Characterization of the solutions of a $2b$-parabolic operator
with initial data in BMO

by A. GRIMALDI and F. RAGNEDEA (Cagliari)

Abstract. In this paper we show that the solution $u(x, y)$ of $2b$-parabolic equation
$H_b u = \sum_{|x|=2b} a_x D^x u - \partial u/\partial t = 0$, satisfying a suitable integral condition, turns out to be of the
form $u(x, y) = [G(\cdot, t) * f](x)$ with $f \in BMO(\mathbb{R}^n)$ and $G(x, t)$ the fundamental solution associated to $H_b$.

Introduction. Letting $H_b = \sum_{|x|=2b} a_x D^x - (\partial/\partial t)$ the $2b$-parabolic operator
on $\mathbb{R}^{n+1} = \mathbb{R}^n \times (0, +\infty)$, we wish to characterize the solutions $u(x, t)$ of the
equation $H_b u = 0$, which for $t = 0$ have initial values in BMO($\mathbb{R}^n$). If we identify two solutions $u_1$ and $u_2$ which differ by a polynomial in $x$ of
degree $\leq b - 1$, each $u(x, t)$ satisfying a suitable integral condition turns out
to be of the form
$$u(x, t) = [G(\cdot, t) * f](x)$$
with $f \in BMO(\mathbb{R}^n)$ and $G(x, t)$ the "fundamental solution" associated to $H_b$.
For $b = 1$, we reobtain the results of E. B. Fabes and U. Neri [3]. The
methods used by us extend, in a natural way, those employed in [3].

In Section 1, we define the space $T^b MO(\mathbb{R}^{n+1})$ of all those solutions of
$H_b u = 0$ satisfying a given global condition (modulo polynomials as above).
Taking momentarily for granted certain estimates on the $D^x u$, we then prove
the characterization $T^b MO = G * BMO$. In Section 2, we give the proofs of the
estimates on the $D^x u$.

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matter of this paper.

1. Definitions and main result. Let us consider the equation

$$H_b u = \sum_{|x|=2b} a_x D^x u - (\partial u/\partial t) = 0$$

(1)

where the $a_x$ are constants and $D^x = (-i)^{|\alpha|}(\partial^{\alpha_1}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n})$, $|\alpha| = \alpha_1 +$
We assume that the equation be parabolic: for each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$

$$\Re\left( \sum_{|\alpha|=2b} a_\alpha \sigma^\alpha \right) \leq -\delta |\sigma|^{2b}$$

where $\sigma^\alpha = \sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \ldots \sigma_n^{\alpha_n}$.

If $u_0 \in C_0^\infty (\mathbb{R}^n)$, say, the solution $u(x, t)$ of the Cauchy problem

$$H_b u = 0 \quad \text{on} \quad \mathbb{R}^{n+1}_+,$$

has the form

$$u(x, t) = [G(\cdot, t) * u_0](x),$$

(see [1]). Here, $G$ is the inverse Fourier transform

$$G(x, t) = \mathcal{F}_x^{-1} [Q(\xi, t)](x)$$

and $Q(\xi, t)$ satisfies the bounds

$$|Q(\xi, t)| \leq C \exp[-\delta |\xi|^{2b}]$$

with $C$ and $\delta$ strictly positive reals.

The fundamental solution $G$, and its derivatives, satisfy the following estimates:

$$|D^\alpha G(x, t)| \leq C_\alpha t^{-(n+|\alpha|)/2b} \exp[-c \sum_{k=1}^n |x_k|^{2b/(2b-1)} t^{1/(2b-1)}],$$

$$|D^\alpha G(x, t)| \leq C_\alpha |x|^{-(n+|\alpha|)},$$

$$|\partial t G(x, t)| \leq C t^{-1-n/2b} \exp[-c \sum_{k=1}^n |x_k|^{2b/(2b-1)} t^{1/(2b-1)}],$$

where the constants $c, C$ and $C_\alpha$ are (as always) strictly positive and may differ from line to line. For each $t$, the rapid decrease of $G(x, t)$ and of its derivatives (as $|x| \to \infty$) yields the inversion formula

$$[\mathcal{F}_x G(x, t)](\xi) = Q(\xi, t)$$

on account of (4).

For every $u \in C^\infty (\mathbb{R}^{n+1}_+)$ we associate the function $\gamma = \gamma_u$ on $\mathbb{R}^n$, $0 \leq \gamma_u(x) \leq +\infty$ given by

$$[\gamma(x)]^2 = \int_0^\infty \sum_{|\alpha|=b} |D^\alpha u(x, t)|^2 \, dt.$$

Then, if $u(x, t) = [G(\cdot, t) * f](x)$, with $f \in L^2(\mathbb{R}^n)$ we obtain the inequality

$$\|\gamma\|_2 \leq B \|f\|_2$$

where $B$ depends on $\delta$ of (P).
To prove (7'), we follow the argument of [3], p. 2. Thus, using (twice) Plancherel's formula in the $x$ variables, we have

$$||y||^2_2 = \int_0^\infty \left( \int_0^\infty \sum_{|\xi|=b} |D^x u(x, t)|^2 \, d\tilde{t} \right) \, dx = \int_0^\infty \left( \int_0^\infty \sum_{|\xi|=b} |\mathcal{F}_x D^x u(x, t)|^2 \, dx \right) \, dt$$

$$= \int_0^\infty \left( (2\pi)^{2b} \sum_{|\xi|=b} \int |\xi|^2 |\mathcal{F}_x u(x, t)|^2 \, dx \right) \, dt$$

$$= (2\pi)^{2b} \int_0^\infty \left( \sum_{|\xi|=b} \int |\xi|^2 |\mathcal{F}_x G \ast \mathcal{F}_x f|^2 \, d\xi \right) \, dt.$$ 

