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A difference method of solving the differential equation $y' = h(t, y, y, y')$

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§ 1. In this paper we discuss a method of solving the differential equation

$$y' = h(t, y, y, y'), \tag{1.1}$$

which may be placed between the method of successive iterations

$$u_n'(t) = h\big(t, u_n(t), u_{n-1}(t), u_{n-1}'(t)\big), \quad u_0(t) \equiv 0, \quad 0 \leqslant t < a, \tag{1.2}$$

dealt with in the paper [2] and the Eulerian type method of rectilinear polygons considered in the paper [3].

The method in question is that of retarded argument, the sequence of approximations $y_n(t)$ $(n = 1, 2, 3, \ldots)$ being defined by the formula

$$y_n'(t) = h\left(t, y_n(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right), \quad \text{for} \quad \frac{\delta}{n} \leqslant t < a, \tag{1.3}$$

$$y_n'(t) = h\big(t, y_n(t), 0, d_n\big), \quad \text{for} \quad 0 \leqslant t < \frac{\delta}{n}, \tag{1.4}$$

$$d_{n+1} = h(0, 0, 0, d_n) \quad (n = 1, 2, 3, \ldots), \quad d_1 = 0. \tag{1.5}$$

The connection between the papers [2], [3] and the recent paper may be seen in a following way:

Let

$$t_j = t_j(n) = j \cdot \frac{\delta}{n} \quad (j = 0, 1, 2, \ldots), \tag{1.6}$$

be a sequence of equally spaced points t_j in the interval I': $0 \leqslant t < a$, n being a positive integer and $0 < \delta = \text{const}$. We define successively curvilinear segments of the line $y_n(t)$ in the intervals

$$\Delta_j: \quad t_{j-1} \leqslant t < t_j \quad (j = 1, 2, 3, \ldots). \tag{1.7}$$

First we define a slope $y_n'(0)$ at the initial point $t = 0$ with the aid of (1.5) and then the first curvilinear segment by the differential equation (1.4). Supposing that the segment $y_n(t)$ in the interval $\Delta_j(n)$ (j—fixed,

$j \geqslant 1$) is given, we define the segment $y_n(t)$ in the next interval $\Delta_{j+1}(n)$ as a solution of the differential equation

$$(1.8) \qquad y_n'(t) = h\left(t, y_n(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right) \quad \text{for} \quad t \in \Delta_{j+1}(n),$$

joining the segment $y_n(t)$, $t \in \Delta_{j+1}(n)$, with the preceding one so as to obtain the continuous function $y_n(t)$ for $0 \leqslant t < t_{j+1}$.

The unknown function $y_n(t)$ occurring in equation (1.8) appears only at those places where $y_n(t)$ is printed, $y_n\left(t-\frac{\delta}{n}\right)$ and $y_n'\left(t-\frac{\delta}{n}\right)$ being known functions for $t \in \Delta_{j+1}(n)$.

So we can say that the mecanisme of the iterative method (1.2), supplying successively the functions $u_n(t)$ in the whole interval I', is now modified (cf. (1.8)), and supplies successively curvilinear segments $y_n(t)$ in the intervals $\Delta_j(n)$ $(j = 1, 2, 3, \ldots)$.

This similarity enables us to modify the majorant functions $z_n(t)$ of the paper [2] so that they can be used in the present paper, and in paper [3] on rectilinear polygons $p_n(t)$:

$$(1.9) \qquad p_n'(t) = h\big(t_j, p_n(t_j), p_n(t_{j-1}), p_n'(t_{j-1})\big) \quad \text{for} \quad t \in \Delta_{j+1}(n),$$

$$(1.10) \qquad p_n'(t) = h(0, 0, 0, d_n) \quad \text{for} \quad t \in \Delta_1(n),$$

$$(1.11) \qquad d_{n+1} = h(0, 0, 0, d_n) \quad (n = 1, 2, 3, \ldots), \quad d_1 = 0.$$

In the present paper we solve the equation (1.1) in a complete Banach space with a homogeneous norm, the case of n differential equations for n real functions being included without complicated calculations. We solve the problem of location of $y_n(t)$ and $y_n'(t)$ and prove the almost uniform convergence to the unique solution of (1.1). In addition we derive two error estimates for $y_n(t)$, the second with the aid of the values $y_n(t) - y_n\left(t-\frac{\delta}{n}\right)$ and $y_n'(t) - y_n'\left(t-\frac{\delta}{n}\right)$.

The method of proofs is similar to that of the paper [3].

§ 2. We shall use three well known theorems quoted in the paper [2] as:

Theorem A on differential inequalities (cf. T. Ważewski [7]);

Theorem B on the relation

$$\bar{D}_+\|\varphi(t)\| \leqslant \|\varphi'(t)\|,$$

$\bar{D}_+$ being the right upper derivative in a Banach space with a homogeneous norm (cf. T. Ważewski [5]), and

Theorem D on the solution of the differential equation $u' = g(t, u)$ in a Banach space with the aid of successive approximations (cf. Lusternik, Sobolev [4]).

§ 3. Throughout the rest of the paper we shall use the following assumptions and notations:

ASSUMPTIONS H. 1) Suppose that the function $h(t, u, v, w)$ is defined and continuous for $(t, u, v, w) \in \omega$, where

$$(3.1) \qquad \omega: \quad 0 \leqslant t < a, \quad \|u\| < b, \quad \|v\| < b, \quad \|w\| < c,$$

$(a \leqslant +\infty,\ b \leqslant +\infty,\ c \leqslant +\infty)$.

2) The values of $h(t, u, v, w)$ are in the Banach space B with a homogeneous norm:

$$h(t, u, v, w) \in B \quad \text{for} \quad (t, u, v, w) \in \omega .$$

3) The Lipschitz condition

$$(3.2) \qquad \|h(\bar{t}, \bar{u}, \bar{v}, \bar{w}) - h(t, u, v, w)\| \leqslant K\cdot|\bar{t}-t| + M\cdot\|\bar{u}-u\| + N\cdot\|\bar{v}-v\| + L\cdot\|\bar{w}-w\|,$$

holds for $(\bar{t}, \bar{u}, \bar{v}, \bar{w}) \in \omega$, $(t, u, v, w) \in \omega$, with some non-negative constants K, M, N, and the constant L satisfying

$$(3.3) \qquad 0 \leqslant L < 1 .$$

We suppose also that

$$(3.4) \qquad \|h(0, 0, 0, 0)\| \leqslant P \quad (P < +\infty),$$

where the constant P satisfies the condition

$$(3.5) \qquad P < c\cdot(1-L)^2 .$$

Denote by $s(t)$ the solution of the non-homogeneous linear differential equation

$$(3.6) \qquad s'(t) = \frac{M+N}{1-L}\cdot s(t) + \frac{\bar{h}(t)}{1-L} + \frac{L\cdot P}{(1-L)^2},$$

satisfying the initial condition $s(0) = 0$, and let

$$(3.7) \qquad \bar{h}(t) = \max_{0\leqslant t'\leqslant t} \|h(t', 0, 0, 0)\| .$$

Let I' be the greatest interval contained in the interval I: $0 \leqslant t < a$,

$$(3.8) \qquad I': \quad 0 \leqslant t < \alpha \quad (\alpha < a),$$

such that

$$(3.9) \qquad s(t) < b, \quad s'(t) < c \quad \text{for} \quad t \in I' .$$

The existence of the interval I' follows from the theorem on continuation of the solutions of differential equations.

Suppose that

$$(3.10) \qquad \delta = \begin{cases} \text{arbitrary number if } \alpha = +\infty, \\ \alpha \text{ if } \alpha < +\infty, \end{cases}$$

and denote by Δ_j the intervals

$$(3.11) \qquad \Delta_j: \quad t_{j-1} \leqslant t < t_j \qquad (j = 1, 2, 3, \ldots),$$

with end-points

$$(3.12) \qquad t_j = t_j(n) = j \cdot \frac{\delta}{n} \qquad (j = 0, 1, 2, \ldots),$$

n being an arbitrary fixed positive integer.

§ 4. Now we shall give without proofs Lemma 1, connected with some estimations for a sequence d_n, and Lemma 2 on functions $z_n(t)$, $\tau_0 \leqslant t < +\infty$, simplifying the proof of Lemma 3 (cf. Fig. 1).

