple proof): *for each countable set* $\{f_1, f_2, \ldots\}$ *of unary operations on an infinite set, there exist two unary operations* g *and* h *such that each* f_n *can be composed of them.*

Analogously, (p) $\Leftrightarrow$ (o) follows from a result of Łoś [6]: *for each countable set* $\{g_1, g_2, \ldots\}$ *of operations on an infinite set, there exists one binary operation* g *such that each* g_n *is a composition of* g.

It follows from the last equivalence that there are no minimal families with property (o).

Let us observe that if F has property (o), then $|F| = 2^{|A|}$ (A infinite). The next theorem shows that the converse is not true.

THEOREM 3. *If* A *is uncountable,* $C \subset A$, $|C| < |A|$, *and*

$$F = \{f \in O^{(1)}: f(A) \subset C\},$$

then F *has not property* (o).

Proof. Suppose that there is a binary operation h such that the algebra $(A, F \cup \{h\})$ is complete. Let F_k denote the set of all unary operations f such that k is the minimal number of occurrences of h in any composition of algebraic operations equal to f. Let G_0 consist of all constant mappings ranging over C and of the trivial operation $e_1^{(1)}$. Putting

$$G_{k+1} = \{h(g_1, g_2): g_1, g_2 \in G_k\} \cup G_k \quad \text{for } k \geqslant 0,$$

it is easy to see that

$$|G_k| \leqslant |C| + \aleph_0 \quad \text{and} \quad f \subset \bigcup_{g \in G_k} g \quad \text{for each } f \in F_k$$

(this inclusion is understood as the usual inclusion of binary relations). Put now $G = G_0 \cup G_1 \cup \ldots$ and $B_a = \{g(a): g \in G\}$ for $a \in A$. Note that $|B_a| \leqslant |G| < |A|$ and $f(a) \in B_a$ for each algebraic unary operation f, which contradicts the assumption of the completeness.

The next theorem gives a sufficient condition for property (o_1).

THEOREM 4. *Let* A *be an infinite set and* F *a family of unary operations on* A. *Suppose that there exist a subset* $B \subset A$, $|B| = |A|$, *and a family* $\{A_a\}_{a \in A}$ *of pairwise disjoint subsets of* A *such that, for every family* $\{B_a\}_{a \in A}$

of pairwise disjoint subsets of B, there is an operation $f \in \boldsymbol{F}$ with $f(B_a) \subset A_a$ for all $a \in A$. Then there exist two unary operations g and h such that each unary operation p can be obtained as gfh for some $f \in \boldsymbol{F}$.

Proof. Let g be an operation satisfying $g(A_a) = \{a\}$ and let $h\colon A \to B$ be a one-to-one mapping. Suppose that p is an arbitrary unary operation. Let $B_a = h(p^{-1}(\{a\}))$, $a \in A$. There exists an $f \in \boldsymbol{F}$ such that $f\big(h(p^{-1}(\{a\}))\big) \subset A_a$. Therefore

$$g\Big(f\big(h(p^{-1}(\{a\}))\big)\Big) \subset g(A_a) = \{a\} \quad \text{and} \quad gfh = p,$$

which completes the proof.

Let us observe that the family $\mathscr{G}_A$ of all permutations of the set A satisfies the assumption of Theorem 4. Indeed, it suffices to take a partition $\{A_a\}_{a \in A}$ with $|A_a| = |A|$ and a subset B such that $|B| = |A \setminus B|$. Hence we get

COROLLARY 1. *If A is infinite, then*

(i) *there exist two unary operations g and h such that each unary operation p can be obtained as gfh for some permutation $f \in \mathscr{G}_A$.*

Consequently,

(ii) *$\mathscr{G}_A$ has property* (o_1).

Consider now the semigroup $\mathscr{T}_A$ of all transformations (i.e., unary operations) of A. Corollary 1 (i) says that $\mathscr{G}_A \cup \{g, h\}$ is a set of generators of $\mathscr{T}_A = g\mathscr{G}_A h$.

Denote by $\boldsymbol{D}$ the family of all transformations f with $|f(A)| = |A|$, i.e. the Green $\mathscr{D}$-class containing the identity mapping (see, e.g., [3]). There are $2^{|A|}$ disjoint Green's $\mathscr{H}$-classes in $\boldsymbol{D}$ and $\mathscr{G}_A$ is one of them. Hence, for each such $\mathscr{H}$-class $\boldsymbol{H}$, there exist two transformations g' and h' such that $\mathscr{G}_A = g'\boldsymbol{H}h'$ (see [3], Theorem 2.3). Thus we get

COROLLARY 2. *If A is infinite, then for each $\mathscr{H}$-class $\boldsymbol{H}$ contained in $\boldsymbol{D}$ there exist two transformations g and h such that $\mathscr{T}_A = g\boldsymbol{H}h$.*

This corollary gives $2^{|A|}$ disjoint families with property (o_1).

Observe that $\mathscr{T}_A \setminus \boldsymbol{D}$ is an ideal of $\mathscr{T}_A$ and it does not satisfy (o_1). Hence $\mathscr{H}$-classes contained in $\boldsymbol{D}$ are the only ones satisfying (o_1).

We have obtained a lot of families of unary operations with property (o_1) and, obviously, with (o), but the answer to the following problem remains unknown:

PROBLEM 2. Which families of unary operations on an infinite set satisfy properties (o_1) and (o)? (**P 886**)

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