Since $\mathcal{F}_x G = Q$, we see that

$$||y||^2_2 = (2\pi)^{2b} \int_0^\infty \left( \sum_{|\xi|=b} \int |\xi|^2 |Q|^2 |\hat{f}|^2 \, d\xi \right) \, dt$$

$$= (2\pi)^{2b} \int |\hat{f}|^2 \left( \int \sum_{|\xi|=b} |Q(\xi, t)|^2 \, dt \right) \, d\xi$$

$$\leq C \int |\hat{f}|^2 \left( \int \sum_{|\xi|=b} e^{-2|\xi|^2} \, dt \right) \, d\xi.$$ 

Using the change $s = t |\xi|^{2b}$ we conclude

$$||y||^2_2 \leq C \int |\hat{f}|^2 \left( \int e^{-2bs} \, ds \right) \, d\xi \leq B^2 ||\hat{f}||^2_2.$$ 

The space $T^b \text{MO}$. Denoting by $Q_\delta \subset R^n$ the cube centered at $x_0$ having sides (parallel to the axes) of length $\delta$, we associate with each $u(x, t)$ on $R^{n+1}_+ \times$ the "average" $[u]_{x_0, \delta}$ where

$$[u]_{x_0, \delta}^2 = |Q_\delta|^{-1} \int_{Q_\delta} \sum_{|\xi|=b} |D^x u(x, t)|^2 \, dt \, dx$$

and

$$||u||_{\ast \ast} = \sup \{ [u]_{x_0, \delta}: x_0 \in R^n, \ \delta > 0 \}.$$ 

The linear space of all solutions $u(x, t)$ of (1) such that $||u||_{\ast \ast} < \infty$ will be denoted by $T^b \text{MO} = T^b \text{MO}(R^{n+1}_+)$. In $T^b \text{MO}$ we identify two solutions $u_1$ and $u_2$ which differ by a polynomial in $x$ (independent of $t$) of degree $\leq b - 1$. In this way, the map $u \rightarrow ||u||_{\ast \ast}$ becomes a norm.

Recall that a locally integrable function $f$ is in $\text{BMO} = \text{BMO}(R^n)$ if it has bounded mean oscillation on $R^n$:

$$\|f\|_{\ast} = \sup \{ |Q_\delta|^{-1} \int_{Q_\delta} |f(x) - f_{Q_\delta}| \, dx: x_0 \in R^n, \ \delta > 0 \} < +\infty$$

and

$$\|f\|_{\ast \ast} = \sup \{ |Q_\delta|^{-1} \int_{Q_\delta} |f(x)| \, dx: x_0 \in R^n, \ \delta > 0 \} < +\infty.$$
where \( f_{Q_\delta} \) is the integral average (or mean value) of \( f \) on \( Q_\delta \). With the identification \( f \sim f + C \) (for any constant \( C \)), \( f \to \|f\|_* \) becomes a norm. It is well known that the \( L^1 \)-averages above (see [6]) may be replaced by equivalent \( L^2 \)-averages \([f]_{x_0,\delta}\) where

\[
[f]_{x_0,\delta}^2 = |Q_\delta|^{-1} \int_{Q_\delta} |f(x) - f_{Q_\delta}|^2 \, dx.
\]

If we denote by \( G \ast \text{BMO} = \{u = G \ast f : f \in \text{BMO}\} \), our result can be stated as follows:

**Theorem.** \( T^b \text{MO} = G \ast \text{BMO} \), with equivalence of norms.

We shall prove the theorem through several lemmas.

**Lemma 1.1.** \( G \ast \text{BMO} \subset T^b \text{MO} \) and the inclusion is continuous.

**Proof.** We fix any cube \( Q_\delta \subset \mathbb{R}^n \) centered at \( x_0 \) having sides of length \( \delta = 4h \). By translation invariance of \( \|u\|_{\ast \ast} \), we may suppose that \( x_0 = 0 \). We denote by \( \chi_\delta \) the characteristic function of \( Q_\delta \) and we choose any \( f \in \text{BMO}(\mathbb{R}^n) \). We have that

\[
(f - f_{Q_\delta}) = (f - f_{Q_\delta}) \chi_\delta + (f - f_{Q_\delta}) (1 - \chi_\delta) = f_1 + f_2
\]

and we let

\[
u_1 = G \ast f_1 \quad \text{and} \quad \nu_2 = G \ast f_2.
\]

Since \( f_1 \in L^2(\mathbb{R}^n) \), we associate with \( \nu_1 \) by (7) the corresponding function \( \gamma_1 \).