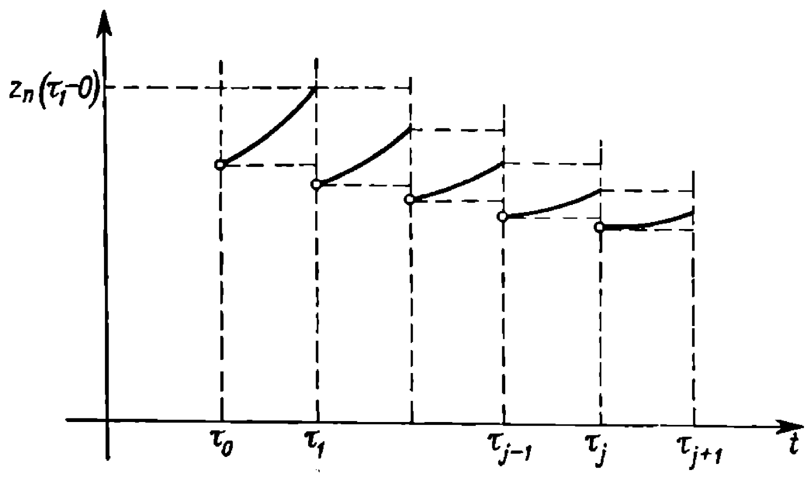

Fig. 1. The graph of the functions $z_n(t)$ (cf. Lemma 2)

These lemma can be proved in the same way as Lemmas 1 and 2 in the paper [3].

LEMMA 1. *Let us suppose that the function $h(t, u, v, w)$ fulfils assumptions* H.

Under these assumptions:

1° *All terms of the sequence*

$$(4.1) \qquad d_{n+1} = h(0, 0, 0, d_n) \quad (n = 0, 1, 2, \ldots), \qquad d_0 = 0$$

are defined, the sequence d_n converges and a limit $d = \lim_{n\to\infty} d_n$ is a unique solution of the equation $d = h(0, 0, 0, d)$.

2° *The inequalities*

$$(4.2) \qquad \|d_p - d_q\| \leqslant 2 \cdot L^{n-1} \cdot \frac{P}{1-L}, \quad \|d_n\| \leqslant \frac{P}{1-L} \qquad (n = 1, 2, 3, \ldots),$$

are satisfied for arbitrary positive integers n, p, q, such that $p \geqslant n$, $q \geqslant n$.

3° *The error estimates of the form*

$$\|d_p - d\| \leqslant 2 \cdot L^{n-1} \cdot \frac{P}{1-L}, \tag{4.3}$$

hold for $p \geqslant n$ $(n = 1, 2, 3, \ldots)$.

LEMMA 2. *Suppose that the real-valued function* $z_n(t)$ *fulfils the following assumptions in the intervals* θ_j

$$\begin{aligned} &\theta_j: \quad \tau_{j-1} \leqslant t < \tau_j \qquad (j = 1, 2, 3, \ldots), \\ &\tau_j = \tau_j(n) = \tau_0 + j \cdot \frac{\delta}{n} \qquad (j = 0, 1, 2, \ldots): \end{aligned} \tag{4.4}$$

1° *The function* $z_n(t)$ *defined in the interval* θ_1, *as well as the derivative* $z_n'(t)$ *are increasing functions in* θ_1 *and satisfy conditions* $z_n(\tau_0) \geqslant 0$ *and*

$$z_n(t) \geqslant 0, \quad z_n'(t) \geqslant 0 \quad \text{for} \quad t \in \theta_1, \tag{4.5}$$

$$z_n'(t) \geqslant M \cdot z_n(t) + N \cdot z_n(t) + L \cdot z_n'(t) + f_n(t) \quad \text{for} \quad t \in \theta_1, \tag{4.6}$$

where $f_n(t)$ *is a real-valued function increasing in each interval* θ_j $(j = 1, 2, 3, \ldots)$, *and*

$$f_n\left(t - \frac{\delta}{n}\right) = f_n(t) \quad \text{for} \quad \tau_1 \leqslant t < +\infty. \tag{4.7}$$

2° *The function* $z_n(t)$ *is the solution* $\zeta = z_n(t)$ *of the linear non-homogeneous equation*

$$\zeta'(t) = M \cdot \zeta(t) + N \cdot z_n\left(t - \frac{\delta}{n}\right) + L \cdot z_n'\left(t - \frac{\delta}{n}\right) + f_n(t), \tag{4.8}$$

for $t \in \theta_{j+1}(n)$ $(j = 1, 2, 3, \ldots)$, *and satisfies the initial condition*

$$\zeta(\tau_j) = z_n(\tau_j) \geqslant 0. \tag{4.9}$$

Thus, (4.8) *becomes*

$$z_n'(t) \equiv M \cdot z_n(t) + N \cdot z_n\left(t - \frac{\delta}{n}\right) + L \cdot z_n'\left(t - \frac{\delta}{n}\right) + f_n(t), \tag{4.10}$$

identically for $t \in \theta_{j+1}(n)$ $(n = 1, 2, 3, \ldots)$.

3° *Initial values satisfy the inequalities*

$$z_n(\tau_j) \geqslant z_n(\tau_{j+1}) \geqslant 0 \quad (j = 0, 1, 2, \ldots). \tag{4.11}$$

Under these assumptions the function $z_n(t)$ *and its derivative* $z_n'(t)$ *are increasing for* $t \in \theta_j(n)$ $(j = 1, 2, 3, \ldots)$, *and satisfy the conditions*

$$\begin{aligned} &z_n\left(t - \frac{\delta}{n}\right) \geqslant z_n(t) \geqslant 0, \\ &z_n'\left(t - \frac{\delta}{n}\right) \geqslant z_n'(t) \geqslant 0 \quad \text{for} \quad t \in \theta_{j+1} \quad (j = 1, 2, 3, \ldots), \end{aligned} \tag{4.12}$$

$$z_n'(t) \geqslant M \cdot z_n(t) + N \cdot z_n(t) + L \cdot z_n'(t) + f_n(t) \quad \text{for} \quad t \in \theta_j \ (j = 1, 2, 3, \ldots), \tag{4.13}$$

and

$$z_n(t) < z_n(\tau_1 - 0), \quad z_n'(t) < z_n'(\tau_1 - 0) \quad \text{for} \quad \tau_0 \leqslant t < +\infty. \tag{4.14}$$

Remark 1. Inequality (4.12) can be interpreted in a following way: the translation of the graph of the segment $z_n(t)$, $t \in \theta_j$, parallel to t-axis yields the graph of $z_n\left(t-\frac{\delta}{n}\right)$ for $t \in \theta_{j+1}$. According to (4.12) the segment $z_n(t)$ for $t \in \theta_{j+1}$ is under the segment $z_n(t)$ for $t \in \theta_j$, consequently the function $z_n(t)$ satisfies the inequality $0 \leqslant z_n(t) < z_n(\tau_1-0)$ in the whole interval $\tau_0 \leqslant t < +\infty$. A similar result can be rewritten in the case of derivative $z_n'(t)$.

Remark 2. It may be noted that the unknown function $z_n(t)$ occurring in equation (4.8) appears only at those places where the letter ζ is printed. Consequently, equation (4.8) is a linear non-homogeneous equation, since

$$N\cdot z_n\left(t-\frac{\delta}{n}\right)+L\cdot z_n'\left(t-\frac{\delta}{n}\right)+f_n(t)$$

is a known function for $t \in \theta_{j+1}(n)$.

§ 5. Now we shall prove Lemma 3 connected with some properties of the functions $r_n(t)$, $R_n(t)$, $s_n(t)$, $S_n(t)$ (cf. Figs. 2 and 3).

The functions $s_n(t)$ are used to obtain the estimations for the differences

$$y_n(t)-y_n\left(t-\frac{\delta}{n}\right) \quad \text{and} \quad y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)$$

(cf. (8.24) and Lemma 6).

The initial values of these differences, i.e. the values $y_n(t_j)-y_n(t_{j-1})$ can be estimated with the aid of functions $r_n(t)$ (cf. Lemma 5 and (7.26)).

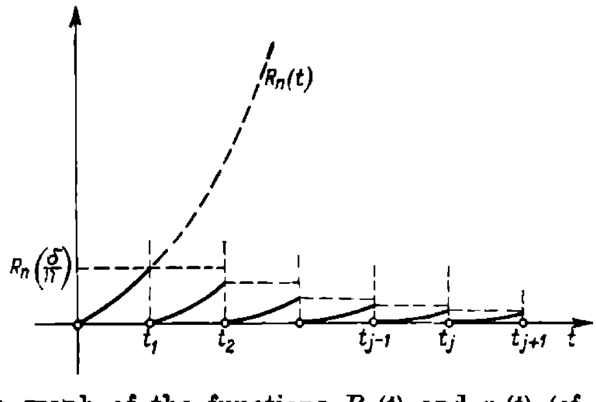

Fig. 2. The graph of the functions $R_n(t)$ and $r_n(t)$ (cf. Lemma 3)

In addition, $R_n(t)$ and $r_n(t)$ supply the precise location of $y_n(t)$ and of each element of $y_n(t)$ respectively (cf. Lemma 4).

Let θ be an arbitrary prescribed interval

$$\theta: \quad 0 \leqslant t < \tau \ (\tau < a),$$

bounded and contained in the interval I'.

LEMMA 3. *Suppose that the real-valued functions $r_n(t)$ satisfy the following conditions in the interval θ:*

1° *$r_n(t) = R_n(t)$ for $t \in \Delta_1(n)$, where $R_n(t)$ denotes the solution of the linear non-homogeneous differential equation*

$$R_n'(t) = \frac{M+N}{1-L}\cdot R_n(t) + \frac{C_n(t)}{1-L} \quad \text{for} \quad t \in \theta, \tag{5.1}$$

satisfying the initial condition $R_n(0) = 0$, and

$$C_n(t) = \gamma\cdot t + \gamma_n \quad \text{for} \quad t \in \theta, \tag{5.2}$$
$$\gamma = (M+N)\cdot s'(\tau) + K; \qquad \gamma_n = 2\cdot L^n\cdot \frac{P}{1-L}.$$

In these formulas $s(t)$ is the solution of linear equation (3.6).

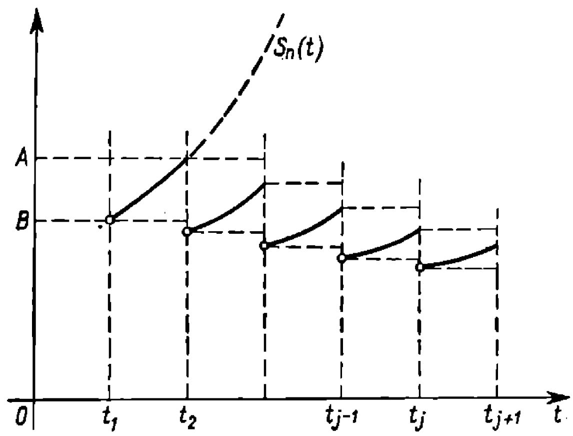

Fig. 3. The graph of the functions $s_n(t)$ and $S_n(t)$ (cf. Lemma 3)

$$OA = S_n\left(2\cdot\frac{\delta}{n}\right), \quad OB = R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n}\cdot s'(\tau)$$

2° *$r_n(t)$, $t \in \Delta_{j+1}(n)$ $(j \geqslant 1)$, is the solution $\varrho = r_n(t)$ of the linear non-homogeneous equation*

$$\varrho'(t) = M\cdot\varrho(t) + N\cdot r_n\left(t-\frac{\delta}{n}\right) + L\cdot r_n'\left(t-\frac{\delta}{n}\right) + c_n(t), \tag{5.3}$$

for $t \in \Delta_{j+1}(n)$ $(j = 1, 2, 3, \ldots)$, where

$$c_n(t) = \gamma\cdot(t-t_j) + \gamma_n \quad \text{for} \quad t \in \Delta_{j+1}(n) \quad (j = 1, 2, 3, \ldots), \tag{5.4}$$

and fulfils the initial condition $\varrho(t_j) = r_n(t_j) = 0$ $(j = 1, 2, 3, \ldots)$.

Thus (5.3) *becomes*

$$r_n'(t) \equiv M\cdot r_n(t) + N\cdot r_n\left(t-\frac{\delta}{n}\right) + L\cdot r_n'\left(t-\frac{\delta}{n}\right) + c_n(t), \tag{5.5}$$

identically for $t \in \Delta_{j+1}(n)$.

Suppose that the real-valued functions $s_n(t)$ satisfy in the interval θ the following conditions:

3° $s_n(t) = S_n(t)$ *for* $t \in \Delta_2(n)$, *where* $S_n(t)$ *is the solution of the linear non-homogeneous differential equation*

$$S_n'(t) = \frac{M+N}{1-L}\cdot S_n(t) + \frac{a_n}{1-L} \quad for \quad \frac{\delta}{n} \leqslant t < \tau, \tag{5.6}$$

where

$$a_n = N\cdot R_n\left(\frac{\delta}{n}\right) + L\cdot R_n'\left(\frac{\delta}{n}\right) + C_n\left(\frac{\delta}{n}\right), \tag{5.7}$$

and satisfies the initial condition for $t = \delta/n$:

$$S_n\left(\frac{\delta}{n}\right) = R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n}\cdot s'(\tau). \tag{5.8}$$

4° $s_n(t)$, $t \in \Delta_{j+1}(n)$ $(j \geqslant 2)$, *is the solution* $\sigma = s_n(t)$ *of the linear non-homogeneous equation*

$$\sigma'(t) = M\cdot\sigma(t) + N\cdot s_n\left(t-\frac{\delta}{n}\right) + L\cdot s_n'\left(t-\frac{\delta}{n}\right) + K\cdot\frac{\delta}{n}, \tag{5.9}$$

for $t \in \Delta_{j+1}(n)$ $(j = 2, 3, \ldots)$, *and fulfils the initial condition*

$$\sigma(t_j) = s_n(t_j) = r_n(t_j - 0) + \frac{\delta}{n}\cdot s'(\tau) \quad (j = 2, 3, \ldots). \tag{5.10}$$

Thus, we have

$$s_n'(t) \equiv M\cdot s_n(t) + N\cdot s_n\left(t-\frac{\delta}{n}\right) + L\cdot s_n'\left(t-\frac{\delta}{n}\right) + K\cdot\frac{\delta}{n}, \tag{5.11}$$

identically for $t \in \Delta_{j+1}(n)$.

Under these assumptions the functions $r_n(t)$ *and* $s_n(t)$ *satisfy the conditions of monotonicity*

$$S_p\left(2\cdot\frac{\delta}{p}\right) \leqslant S_n\left(2\cdot\frac{\delta}{n}\right), \quad S_p'\left(2\cdot\frac{\delta}{p}\right) \leqslant S_n'\left(2\cdot\frac{\delta}{n}\right), \quad a_p \leqslant a_n, \quad for \quad p \geqslant n, \tag{5.12}$$

the conditions of uniform boundedness

$$0 \leqslant r_n(t) < R_n\left(\frac{\delta}{n}\right), \quad 0 \leqslant r_n'(t) < R_n'\left(\frac{\delta}{n}\right) \quad for \quad t \in \theta, \tag{5.13}$$

$$0 \leqslant s_n(t) < S_n\left(2\cdot\frac{\delta}{n}\right), \quad 0 \leqslant s_n'(t) < S_n'\left(2\cdot\frac{\delta}{n}\right) \quad for \quad t_1(n) \leqslant t < \tau, \tag{5.14}$$

and the conditions of convergence

$$R_n\left(\frac{\delta}{n}\right) \to 0, \quad R_n'\left(\frac{\delta}{n}\right) \to 0, \quad S_n\left(2\cdot\frac{\delta}{n}\right) \to 0, \quad S_n'\left(2\cdot\frac{\delta}{n}\right) \to 0, \quad as \quad n \to +\infty, \tag{5.15}$$

$$r_n(t) \Rightarrow 0, \quad r_n'(t) \Rightarrow 0, \quad s_n(t) \Rightarrow 0, \quad s_n'(t) \Rightarrow 0, \quad a_n \to 0, \quad as \quad n \to +\infty \tag{5.16}$$

for $t \in \theta$.

Proof. We verify first the conditions of monotonicity (5.12). The differential equations (5.1) and (5.6) are linear and can be solved in quadratures, which gives

$$(5.17)\qquad R_n(t) = \frac{1}{1-L}\cdot\left[\left(\frac{\gamma}{k^2}+\frac{\gamma_n}{k}\right)\cdot(e^{kt}-1-kt)+\gamma_n\cdot t\right] \quad \text{for} \quad t\in\theta\,,$$

$$(5.18)\qquad S_n(t) = \left[R_n(t_1)+\frac{\delta}{n}\cdot s'(\tau)\right]\cdot e^{k(t-t_1)}+\frac{a_n}{M+N}\left[e^{k(t-t_1)}-1\right],$$

for $\delta/n \leqslant t \leqslant \tau$, where $t_1 = t_1(n) = \delta/n$ and $k = (M+N)/(1-L)$.