On account of (7) and (8), it follows that

\[
h^n \|\nu_1\|_{0,h}^2 \leq \|\gamma_1\|_2^2 \leq B^2 \|f_1\|_2^2 = B^2 (4h)^n \|f\|_{0,4h}^2.
\]

Hence \( \|\nu_1\|_{\ast \ast} \leq B \|f\|_* \).

In order to estimate \( \|\nu_2\|_{\ast \ast} \) we observe that if \( x \in Q_\delta \) and \( y \in \mathbb{R}^n - Q_{4h} \) we have \(|x - y| \geq 3h/2\) and \(|x - y| \approx |y|\) for all large \(|y|\). Moreover, for \(|x| = b\), we have

\[
|D^a u_2(x, t)| \leq 2C_a \int_{\mathbb{R}^n - Q_\delta} |f(y) - f_{Q_\delta}| \left[ h^{n+b} + |x - y|^{n+b} \right]^{-1} \, dy.
\]

Letting now \( \eta = y/h \) and integrating on all of \( \mathbb{R}^n \) (see [5] or [4]), we deduce that

\[
|D^a u_2(x, t)| \leq Ch^{-b} \|f\|_*
\]

since the BMO-norm is dilatation-invariant. Consequently,

\[
\|u_2\|_{0,h}^2 \leq C^2 h^{-2b} \|f\|^2_* h^{2b}
\]

and hence \( \|u_2\|_{\ast \ast} \leq C \|f\|_* \). This completes the proof of Lemma 1.1. To
show the inverse inclusion, we shall make use of some estimates for $D^\theta u$ whose proof is deferred to § 2.

**Lemma 1.2.** There is a constant $C > 0$ such that for all $u \in T^k \text{MO}$, $|\beta| \geq b$ and all $(x, t) \in \mathbb{R}_+^{n+1}$,

\begin{align}
|D^\theta u(x, t)| & \leq Ct^{-1/2} - 2|\beta|^{-2b} ||u||_{**}, \\
|\langle \hat{r} / \hat{t} \rangle u(x, t)| & \leq Ct^{-1} ||u||_{**}.
\end{align}

**Lemma 1.3.** Let $u \in T^k \text{MO}$. If $k$ is any positive integer, then the convolution $G * u(\cdot, k^{-2})$ exists everywhere on $\mathbb{R}_+^{n+1}$.

**Proof.** Let us consider the Taylor expansion of $u$ about $y = 0$: $u(y, t) = P(y, t) + (\text{error term of order } b)$ where

$$P(y, t) = u(0, t) + \sum_{|\gamma| \leq b - 1} c_\gamma D^\gamma u(0, t) y^\gamma.$$ 

Replacing (as we may do) $u$ by the representative $u - P$, we have

$$\left| \int_{\mathbb{R}^n} G(x - y, t) u(y, k^{-2}) \, dy - \int_{\mathbb{R}^n} G(x - y, t) P(y, k^{-2}) \, dy \right|$$

$$\leq M \int_{\mathbb{R}^n} \sum_{|\gamma| = b} |y^\gamma D^\gamma u(\theta y, k^{-2})| \, dy \leq C ||u||_{**}$$

on account of the estimates (5) and (10), where the constant $C$ depends only on $k$ and $0 < \theta < 1$.

**Lemma 1.4.** Let $u \in T^k \text{MO}$. Then, for each $k \in \mathbb{N}$ and all $(x, t) \in \mathbb{R}_+^{n+1}$

$$[G * u(\cdot, k^{-2})](x) = u(x, t + k^{-2}).$$

Moreover, setting from now on $u_k(x, t) = u(x, t + k^{-2})$, we have the uniform estimates

$$||u_k||_{**} \leq C ||u||_{**}.$$ 

**Proof.** Both sides of (11) are solutions on $\mathbb{R}_+^{n+1}$ of (1) having pointwise limit $u(x, k^{-2})$ as $t \to 0^+$. Now, for all $(x, t) \in \mathbb{R}_+^{n+1}$, estimates (10) show that

$$\sum_{|\beta| = b} |D^\theta u(x, t + k^{-2})| \leq C (t + k^{-2})^{-1/2} ||u||_{**}.$$ 

Hence, for each $k$, $\sum_{|\beta| = b} D^\theta u(x, t + k^{-2})$ is a bounded function on $\mathbb{R}_+^{n+1}$ and as $t \to 0^+$ with pointwise limit $\sum_{|\beta| = b} D^\theta u(x, k^{-2})$. Therefore, by the unique-
ness of the solution of the Cauchy problem on $\mathbb{R}_+^{n+1}$ with trace $\sum_{|\beta|=b} D^\beta u(x, k^{-2})$ on $\mathbb{R}^n$ (see [1]) we have that

$$\sum_{|\beta|=b} D^\beta u(x, t + k^{-2}) = [G(\cdot, t) * \sum_{|\beta|=b} D^\beta u(\cdot, k^{-2})](x)$$

$$= \sum_{|\beta|=b} D^\beta [G(\cdot, t) * u(\cdot, k^{-2})]$$

which shows that $u(x, t + k^{-2})$ and $G * u(x, k^{-2})$ are equal up to a polynomial in $x$ of degree $\leq b - 1$. Since they have the same limit as $t \to 0^+$, this polynomial is zero.

Estimate (12) is then verified exactly like Lemma 1.4 of [2], by examining $[u_k]_{d_0}$ separately: for $\delta^b \geq 1/k$ and $\delta^b < 1/k$.

**Lemma 1.5.** Let $u \in T^b$ MO and set $f_k = u(x, k^{-2})$. We prove that

$$\|f_k\|_{\ast} \leq C \|u\|_{\ast}$$

**Proof.** Given $u \in T^b$ MO and setting

$$u_k(x, t) = u(x, t + k^{-2}),$$

we consider the identity

$$\frac{\partial}{\partial t} [u_k - a_k]^2 = 2[u_k - a_k] \frac{\partial}{\partial t} u_k = 2[u_k - a_k] \sum_{|\beta|=b} a_\beta D^\beta u_k$$

where the $a_k$ are polynomials in $x$ of degree $\leq b - 1$, to be determined later. Integrating the left-hand side of (13), we have

$$\int \left( \int \left( \frac{\partial}{\partial t} [u_k - a_k]^2 \right) dt \right) dx = \int \left[ u_k(x, \delta^{2b}) - a_k \right]^2 - \int \left[ u_k(x, 0) - a_k \right]^2 dx.$$

Hence

$$\int \left[ u_k(x, 0) - a_k \right]^2 dx$$

$$= \int \left[ u_k(x, \delta^{2b}) - a_k \right]^2 dx - 2 \int \sum_{|\beta|=2b} a_\beta D^\beta u_k dt dx.$$

We now suppose, as we may do, that $Q_\delta = \{ x : |x - x_0| \leq \delta \}$ and we evaluate each term of the last equality choosing the polynomials

$$a_k = \sum_{|\alpha| \leq b - 1} c_\alpha D^\alpha u_k(x_0, \delta^{2b})(x - x_0)^\alpha$$

where $c_\alpha$ are the Taylor coefficients. We obtain then

$$\int \left[ u_k(x, \delta^{2b}) - a_k \right]^2 dx = \sum_{|\alpha| \leq b - 1} c_\alpha D^\alpha u_k(x_0, \delta^{2b})(x - x_0)^\alpha$$

$$\leq c \int |x - x_0|^{2b} \left[ \sum_{|\alpha| = b} D^\alpha u_k(x, \delta^{2b}) \right]^2 dx.$$
so, by (10), we may write

\[
\int_{Q_\delta} |u_k(x, \delta^{2b} - a_k)^2 |^2 \ dx \leq C \|u\|_{\infty}^2 \int_{Q_\delta} |x - x_0|^{2b(\delta^{2b} + k^{-2})^{-1}} \ dx
\]

\[
\leq C \|u\|_{\infty}^2 \delta^{-2b} \int_{Q_\delta} |x - x_0|^{2b} \ dx \leq C \|u\|_{\infty}^2 |Q_\delta|
\]

with the constant \(C\) which is independent of \(Q_\delta\) and \(k\). We consider now the terms of the form

\[
\int_{Q_\delta} \int_0^{\delta^{2b}} \left[ u_k(x, t) - a_k \right] D^\beta u_k(x, t) \ dt \ dx, \quad |\beta| = 2b.
\]

Adding and subtracting \(u_k(x, \delta^{2b})\), we may write

\[
\left| \int_{Q_\delta} \int_0^{\delta^{2b}} \left[ u_k(x, t) - a_k \right] D^\beta u_k(x, t) \ dt \ dx \right|
\]

\[
\leq \left| \int_{Q_\delta} \int_0^{\delta^{2b}} \left[ u_k(x, t) - u_k(x, \delta^{2b}) \right] D^\beta u_k(x, t) \ dt \ dx \right| + \left| \int_{Q_\delta} \int_0^{\delta^{2b}} \left[ u_k(x, \delta^{2b}) - \sum_{|\alpha| < b-1} c_\alpha (x - x_0)^\alpha D^\alpha u_k(x_0, \delta^{2b}) \right] D^\beta u_k(x, t) \ dt \ dx \right|
\]

\[
= |I_1(k)| + |I_2(k)|.
\]

Let us introduce the Taylor remainder of order \(b\):

\[
R_b(x) = u_k(x, \delta^{2b}) - \sum_{|\alpha| < b-1} c_\alpha (x - x_0)^\alpha D^\alpha u_k(x_0, \delta^{2b}).
\]

For every \(x \in Q_\delta\) and \(|\mu| \geq 0\) we claim that

(14) \quad |D^\mu R_b(x)| \leq C \delta^{-\mu} \|u\|_{\infty}.