According to (5.17) we have $R_n(t) \geqslant 0$ for $t\in\theta$, whence $R'_n(t) \geqslant 0$ for $t\in\theta$ as can be seen from (5.1), i.e. $R_n(t)$ is an increasing function for $t\in\theta$. Therefore it follows from (5.1) that the derivative $R'_n(t)$ is also an increasing function for $t\in\theta$ and (cf. (5.7)):

$$(5.19)\qquad a_p \leqslant a_n \quad \text{for} \quad p\geqslant n\,,$$

which proves the third part of the desired formula (5.12).

In addition (5.17) and (5.1) imply

$$(5.20)\qquad R_n\left(\frac{\delta}{n}\right)\to 0\,, \quad R'_n\left(\frac{\delta}{n}\right)\to 0, \quad \text{as} \quad n\to+\infty\,,$$

whence

$$(5.21)\qquad a_n\to 0\,, \quad \text{as} \quad n\to+\infty\,.$$

The formula (5.18) for $S_n(t)$ and the inequality (5.19) imply

$$(5.22)\qquad S_p\left(2\cdot\frac{\delta}{p}\right) \leqslant S_n\left(2\cdot\frac{\delta}{n}\right) \quad \text{for} \quad p\geqslant n\,.$$

On the other hand, according to (5.22), (5.19) and (5.6) we obtain

$$S'_p\left(2\cdot\frac{\delta}{p}\right) \leqslant S'_n\left(2\cdot\frac{\delta}{n}\right) \quad \text{for} \quad p\geqslant n\,,$$

which means that the conditions of monotonicity (5.12) are satisfied.

We shall now deal with the conditions of the uniform boundedness (5.13) and (5.14). To this end we observe first that the functions $r_n(t)$ satisfy all assumptions of Lemma 2. In fact, the function

$$f_n(t) = c_n(t) = \gamma\cdot(t-t_j)+\gamma_n \quad (j = 0, 1, 2, \ldots), \quad t\in\Delta_{j+1}(n)\,,$$

satisfies the condition (4.7), and according to (5.17) and (5.1) we have in the first interval $\Delta_1(n)$

$$r_n(t)\geqslant 0\,, \quad r'_n(t)\geqslant 0 \quad \text{for} \quad t\in\Delta_1(n)\,,$$
$$r'_n(t)\geqslant M\cdot r_n(t)+N\cdot r_n(t)+L\cdot r'_n(t)+c_n(t) \quad \text{for} \quad t\in\Delta_1(n)\,.$$

Thus, by formula (4.14) of Lemma 2 we obtain the conditions of the uniform boundedness (5.13).

We shall now prove the conditions (5.14). To this end let us observe that $r_n(t)$ satisfy the inequalities

$$r_n\left(t-\frac{\delta}{n}\right) \geqslant r_n(t) \geqslant 0, \qquad r_n'\left(t-\frac{\delta}{n}\right) \geqslant r_n'(t) \geqslant 0,$$

for $t \in \Delta_{j+1}(n)$ $(j = 1, 2, 3, \ldots)$, which immediately gives

$$r_n(t_j-0) \geqslant r_n(t_{j+1}-0) \geqslant 0 \qquad (j = 1, 2, 3, \ldots). \tag{5.23}$$

From (5.23) it follows that the initial values (5.10) of the functions $s_n(t)$ satisfy assumption 3° of Lemma 2. The functions $s_n(t)$ satisfy also assumptions 1° of Lemma 2, since (4.7) holds for $f_n(t) = K\cdot\frac{\delta}{n}$. Furthermore from (5.18) we obtain $s_n(t) \geqslant 0$ for $t \in \Delta_2(n)$, hence the equation (5.6) implies $s_n'(t) \geqslant 0$ for $t \in \Delta_2(n)$, which means that $s_n(t)$ is an increasing function for $t \in \Delta_2(n)$. Therefore it follows from (5.6) that the derivative $s_n'(t)$ is also an increasing function for $t \in \Delta_2(n)$, and

$$s_n'(t) \geqslant M\cdot s_n(t) + N\cdot s_n(t) + L\cdot s_n'(t) + K\cdot\frac{\delta}{n} \quad \text{for} \quad t \in \Delta_2(n).$$

Hence the functions $s_n(t)$ satisfy assumptions 1° of Lemma 2. The functions $s_n(t)$ satisfy also assumptions 2° of Lemma 2 in view of (5.9); thus by formula (4.14) of Lemma 2 we obtain the conditions of the uniform boundedness (5.14).

The conditions of convergence (5.16) follow immediately from (5.13), (5.14) and (5.15); consequently, all that remains to be proved is that relations (5.15) are satisfied.

The first and second part of (5.15) is identical with (5.20), therefore place $t = 2\cdot\frac{\delta}{n}$ in the formula (5.18) and (5.6). Then we obtain

$$\begin{aligned} S_n\left(2\cdot\frac{\delta}{n}\right) &= \left[R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n}\cdot s'(\tau)\right]\cdot e^{k\cdot\frac{\delta}{n}} + \frac{a_n}{M+N}\cdot\left(e^{k\cdot\frac{\delta}{n}}-1\right) \to 0, \\ S_n'\left(2\cdot\frac{\delta}{n}\right) &= k\cdot S_n\left(2\cdot\frac{\delta}{n}\right) + \frac{a_n}{1-L} \to 0, \quad \text{as} \quad n\to+\infty,\ k = \frac{M+N}{1-L}, \end{aligned} \tag{5.24}$$

which means that conditions of convergence (5.15) are fulfilled.

This completes the proof of Lemma 3.

Remark 3. The functions $r_n(t)$ and $s_n(t)$ satisfy all assumptions of Lemma 2; therefore in particular

$$\begin{aligned} r_n\left(t-\frac{\delta}{n}\right) &\geqslant r_n(t) \geqslant 0, \qquad r_n'\left(t-\frac{\delta}{n}\right) \geqslant r_n'(t) \geqslant 0, \\ s_n\left(t-\frac{\delta}{n}\right) &\geqslant s_n(t) \geqslant 0, \qquad s_n'\left(t-\frac{\delta}{n}\right) \geqslant s_n'(t) \geqslant 0, \end{aligned} \tag{5.25}$$

for $t \in \Delta_{j+1}(n)$ $(j = 1, 2, 3, \ldots)$.

These inequalities are in connection with a certain property of estimates used in § 7 (cf. Lemma 4).

§ 6. Now we shall give a theorem connected with the existence and location of the functions $y_n(t)$ and of derivatives $y_n'(t)$ in some interval common for all $y_n(t)$.

THEOREM 1. *Suppose that the function $h(t,u,v,w)$ fulfils assumptions* H.

Under these assumptions the continuous functions $y_n(t)$ satisfying conditions

$$y_n'(t) = h\big(t, y_n(t), 0, d_n\big) \quad \textit{for} \quad t \in \Delta_1(n),\ y_n(0) = 0\,,$$

$$(6.1) \qquad y_n'(t) = h\left(t, y_n(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right) \quad \textit{for} \quad \frac{\delta}{n} \leqslant t < a\,,$$

$$d_{n+1} = h(0,0,0,d_n) \quad (n = 1,2,3,\ldots), \quad d_1 = 0\,,$$

are defined in a common interval I': $0 \leqslant t < a$ and satisfy the inequalities

$$(6.2) \qquad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \textit{for} \quad t \in I'\,,$$

where $s(t)$, $t \in I'$, denotes the solution of the non-homogeneous linear equation (3.6) *and* $s(0) = 0$.

Proof. We can easily verify that the function $y_n(t)$ is defined in the first interval $\Delta_1(n)$, and satisfies the inequalities

$$(6.3) \qquad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_1(n)\,.$$

Proceeding by induction let us suppose that the function $y_n(t)$ is defined in the interval $0 \leqslant t < t_j$ $(j \geqslant 1)$, and fulfils the conditions

$$(6.4) \qquad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \text{for} \quad 0 \leqslant t < t_j \ (j \geqslant 1)\,.$$

We prove that the function $y_n(t)$ exists in the interval $0 \leqslant t < t_{j+1}$ and

$$(6.5) \qquad \|y_n(t)\| \leqslant s(t), \quad \|y_n'(t)\| \leqslant s'(t) \quad \text{for} \quad 0 \leqslant t < t_{j+1}\,.$$