If \(|\mu| = 1\) we obtain

\[
D^\mu R_b(x) = D^\mu u_k(x, \delta^{2b}) - \sum_{|\alpha| < b-2} c_\alpha (x - x_0)^\alpha D^{\mu + \alpha} u_k(x_0, \delta^{2b})
\]

which is the remainder of order \(b-1\) of \(D^\mu u_k(x, \delta^{2b})\).

The estimates (10) show that \((x - x_0)^\alpha D^{\mu + \alpha} u_k\) is dominated by \(\delta^{-\mu} \|u\|_{\infty}\). By iteration we obtain (14).

First, we estimate \(I_2(k)\). For a fixed \(t\) between 0 and \(\delta^{2b}\), integrating by parts in \(x\), we have

\[
|I_2(k)| = \left| \int_{0}^{\delta^{2b}} \int_{Q_\delta} R_b(x) D^{\beta_1} u_k(x, t) \ dx \ dt \right|
\]

\[
\leq \left| \int_{0}^{\delta^{2b}} \int_{S_{\delta}} |R_b(x)| D^{\beta_1} u_k(x, t) \ dx \ dt \right| + \left| \int_{0}^{\delta^{2b}} \int_{Q_\delta} D^{\beta_1} R_b(x) D^{\beta_1} u_k(x, t) \ dx \ dt \right|
\]

where \(|\beta_1| = 2b - 1\), \(|\mu_1| = 1\) and \(S_{\delta} = \{x : |x - x_0| = \delta\} \).
From estimates (10) and (14) it follows that

$$|I_2(k)| \leq C \|u\|_{\infty}^2 \int_{S_a} \int_0^{\delta^{2b}} \int_0^{(t+k^{-2})^{-1} + (1/2b)} dt \, d\sigma + \int_{Q_a} \int_0^{\delta^{-1}} \int_0^{(t+k^{-2})^{-1} + (1/2b)} dt \, dx \]$$

$$\leq C \|u\|_{\infty}^2 \|S_a\| + \delta^{-1} \|Q_a\|.$$  

Hence

$$|I_2(k)| \leq C \|u\|_{\infty}^2 \|Q_a\|.$$  

Similarly we estimate $I_1(k)$ as follows:

$$|I_1(k)| \leq \int_0^{\delta^{2b}} \int_0^{\delta^{-1}(t)} \left|D^{\theta_1} u_k(x, t) - D^{\theta_1} u_k(x, t)\right| \, dx \, dt + \int_0^{\delta^{2b}} \int_0^{\delta^{-1}(t)} \left|D^{\theta_2} u_k(x, t)\right| \, dx \, dt$$

$$= |I_{11}(k)| + |I_{12}(k)|.$$  

For the first addendum, if $\delta^{b} \leq 1/k$, we see that

$$|I_{11}(k)| \leq C \|u\|_{\infty}^2 \int_{S_a} \int_0^{\delta^{2b}} \left(\delta^{2b} - t\right) (t+k^{-2})^{-1} \left(1 + (1/2b)\right) d\sigma \, dt$$

$$\leq C \|u\|_{\infty}^2 \|Q_a\|$$

since $\delta^{2b} - t \leq \delta^{2b} + t \leq k^{-2} + t$.

If $\delta^{b} > 1/k,$

$$|I_{11}(k)| \leq \int_{S_a} \int_0^{\delta^{2b} + 1/k} \left\{ \int_0^{(t+k^{-2})^{-1} + (1/2b)} |D^{\theta_1} u_k(x, t)| \, dt \, d\sigma \right\}$$

$$\leq C \|u\|_{\infty}^2 \int_{S_a} \left\{ \int_0^{\delta^{2b} + 1/k} \log \frac{\delta^{2b} + k^{-2}}{t+k^{-2}} \left(1 + (1/2b)\right) dt \, d\sigma \right\}$$

and with an easy computation we deduce that

$$\left|\int_0^{\delta^{2b} + 1/k} \log \frac{\delta^{2b} + k^{-2}}{t+k^{-2}} \left(1 + (1/2b)\right) dt \right| \leq C \cdot \delta.$$  

Therefore, for both the cases, we can conclude that

$$|I_{11}(k)| \leq C \|u\|_{\infty}^2 \|Q_a\|.$$  

In order to estimate $I_{12}(k)$, integrating by parts, we see that

$$|I_{12}(k)| \leq \int_{S_a} \int_0^{\delta^{2b}} \left|D^{\theta_1} u_k(x, t) - D^{\theta_1} u_k(x, \delta^{2b})\right| \left|D^{\theta_2} u_k(x, t)\right| \, d\sigma \, dt +$$
\[
+ \left| \int_0^{\delta_{2b}} \int_{Q_{\delta}} \left[ D^{\mu_1} u_k(x, t) - D^{\mu_2} u_k(x, \delta_{2b}) \right] D^{\beta_2} u_k(x, t) \, dx \, dt \right|
= |I_{13}(k)| + |I_{14}(k)|; \quad |\beta_2| = 2b - 2, \ |\mu_2| = 2.
\]

If \( \delta^b \leq 1/k \), since \( u_k \) are solutions of (1), we have
\[
|I_{13}(k)| \leq \int_0^{\delta_{2b}} \int_{Q_{\delta}} \left| (\delta_{2b} - t) (d/dt) D^{\mu_1} u_k(x, \theta(\delta_{2b} - t)) \right| |D^{\beta_2} u_k(x, t)| \, d\sigma \, dt
\]
\[
= \int_0^{\delta_{2b}} \int_{Q_{\delta}} \left| (\delta_{2b} - t) \sum_{|\varepsilon| = 2b} a_\varepsilon D^{\varepsilon_{\mu_1}} u_k(x, \theta(\delta_{2b} - t)) \right| |D^{\beta_2} u_k(x, t)| \, d\sigma \, dt.
\]

For every addendum of the last integral we have
\[
a_\varepsilon \int_0^{\delta_{2b}} \int_{Q_{\delta}} (\delta_{2b} - t) |D^{\varepsilon_{\mu_1}} u_k(x, \theta(\delta_{2b} - t))| |D^{\beta_2} u_k(x, t)| \, d\sigma \, dt
\]
\[
\leq C \|u\|^2_{\mathcal{H}} \|S_\delta\| \int_0^{\delta_{2b}} (t + k^{-2})^{-1} (t + k^{-2})^{-1 + (1/2b)} (t + k^{-2})^{-1 + (1/2b)} \, dt.
\]

Hence, reasoning as for \( I_{11}(k) \), we see that last expression is dominated by
\[
|u|^2_{\mathcal{H}} \|S_\delta\| \int_0^{\delta_{2b}} (t + k^{-2})^{-1 + (1/2b)} \, dt \leq |u|^2_{\mathcal{H}} \|Q_\delta\|.
\]

Therefore, if \( \delta^b \leq 1/k \) we have
\[
|I_{13}(k)| \leq C \|u\|^2_{\mathcal{H}} \|Q_\delta\|.
\]

If \( \delta^b > 1/k \),
\[
|I_{13}(k)| \leq \int_0^{\delta_{2b}} \int_{Q_{\delta}} \left\{ \int_{t + k^{-2}}^{t + k^{-2}} |(d/ds) D^{\mu_1} u_k(x, s)| \, ds \right\} |D^{\beta_2} u_k(x, t)| \, d\sigma \, dt
\]
\[
= \int_0^{\delta_{2b}} \int_{Q_{\delta}} \left\{ \sum_{|\varepsilon| = 2b} a_\varepsilon \left( \int_{t + k^{-2}}^{t + k^{-2}} |D^{\varepsilon_{\mu_1}} u_k(x, s)| \, ds \right) D^{\beta_2} u_k(x, t) \right\} \, d\sigma \, dt.
\]

For each term of the sum, we may write
\[
\int_0^{\delta_{2b}} \int_{Q_{\delta}} \left| (\delta_{2b} - t) (d/ds) D^{\mu_1} u_k(x, s) \right| |D^{\beta_2} u_k(x, t)| \, d\sigma \, dt
\]
\[
\leq c \|u\|^2_{\mathcal{H}} \|S_\delta\| \int_0^{\delta_{2b}} (t + k^{-2})^{-1 + (1/2b)} \left( t + k^{-2} \right)^{\delta_{2b} + k^{-2}} \, dt
\]
\[
\leq C \|u\|^2_{\mathcal{H}} \|S_\delta\| \int_0^{\delta_{2b}} (t + k^{-2})^{-1 - (2/2b)} (\delta_{2b} + k^{-2})^{1/2b} \, dt +
\]
\[ \int_0^{\delta^{2b}} \left( t + k^{-2} - 1 + (2/2b)(t + k^{-2}) - 1/2b \right) dt \leq C \|u\|^2_{\infty} |Q_\delta| \left( (\delta^{2b} + k^{-2})^{2/2b} - (1/2b) + (k^{-2})^{2/2b} \right) \]
\[ + (k^{-2})^{2/2b} (\delta^{2b} + k^{-2})^{1/2b} + (\delta^{2b} + k^{-2})^{1/2b} + (k^{-2})^{1/2b} \]
\[ \leq C \|u\|^2_{\infty} |Q_\delta| \cdot \delta. \]

In order to estimate
\[ \int_0^{\delta^{2b}} \int_{Q_\delta} \left| [D^{\mu_s} u_k(x, t) - D^{\mu_s} u_k(x, \delta^{2b})] D^{\mu_s} u_k(x, t) \right| dx \, dt \]
let us make an integration by parts. We can estimate the term relative to the surface integral as in the previous case and for the term relative to the volume integral we make a new integration by parts. If we iterate this process \( b \) times we obtain the volume integral
\[ \int_0^{\delta^{2b}} \int_{Q_\delta} \left( D^{\mu_s} u_k(x, t) - D^{\mu_s} u_k(x, \delta^{2b}) \right) D^{\mu_s} u_k(x, t) \, dx \, dt \equiv J \]
where \( |\beta| = 2b - 2 \) and \( |\mu| = b \).