In fact, consider the differential equation

$$(6.6) \qquad \eta'(t) = h\left(t, \eta(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right) \quad \text{for} \quad t \in \Delta_{j+1}(n)\,,$$

the initial condition

$$(6.7) \qquad \eta(t_j) = y_n(t_j)\,,$$

and the sequence of successive approximations

$$\eta_0(t) \equiv y_n(t_j) \quad \text{for} \quad t \in \Delta_{j+1}(n),$$

$$\eta'_{i+1}(t) = h\left(t, \eta_i(t), y_n\left(t-\frac{\delta}{n}\right), y'_n\left(t-\frac{\delta}{n}\right)\right) \quad \text{for} \quad t \in \Delta_{j+1}(n), \tag{6.8}$$

$$\eta_i(t_j) = y_n(t_j) \quad (i = 0, 1, 2, \ldots).$$

The first approximation $\eta_0(t) \equiv y_n(t_j)$ satisfies the inequalities

$$\|\eta_i(t)\| \leqslant s(t), \quad \|\eta'_i(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n), \tag{6.9}$$

therefore suppose that the approximation $\eta_i(t)$ satisfies (6.9). Then the next approximation $\eta_{i+1}(t)$ fulfils the conditions

$$\|\eta_{i+1}(t)\| \leqslant s(t), \quad \|\eta'_{i+1}(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n), \tag{6.10}$$

because of Theorem A, since $\|\eta_{i+1}(t)\|$ satisfies the differential inequality

$$\begin{aligned}
\|\eta'_{i+1}(t)\| &\leqslant \left\|h\left(t, \eta_i(t), y_n\left(t-\frac{\delta}{n}\right), y_n\left(t-\frac{\delta}{n}\right)\right) - h(t, 0, 0, 0)\right\| + \|h(t, 0, 0, 0)\| \\
&\leqslant M \cdot \|\eta_i(t)\| + N \cdot \left\|y_n\left(t-\frac{\delta}{n}\right)\right\| + L \cdot \left\|y'_n\left(t-\frac{\delta}{n}\right)\right\| + \bar{h}(t) \\
&\leqslant M \cdot s(t) + N \cdot s\left(t-\frac{\delta}{n}\right) + L \cdot s'\left(t-\frac{\delta}{n}\right) + \bar{h}(t) \\
&\leqslant M \cdot s(t) + N \cdot s(t) + L \cdot s'(t) + \bar{h}(t) + \frac{L \cdot P}{1-L},
\end{aligned}$$

i.e. the inequality (cf. Theorem B):

$$\bar{D}_+\|\eta_{i+1}(t)\| \leqslant M \cdot s(t) + N \cdot s(t) + L \cdot s'(t) + \bar{h}(t) + \frac{L \cdot P}{1-L} \quad \text{for} \quad t \in \Delta_{j+1}(n),$$

the function $s(t)$ is the solution of (3.6), and the relation

$$\|\eta_{i+1}(t_j)\| \leqslant s(t_j),$$

holds for the initial values (cf. (6.4) and (6.7)).

Consequently (6.9) holds for all $i = 0, 1, 2, \ldots$, because of the principle of finite induction, whence the solution $\eta(t)$ of the equation (6.6) exists in view of Theorem D and satisfies the conditions

$$\|\eta(t)\| \leqslant s(t), \quad \|\eta'(t)\| \leqslant s'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n). \tag{6.11}$$

We define now $y_n(t) = \eta(t)$ for $t \in \Delta_{j+1}(n)$, and we obtain

$$\|y_n(t)\| \leqslant s(t), \quad \|y'_n(t)\| \leqslant s'(t) \quad \text{for} \quad 0 \leqslant t < t_{j+1},$$

because of (6.4) and (6.11), which completes the proof of (6.5).

The principle of finite induction, (6.3), (6.4) and (6.5) imply that the estimations (6.2) hold in the whole interval I', and this completes the proof of Theorem 1.

§ 7. Now we shall give the precise location of the functions $y_p(t)$ and their derivatives $y'_p(t)$ $(p \geqslant n)$, $t \in \theta$, with respect to the tangent $y(t) = t \cdot y'_p(0)$, $t \in \theta$, at the origin. Here the functions $R_n(t)$ of Lemma 3 will be used.

We shall give also the precise location of the curvilinear segment $y_n(t)$, $t \in \Delta_{j+1}(n)$, with respect to the tangent

$$y(t) = y_n(t_j) + (t - t_j) \cdot y'_n(t_j) \qquad \text{for} \qquad t \in \Delta_{j+1}(n),$$

at the point $(t_j, y_n(t_j))$ with the aid of functions $r_n(t)$ of Lemma 3.

It seems to be worth observing that there is a certain characteristic property connected with the difference method (6.1) namely estimates (7.2) and (7.19) are decreasing when the interval $\Delta_j(n)$ is replaced by the next interval $\Delta_{j+1}(n)$ (cf. property (5.25)).

LEMMA 4. *Let us suppose that the function $h(t, u, v, w)$ satisfies assumptions* H, *consider the sequence $y_n(t)$ defined by* (6.1), *and assume that θ, $0 \leqslant t < \tau$ $(\tau < a)$, is an arbitrary prescribed interval, bounded and contained in the interval I'.*

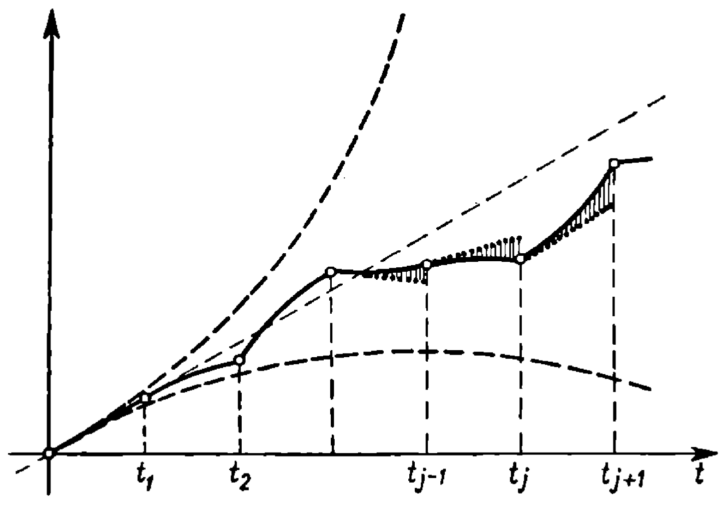

Fig. 4. The precise location of $y_n(t)$ with the aid of the function $R_n(t)$ (also the precise location of segments of $y_n(t)$, cf. Lemma 4)

Under these assumptions the functions $y_n(t)$ fulfil the inequalities

$$\text{(7.1)} \qquad \begin{aligned} &\|y_p(t) - t \cdot y'_p(0)\| \leqslant R'_n(t), \\ &\|y'_p(t) - y'_p(0)\| \leqslant R'_n(t) \qquad \textit{for} \qquad t \in \theta,\ p \geqslant n, \end{aligned}$$

in the interval θ, *and*

$$\begin{aligned}\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y'_n(t_j)\| &\leqslant r_n(t)\,,\\ \|y'_n(t)-y'_n(t_j)\| &\leqslant r'_n(t)\,,\end{aligned}\tag{7.2}$$

in the intervals $\Delta_{j+1}(n)\cdot\theta$ $(j=0,1,2,\ldots)$, *the functions* $R_n(t)$ *and* $r_n(t)$ *being defined in the part* 1° *and* 2° *of Lemma* 3.

Proof. We prove first that (7.1) are fulfilled in the first interval $\Delta_1(p)$ $(p\geqslant n)$, choosing n so as to obtain $\delta/n<\tau$.

In fact, all functions $y_p(t)$ $(p=1,2,3,\ldots)$ exist in the common interval I' in view of Theorem 1, whence from (6.1) and (4.2) it follows

$$\begin{aligned}\|y'_p(t)-y'_p(0)\| &= \big\|h\big(t,y_p(t),0,d_p\big)-h(0,0,0,d_p)\big\|\leqslant K\cdot t+M\cdot\|y_p(t)\|\\ &\leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\|+N\cdot R_n(t)+L\cdot R'_n(t)+\\ &\quad+[(M+N)\cdot\|y'_p(0)\|+K]\cdot t+2\cdot L^n\cdot\frac{P}{1-L}\qquad\text{for}\quad t\in\Delta_1(p)\,.\end{aligned}\tag{7.3}$$

But $\|y'_p(0)\|\leqslant s'(\tau)$ in virtue of (6.2), whence the function $\|y_p(t)-t\cdot y'_p(0)\|$ satisfies the differential inequality (cf. Theorem B):

$$\bar{D}_+\|y_p(t)-t\cdot y'_p(0)\|\leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\|+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t)\tag{7.4}$$

for $t\in\Delta_1(p)$, the function $\lambda=R_n(t)$ satisfies the differential equation (cf. (5.1)):

$$\lambda'(t)=M\cdot\lambda(t)+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t)\qquad\text{for}\quad t\in\theta\,,\tag{7.5}$$

and initial values are equal:

$$R_n(0)=\|y_p(0)-0\cdot y'_p(0)\|=0\,.$$

Therefore Theorem A implies

$$\|y_p(t)-t\cdot y'_p(0)\|\leqslant R_n(t)\qquad\text{for}\quad t\in\Delta_1(p)\ (p\geqslant n)\,.\tag{7.6}$$

In addition, from (7.3), (7.5) and (7.6) we obtain

$$\begin{aligned}\|y'_p(t)-y'_p(0)\| &\leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\|+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t)\\ &\leqslant M\cdot R_n(t)+N\cdot R_n(t)+L\cdot R'_n(t)+C_n(t)=R'_n(t)\,,\end{aligned}\tag{7.7}$$

for $t\in\Delta_1(p)$ $(p\geqslant n)$, which means that the inequalities

$$\begin{aligned}\|y_p(t)-t\cdot y'_p(0)\| &\leqslant R_n(t)\,,\\ \|y'_p(t)-y'_p(0)\| &\leqslant R'_n(t)\,,\end{aligned}\tag{7.8}$$

hold in the interval $\Delta_1(p)$ $(p\geqslant n)$.