By the elementary inequality \( ab \leq a^2 + b^2 \), \( J \) is majorized by
\[ \int_0^{\delta^{2b}} \int_{Q_\delta} \left( D^{\mu_s} u_k(x, t) \right)^2 dx \, dt + \int_0^{\delta^{2b}} \int_{Q_\delta} \left( D^{\mu_s} u_k(x, \delta^{2b}) \right)^2 dx \, dt + \]
\[ + \int_0^{\delta^{2b}} \int_{Q_\delta} \left( D^{\mu_s} u_k(x, t) \right)^2 dx \, dt. \]

For the second integral, we obtain
\[ \int_0^{\delta^{2b}} \int_{Q_\delta} \left( D^{\mu_s} u_k(x, \delta^{2b}) \right)^2 dx \, dt \leq C \|u\|^2_{\infty} |Q_\delta| \int_0^{\delta^{2b}} \left( (\delta^{2b} + k^{-2})^{-1/2} \right)^2 dt \]
\[ \leq C \|u\|^2_{\infty} |Q_\delta|. \]

The first and third integrals are dominated by \( \|u\|^2_{\infty} |Q_\delta| \) in view of the definition of \( T^b \) MO. Hence the lemma is proved.

The proof of our theorem may then be concluded as follows. For any fixed \( t > 0 \) and \( |\alpha| = b \) the function \( x \mapsto D^\alpha G(x, t) \) belongs to the space \( H^1 = H^1(\mathbb{R}^n) \) where \( H^1 = \{ f \in L^1 : R_j(f) \in L^1, j = 1, 2, \ldots, n \} \) and \( R_j \) denotes the Riesz transform (see [7]). For this proof the reader can see Lemma 1.5 of [2]. Now, for each \( (x, t) \in \mathbb{R}^{n+1} \) and \( k \in N \), formula (11) implies that
\[ [(D^\alpha G) \ast f_k(\cdot)](x) = D^\alpha u(x, t + k^{-2}). \]
On account of the duality BMO = (H^1)^*, reasoning as in [3] we deduce the existence of some f ∈ BMO such that

\[(D^a G * f)(x) = D^a u(x, t) \quad \text{for any } |x| = b.\]

Hence \(u(x, t) = G * f\) modulo a polynomial in \(x\) of degree \(\leq b - 1\), that is, \(u = G * f\) as element of \(T^b\) MO. The proof of the theorem is complete.

2. Proof of the pointwise estimates (10) and (10'). We denote by \(H_b\) the operator \(\sum |\beta| = 2b a_\beta D^\beta \frac{\partial}{\partial t}\). Fixing any \((x_0, t_0) \in \mathbb{R}^{n+1}_+\) we choose some \(\phi \in C_0^\infty (\mathbb{R}^n)\) such that, \(0 \leq \phi(x) \leq 1,\)

\[
\phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}
\]

and \(|D^\beta \phi(x)| < K\), say, if \(|\beta| = 2b\).

We then choose \(\psi \in C^\infty (\mathbb{R})\) with \(0 \leq \psi(t) \leq 1,\)

\[
\psi(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 1 & \text{if } t \geq 2 \end{cases}
\]

for some absolute constant \(K > 0\).

If \(\varepsilon, r\) are \(> 0\) we let \(\phi_r = \phi(x/r), \psi_\varepsilon(t) = \psi(t/\varepsilon)\) and we examine the smooth function

\[v(x, t) = \phi_r(x - x_0) \psi_\varepsilon(t) D^a u(x, t), \quad \text{where } |x| \geq b.\]

For all \((x, t) \in \mathbb{R}^{n+1}_+\) we have the representation formula (see [1], [3])

\[v(x, t) = \int_0^t \int_{\mathbb{R}} G(x - y, t - s) [H_b v](y, s) \, dy \, ds\]

where \(G(x, t)\) is the fundamental solution associated to \(H_b u = 0\). Moreover, if \(0 < \varepsilon \leq t_0/\varepsilon\), on account of the previous definitions we have

\[v(x_0, t_0) = D^a u(x_0, t_0).\]

Consequently

\[D^a u(x_0, t_0) = \int_0^{t_0} \int_{\mathbb{R}} G(x_0 - y, t_0 - s) [H_b v](y, s) \, dy \, ds.\]

We observe that

\[H_b v = H_b [\phi_r(x - x_0) \psi_\varepsilon(t) D^a u] = \sum_{|\beta| = 2b} a_\beta \psi_\varepsilon(t) \left[ \Sigma^\beta D^\gamma_1 \phi_r \right] D^\gamma_2 (D^a u) - \phi_r \frac{\partial}{\partial t} \psi_\varepsilon(t) D^a u - \phi_r \psi_\varepsilon(t) \left( \frac{\partial}{\partial t} \right) D^a u\]
where the $\Sigma^\beta$ means that the sum is taken for all $\gamma_1$ and $\gamma_2$ such that $\gamma_1 + \gamma_2 = \beta$. Since $H_b(D^2 u) = 0$, we have

\begin{equation}
H_b v = \sum_{|\beta| = 2b} a_\beta \psi_\epsilon(t) (\Sigma^\beta D^{\gamma_1} \varphi_r D^{\gamma_2} (D^2 u) - \varphi_r (\partial_x / \partial t) \psi_\epsilon D^2 u)
\end{equation}

where the $\Sigma^\beta$ is taken for all $\gamma_1 + \gamma_2 = \beta$ but $\gamma_1 \neq 0$.