We shall verify now that the relation (7.1) hold in the interval $\delta/p \leqslant t < \tau$ $(p \geqslant n)$. In fact, (6.1), (4.2), (7.8) and the monotonicity of $R_n(t)$ and $R'_n(t)$ imply that

$$(7.9)\qquad \|y'_p(t)-y'_p(0)\| = \left\|h\left(t, y_p(t), y_p\left(t-\frac{\delta}{p}\right), y'_p\left(t-\frac{\delta}{p}\right)\right)-h(0,0,0,d_p)\right\|$$
$$\leqslant K\cdot t + M\cdot\|y_p(t)\| + N\cdot\left\|y_p\left(t-\frac{\delta}{p}\right)\right\| + L\cdot\left\|y'_p\left(t-\frac{\delta}{p}\right)-d_p\right\|$$
$$\leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\| + N\cdot\left\|y_p\left(t-\frac{\delta}{p}\right)-\left(t-\frac{\delta}{p}\right)\cdot y'_p(0)\right\| +$$
$$+[(M+N)\cdot\|y'_p(0)\|+K]\cdot t + L\cdot\left\|y'_p\left(t-\frac{\delta}{p}\right)-d_{p+1}\right\| + L\cdot\|d_{p+1}-d_p\|$$
$$\leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t),$$

for $\delta/p \leqslant t < \tau$ $(p \geqslant n)$, since $\|y'_p(0)\| \leqslant s'(\tau)$.

Thus, the function $\|y_p(t)-t\cdot y'_p(0)\|$ satisfies the differential inequality

$$(7.10)\qquad \bar{D}_+\|y_p(t)-t\cdot y'_p(0)\| \leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C(t)$$
$$\text{for}\quad \delta/p \leqslant t < \tau\ (p \geqslant n),$$

the function $\lambda = R_n(t)$ satisfies the differential equation (7.5), and initial values for $t = \delta/p$ fulfil the inequality

$$\left\|y_p\left(\frac{\delta}{p}\right)-\frac{\delta}{p}\cdot y'_p(0)\right\| \leqslant R_n\left(\frac{\delta}{p}\right)$$

in view of (7.6), hence from Theorem A we obtain

$$(7.11)\qquad \|y_p(t)-t\cdot y'_p(0)\| \leqslant R_n(t) \quad \text{for} \quad \delta/p \leqslant t < \tau\ (p \geqslant n).$$

In addition, (7.11), (7.9) and (7.5) imply that

$$\|y'_p(t)-y'_p(0)\| \leqslant M\cdot\|y_p(t)-t\cdot y'_p(0)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t)$$
$$\leqslant M\cdot R_n(t) + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t)$$
$$= R'_n(t) \quad \text{for} \quad \delta/p \leqslant t < \tau\ (p \geqslant n).$$

This completes the proof of (7.1) in the whole interval θ: $0 \leqslant t < \tau$ for $p \geqslant n$.

The proof of the inequalities (7.2) in the first interval $\Delta_1(n)$ is completed, since (7.1) for $p = n$ is identical with (7.2) in the first interval $\Delta_1(n)$.

Proceeding by induction suppose that

$$(7.12)\qquad \begin{aligned} &\|y_n(t)-y_n(t_{j-1})-(t-t_{j-1})\cdot y'_n(t_{j-1})\| \leqslant r_n(t), \\ &\|y'_n(t)-y'_n(t_{j-1})\| \leqslant r'_n(t) \quad \text{for} \quad t \in \Delta_j(n)\cdot\theta\ (j \text{ fixed}). \end{aligned}$$

We shall prove that

$$
(7.13)\qquad
\begin{aligned}
&\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|\leqslant r_n(t)\,,\\
&\|y_n'(t)-y_n'(t_j)\|\leqslant r_n'(t)\quad \text{for}\quad t\in \Delta_{j+1}(n)\cdot\theta\,.
\end{aligned}
$$

In fact, from (7.12) and (5.4) it follows that

$$
\begin{aligned}
(7.14)\quad &\|y_n'(t)-y_n'(t_j)\|\\
&=\left\|h\left(t,\,y_n(t),\,y_n\left(t-\frac{\delta}{n}\right),\,y_n'\left(t-\frac{\delta}{n}\right)\right)-h\big(t_j,\,y_n(t_j),\,y_n(t_{j-1}),\,y_n'(t_{j-1})\big)\right\|\\
&\leqslant K\cdot(t-t_j)+M\cdot\|y_n(t)-y_n(t_j)\|+N\cdot\left\|y_n\left(t-\frac{\delta}{n}\right)-y_n(t_{j-1})\right\|+\\
&\quad+L\cdot\left\|y_n'\left(t-\frac{\delta}{n}\right)-y_n'(t_{j-1})\right\|\\
&\leqslant M\cdot\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|+M\cdot(t-t_j)\cdot\|y_n'(t_j)\|+\\
&\quad+N\cdot\left\|y_n\left(t-\frac{\delta}{n}\right)-y_n(t_{j-1})-\left(t-\frac{\delta}{n}-t_{j-1}\right)\cdot y_n'(t_{j-1})\right\|+\\
&\quad+N\cdot\left(t-\frac{\delta}{n}-t_{j-1}\right)\cdot\|y_n'(t_{j-1})\|+L\cdot r_n'\left(t-\frac{\delta}{n}\right)+K\cdot(t-t_j)\\
&\leqslant M\cdot\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|+N\cdot r_n\left(t-\frac{\delta}{n}\right)+L\cdot r_n'\left(t-\frac{\delta}{n}\right)+c_n(t)\,.
\end{aligned}
$$

Thus, the function $\mu(t)=\|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|$ satisfies the differential inequality (cf. Theorem B)

$$
(7.15)\quad \bar{D}_+\mu(t)\leqslant M\cdot\mu(t)+N\cdot r_n\left(t-\frac{\delta}{n}\right)+L\cdot r_n'\left(t-\frac{\delta}{n}\right)+c_n(t)\ \text{ for }\ t\in\Delta_{j+1}(n)\cdot\theta,
$$

the function $\varrho=r_n(t)$ satisfies the differential equation (5.3) and initial values are equal: $\mu(t_j)=r_n(t_j)=0$, whence from Theorem A we obtain

$$
(7.16)\qquad \|y_n(t)-y_n(t_j)-(t-t_j)\cdot y_n'(t_j)\|\leqslant r_n(t)\quad \text{for}\quad t\in\Delta_{j+1}(n)\cdot\theta\,.
$$

In addition, (7.16), (7.14) and (5.3) imply that the second inequality (7.13) holds for $t\in\Delta_{j+1}(n)\cdot\theta$, which completes the proof of (7.13).

By induction relations (7.2) are satisfied in the whole interval θ. This completes the proof of Lemma 4.

We shall now give the estimates for $y_n(t)-y_n(t_j)$ and $y_n'(t)-y_n'(t_j)$ with the aid of functions $r_n(t)$ considered in Lemma 3.

LEMMA 5. *Let us suppose that the function $h(t,u,v,w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6.1).

Under these assumptions

$$
(7.17)\qquad
\begin{aligned}
&\|y_n(t)-y_n(t_{j-1})\|\leqslant r_n(t)+(t-t_{j-1})\cdot s'(\tau)\,,\\
&\|y_n'(t)-y_n'(t_{j-1})\|\leqslant r_n'(t)\quad \textit{for}\quad t\in\Delta_j(n)\cdot\theta\ \ (j=1,2,3,\ldots)\,,
\end{aligned}
$$

and in particular

$$(7.18)\quad \begin{aligned} &\|y_n(t)-y_n(t_{j-1})\| \leqslant R_n\left(\frac{\delta}{n}\right)+\left(\frac{\delta}{n}\right)\cdot s'(\tau), \\ &\|y'_n(t)-y'_n(t_{j-1})\| \leqslant R'_n\left(\frac{\delta}{n}\right) \quad \textit{for} \quad t\in \Delta_j(n)\cdot\theta \ (j=1,2,3,\ldots). \end{aligned}$$

Proof. Inequalities (7.1) and (7.2) imply the second line of the formula (7.17). But

$$\begin{aligned} &\|y_n(t)-y_n(t_{j-1})\| \\ &\quad \leqslant \|y_n(t)-y_n(t_{j-1})-(t-t_{j-1})\cdot y'_n(t_{j-1})\|+(t-t_{j-1})\cdot\|y'_n(t_{j-1})\| \\ &\quad \leqslant r_n(t)+(t-t_{j-1})\cdot s'(\tau) \quad \text{for} \quad t\in\Delta_j(n)\cdot\theta \ (j=1,2,3,\ldots) \end{aligned}$$

which completes the proof of (7.17).

Inequalities (7.18) follow immediately from (7.17) and the conditions of uniform boundedness (5.13).

This completes the proof of Lemma 5.

We shall now give the estimates for $y_n(t)-y_n\left(t-\frac{\delta}{n}\right)$ and $y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)$, with the aid of the functions $s_n(t)$ defined in Lemma 3.

LEMMA 6. *Suppose that the function $h(t,u,v,w)$ satisfies assumptions* H *and consider the sequence $y_n(t)$ defined by* (6.1).

Under these assumptions

$$(7.19)\quad \begin{aligned} &\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant s_n(t), \\ &\left\|y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant s'_n(t) \quad \textit{for} \quad \frac{\delta}{n}\leqslant t<\tau, \end{aligned}$$

and in particular

$$(7.20)\quad \begin{aligned} &\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant S_n\left(2\cdot\frac{\delta}{n}\right), \\ &\left\|y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant S'_n\left(2\cdot\frac{\delta}{n}\right) \quad \textit{for} \quad \frac{\delta}{n}\leqslant t<\tau. \end{aligned}$$

Proof. We prove first that (7.19) holds in the interval $\Delta_2(n)$. In fact, (6.1), (7.18), (4.2) and (5.7) imply

$$\begin{aligned} (7.21)\quad &\left\|y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)\right\| \\ &=\left\|h\left(t,\,y_n(t),\,y_n\left(t-\frac{\delta}{n}\right),\,y'_n\left(t-\frac{\delta}{n}\right)\right)-h\left(t-\frac{\delta}{n},\,y_n\left(t-\frac{\delta}{n}\right),\,0,\,d_n\right)\right\| \\ &\leqslant K\cdot\frac{\delta}{n}+M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+N\cdot\left\|y_n\left(t-\frac{\delta}{n}\right)-y_n(0)\right\|+L\cdot\left\|y'_n\left(t-\frac{\delta}{n}\right)-d_n\right\| \\ &\leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+N\cdot\left[R_n\left(\frac{\delta}{n}\right)+\frac{\delta}{n}\cdot s'(\tau)\right]+L\cdot R'_n\left(\frac{\delta}{n}\right)+L\cdot\|d_{n+1}-d_n\|+K\cdot\frac{\delta}{n} \\ &\leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+N\cdot s_n(t)+L\cdot s'_n(t)+a_n \quad \text{for} \quad t\in\Delta_2(n). \end{aligned}$$

Thus the function $\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|$ fulfils the differential inequality

$$\overline{D}_+\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| + N\cdot s_n(t) + L\cdot s_n'(t) + a_n, \tag{7.22}$$

for $t \in \Delta_2(n)$, the function $s_n(t)$ satisfies the differential equation (5.6)

$$s_n'(t) = M\cdot s_n(t) + N\cdot s_n(t) + L\cdot s_n'(t) + a_n \quad \text{for} \quad t \in \Delta_2(n),$$

and initial values for $t = \delta/n$ satisfy the condition (cf. (7.18) and (5.8))

$$\left\|y_n\left(\frac{\delta}{n}\right) - y_n(0)\right\| \leqslant R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n}\cdot s'(\tau) = s_n\left(\frac{\delta}{n}\right).$$

Hence Theorem A on differential inequalities implies

$$\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant s_n(t) \quad \text{for} \quad t \in \Delta_2(n). \tag{7.23}$$

In addition, from (7.21), (7.23) and (5.6) we obtain

$$\begin{aligned}\left\|y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)\right\| &\leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| + N\cdot s_n(t) + L\cdot s_n'(t) + a_n \\ &\leqslant M\cdot s_n(t) + N\cdot s_n(t) + L\cdot s_n'(t) + a_n = s_n'(t)\end{aligned}$$

for $t \in \Delta_2(n)$, which means that (7.19) hold in the interval $\Delta_2(n)$.

Proceeding by induction suppose that (7.19) hold in the interval $\Delta_j(n)\cdot\theta$ (j fixed).

We shall prove that (7.19) are fulfilled in the next interval $\Delta_{j+1}(n)\cdot\theta$, i.e.

$$\begin{aligned}&\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant s_n(t), \\ &\left\|y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)\right\| \leqslant s_n'(t) \quad \text{for} \quad t \in \Delta_{j+1}(n)\cdot\theta.\end{aligned} \tag{7.24}$$

In fact, from (6.1) and the induction assumption we obtain in the interval $\Delta_{j+1}(n)\cdot\theta$:

$$\begin{aligned}&\left\|y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)\right\| \\ &= \left\|h\left(t, y_n(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right) - h\left(t-\frac{\delta}{n}, y_n\left(t-\frac{\delta}{n}\right), y_n\left(t-2\cdot\frac{\delta}{n}\right), y_n'\left(t-2\cdot\frac{\delta}{n}\right)\right)\right\| \\ &\leqslant K\cdot\frac{\delta}{n} + M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| + N\cdot\left\|y_n\left(t-\frac{\delta}{n}\right)-y_n\left(t-2\cdot\frac{\delta}{n}\right)\right\| + \\ &\qquad\qquad + L\cdot\left\|y_n'\left(t-\frac{\delta}{n}\right)-y_n'\left(t-2\cdot\frac{\delta}{n}\right)\right\| \\ &\leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| + N\cdot s_n\left(t-\frac{\delta}{n}\right) + L\cdot s_n'\left(t-\frac{\delta}{n}\right) + K\cdot\frac{\delta}{n} \quad \text{for} \quad t \in \Delta_{j+1}(n)\cdot\theta.\end{aligned} \tag{7.25}$$

Hence the function $\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|$ satisfies the differential inequality (cf. Theorem B):

$$\bar{D}_+\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+N\cdot s_n\left(t-\frac{\delta}{n}\right)+L\cdot s_n'\left(t-\frac{\delta}{n}\right)+K\cdot\frac{\delta}{n}, \tag{7.26}$$

in the interval $\Delta_{j+1}(n)\cdot\theta$, the function $s_n(t)$ fulfils equation (5.9) in the interval $\Delta_{j+1}(n)\cdot\theta$, and initial values for $t = t_j$ satisfy the condition

$$\|y_n(t_j)-y_n(t_{j-1})\| \leqslant r_n(t_j-0)+\frac{\delta}{n}\cdot s'(\tau) = s_n(t_j),$$

because of (7.17) and (5.10).

Therefore Theorem A implies

$$\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant s_n(t) \quad \text{for} \quad t\in\Delta_{j+1}(n)\cdot\theta. \tag{7.27}$$

Furthermore, using successively (7.25), (7.27) and the equation (5.9) we obtain

$$\left\|y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)\right\| \leqslant M\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+N\cdot s_n\left(t-\frac{\delta}{n}\right)+L\cdot s_n'\left(t-\frac{\delta}{n}\right)+K\cdot\frac{\delta}{n}$$
$$\leqslant M\cdot s_n(t)+N\cdot s_n\left(t-\frac{\delta}{n}\right)+L\cdot s_n'\left(t-\frac{\delta}{n}\right)+K\cdot\frac{\delta}{n} = s_n'(t),$$

for $t\in\Delta_{j+1}(n)\cdot\theta$, which completes the proof of (7.24).

By induction relations (7.19) are satisfied in the whole interval $\delta/n \leqslant t < \tau$.

The conditions (7.20) can be obtained from (7.19) and (5.14), and this completes the proof of Lemma 6.