Now, (15) implies that

\begin{equation}
D^2 u(x_0, t_0) = \sum_{|\beta| = 2b} a_\beta \Sigma^\beta \int_0^{t_0} \int_{\mathbb{R}^n} G(x_0 - y, t_0 - s) \psi_\epsilon D^{\gamma_1} \varphi_r D^{\gamma_2} (D^2 u) dy ds - \\
- \int_0^{t_0} \int_{\mathbb{R}^n} G(x_0 - y, t_0 - s) \varphi_r (\partial_x / \partial t) \psi_\epsilon D^2 u dy ds.
\end{equation}

In (16), we consider first the term

\[ \int_0^{t_0} \int_{\mathbb{R}^n} G(x_0 - y, t_0 - s) \varphi_r (\partial_x / \partial t) \psi_\epsilon D^2 u dy ds \]

\[ \leq CK \varepsilon^{-1} \int_{|x_0 - y| \leq 2r} (t_0 - s)^{-n/2b} |D^2 u| dy ds \]

\[ \leq CK \varepsilon^{-1} (t_0 - 2\varepsilon)^{-n/2b} \int_{|x_0 - y| \leq 2r} |D^2 u| dy ds \]

\[ \leq C_n (t_0 - 2\varepsilon)^{-n/2b} \varepsilon^{-1/2} r^{n/2} \int_0^{t_0} \int_{Q_2r} |D^2 u|^2 dy ds \]

by the Schwarz inequality and where $Q_{2r}$ is the cube with center $x_0$ and side $2r$. Letting $\varepsilon = t_0/4$, $2r = t_0^{1/2b}$, the last expression is majorized by

\[ C_n t_0^{1/2} (r^{-n})^{2b} \int_0^{t_0} \int_{Q_{2r}} |D^2 u|^2 dy ds \leq Ct_0^{1/2} \|u\|_{**} \]

for any $\alpha$ with $|\alpha| = b$.

Let us estimate the terms of the form

\[ \int_0^{t_0} \int_{\mathbb{R}^n} G(x_0 - y, t_0 - s) \psi_\epsilon D^{\gamma_1} \varphi_r D^{\gamma_2} (D^2 u) dy ds. \]

If we integrate by parts in space $|\gamma_2|$ times (where $|\gamma_2| \leq 2b - 1$), we have

\[ \int_0^{t_0} \int_{\mathbb{R}^n} G(x_0 - y, t_0 - s) \psi_\epsilon(t) D^{\gamma_1} \varphi_r D^{\gamma_2} (D^2 u) dy ds \]

\[ \leq \Sigma^{\gamma_2} \int_0^{t_0} \int_{\mathbb{R}^n} |D^\alpha G(x_0 - y, t_0 - s) D^{\gamma_1 + \mu} \varphi_r \psi_\epsilon (D^2 u)| dy ds \]
where $\Sigma^{r_s}$ means that the sum is taken for all $v$ and $\mu$ with $v + \mu = \gamma_2$. Using (5') and remarking that $|D^{\gamma_1} \varphi_\nu| < Kr^{-1} |\nu|$, we may write that

$$\int_0^{t_0} \int |D^s G(x_0 - y, t_0 - s)\psi_\nu D^{\gamma_1} \varphi_\nu (D^s u)| \, dy \, ds$$

$$\leq Kr^{-1} |\nu| \int_0^{t_0} \int |x_0 - y|^{-n - |\nu|} |D^s u| \, dy \, ds$$

$$\leq Kr^{-1} |\nu| \int_0^{t_0} \int |D^s u| \, dy \, ds$$

$$\leq Kr^{-1} |\nu| \int_0^{t_0} \int |D^s u| \, dy \, ds$$

$$\leq Kr^{-1} |\nu| \int_0^{t_0} \int |D^s u|^2 \, dy \, ds \right)^{1/2}$$

$$\leq C_n t_0^{-|\nu|/2b} t_0^{1/2} \left[ \delta^{-n} \int_0^{t_0} \int |D^s u|^2 \, dy \, ds \right]^{1/2} \leq C_n t_0^{-1/2} ||u||_{\infty}$$

where $2r = t_0^{1/2b} = \delta$ and $|\nu| = b$. Hence, for $|\nu| = b$.

(17) $$|D^s u(x_0, t_0)| \leq C t_0^{-1/2} ||u||_{\infty}.$$ 

For $|\nu| > b$, we can obtain the estimate integrating $|\nu| - b$ times by parts and reasoning in the same manner as for $|\nu| = b$. Therefore, we have

(18) $$|D^s u(x, t)| \leq C t_0^{-1/2 - (|\nu| - b)/2b} \quad \text{for any } |\nu| \geq b.$$

References


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