§ 8. Now we shall prove a theorem connected with the existence and uniqueness of the solution.

THEOREM 2. *Let us suppose that the right-hand member of the differential equation*

$$y' = h(t, y, y, y'), \tag{8.1}$$

satisfies assumptions H.

Under these assumptions

1° *The sequence of functions* $y_n(t)$ *defined by the formula*

$$y_n'(t) = h\big(t, y_n(t), 0, d_n\big) \quad \textit{for} \quad t\in\Delta_1(n),\ y_n(0) = 0,$$
$$y_n'(t) = h\left(t, y_n(t), y_n\left(t-\frac{\delta}{n}\right), y_n'\left(t-\frac{\delta}{n}\right)\right) \quad \textit{for} \quad \frac{\delta}{n}\leqslant t < a, \tag{8.2}$$
$$d_{n+1} = h(0,0,0,d_n) \quad (n = 1,2,3,\dots), \quad d_1 = 0,$$

converges almost uniformly in the interval I': $0\leqslant t < a$ *to the unique solution* $y = \varphi(t)$, $t\in I'$, *of equation* (8.1), *satisfying the initial condition* $\varphi(0) = 0$.

2° *The error estimates of the form*

$$(8.3)\qquad \begin{aligned}&\|y_q(t)-\varphi(t)\| \leqslant k_n(t)\,,\\ &\|y_q'(t)-\varphi'(t)\| \leqslant k_n'(t) \qquad (q \geqslant n), \qquad t \in \theta\,,\end{aligned}$$

hold in an arbitrary prescribed interval θ:

$$(8.4)\qquad \theta:\quad 0 \leqslant t < \tau \quad (\tau < a)\,,$$

bounded and contained in the interval I'.

Here the real-valued function $k_n(t)$, $t \in \theta$, *is the solution of the linear equation with constant coefficients*:

$$(8.5)\qquad k_n'(t) = \frac{M+N}{1-L}\cdot k_n(t) + \frac{b_n}{1-L}\,,$$

and satisfies the initial condition $k_n(0) = 0$. *The constant* $b_n \geqslant 0$ *is defined by*

$$(8.6)\qquad b_n = 2\cdot N\cdot S_n\Big(2\cdot\frac{\delta}{n}\Big) + 2\cdot L\cdot S_n'\Big(2\cdot\frac{\delta}{n}\Big) + C_n\frac{\delta}{n}\,,$$

can be computed from (5.2), (5.24) *and* (3.6), *and fulfils the condition*

$$(8.7)\qquad b_n \to 0, \quad \text{as} \quad n \to +\infty\,.$$

Proof. a) We shall prove that the sequence $y_n(t)$ and the sequence of derivatives $y_n'(t)$ satisfy the Cauchy criterion of the almost uniform convergence in the interval I'. For this purpose let us denote by θ: $0 \leqslant t < \tau$ $(\tau < a)$ an arbitrary prescribed interval, bounded and contained in the interval I', and suppose that $\delta/n < \tau$.

We shall verify first that

$$(8.8)\qquad \begin{aligned}&\|y_p(t)-y_q(t)\| \leqslant k_n(t)\,,\\ &\|y_p'(t)-y_q'(t)\| \leqslant k_n'(t), \qquad t \in \Delta_1(q),\ p \geqslant q \geqslant n\,,\end{aligned}$$

in the interval $\Delta_1(q)$: $0 \leqslant t < \delta/q$. In order to see this it is sufficient to verify that

$$(8.9)\qquad \begin{aligned}&\|y_p(t)-y_q(t)\| \leqslant R_n(t)\,,\\ &\|y_p'(t)-y_q'(t)\| \leqslant R_n'(t) \qquad \text{for} \quad t \in \Delta_1(q)\,,\ p \geqslant q \geqslant n\,,\end{aligned}$$

and

$$(8.10)\qquad R_n(t) \leqslant k_n(t), \qquad R_n'(t) \leqslant k_n'(t) \qquad \text{for} \quad t \in \Delta_1(q)\,,$$

where $R_n(t)$ is the solution of the linear equation (5.1).

In fact, in the interval $\Delta_1(p)$: $0 \leqslant t < \delta/p$ we obtain from (8.2) and (4.2):

$$(8.11)\qquad \begin{aligned}\|y_p'(t)-y_q'(t)\| &= \big\|h\big(t, y_p(t), 0, d_p\big) - h\big(t, y_q(t), 0, d_q\big)\big\|\\ &\leqslant M\cdot\|y_p(t)-y_q(t)\| + L\cdot\|d_p - d_q\|\\ &\leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot R_n(t) + L\cdot R_n'(t) + C_n(t)\end{aligned}$$

for $t \in \Delta_1(p)$, $C_n(t)$ being defined by (5.2).

Furthermore, using successively (8.2), (7.1), (4.2) and monotonicity of $R_n(t)$ and $R'_n(t)$ we see that in the interval $\delta/p \leqslant t < \delta/q$:

$$(8.12)\qquad \begin{aligned} &\|y'_p(t)-y'_q(t)\| \\ &= \left\|h\left(t, y_p(t), y_p\left(t-\frac{\delta}{p}\right), y'_p\left(t-\frac{\delta}{p}\right)\right) - h\big(t, y_q(t), 0, d_q\big)\right\| \\ &\leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot\left\|y_p\left(t-\frac{\delta}{p}\right)\right\| + L\cdot\left\|y'_p\left(t-\frac{\delta}{p}\right)-d_q\right\| \\ &\leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot\left\|y_p\left(t-\frac{\delta}{p}\right) - \left(t-\frac{\delta}{p}\right)\cdot y'_p(0)\right\| + \\ &\quad + [(M+N)\cdot\|y'_p(0)\|+K]\cdot t + L\cdot\left\|y'_p\left(t-\frac{\delta}{p}\right)-d_{p+1}\right\| + L\cdot\|d_{p+1}-d_q\| \\ &\leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t)\,. \end{aligned}$$

So (8.11) and (8.12) imply that the relation

$$(8.13)\qquad \|y'_p(t)-y'_q(t)\| \leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t)\,,$$

holds in the whole interval $\Delta_1(q)$: $0 \leqslant t < \delta/q$ $(p \geqslant q \geqslant n)$.

Thus, the function $\|y_p(t)-y_q(t)\|$ satisfies the differential inequality (cf. Theorem B):

$$(8.14)\qquad \bar{D}_+\|y_p(t)-y_q(t)\| \leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t)\,,$$

for $t \in \Delta_1(q)$ $(p \geqslant q \geqslant n)$, the function $\lambda = R_p(t)$ fulfils the differential equation (cf. (5.1)):

$$(8.15)\qquad \lambda'(t) = M\cdot\lambda(t) + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t) \qquad \text{for} \qquad t\in\theta\,,$$

and initial values are equal: $R_n(0) = \|y_p(0)-y_q(0)\| = 0$, whence from Theorem A we obtain

$$(8.16)\qquad \|y_p(t)-y_q(t)\| \leqslant R_n(t) \qquad \text{for} \qquad t\in\Delta_1(q)\ (p \geqslant q \geqslant n)\,.$$

In addition, (8.13), (8.16) and (8.15) imply that

$$(8.17)\qquad \begin{aligned} \|y'_p(t)-y'_q(t)\| &\leqslant M\cdot\|y_p(t)-y_q(t)\| + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t) \\ &\leqslant M\cdot R_n(t) + N\cdot R_n(t) + L\cdot R'_n(t) + C_n(t) = R'_n(t)\,, \end{aligned}$$

for $t\in\Delta_1(q)$ $(p \geqslant q \geqslant n)$, which completes the proof of (8.9).

The inequalities (8.10) can be proved as follows:

From (5.2) and (8.6) we see that

$$(8.18)\qquad C_n(t) \leqslant C_n\left(\frac{\delta}{n}\right) \leqslant b_n \qquad \text{for} \qquad t\in\Delta_1(n)\,.$$

Thus, the function $R_n(t)$ satisfies the differential inequality

$$(8.19)\qquad R'_n(t) \leqslant \frac{M+N}{1-L}\cdot R_n(t) + \frac{b_n}{1-L} \qquad \text{for} \qquad t\in\Delta_1(n)\,,$$

because of (8.18) and (5.1), the function $k_n(t)$ fulfils the differential equation (8.5), and initial values are equal: $R_n(0) = k_n(0) = 0$, whence Theorem A implies

$$R_n(t) \leqslant k_n(t) \quad \text{for} \quad t \in \Delta_1(n). \tag{8.20}$$

In addition, using successively (8.19), (8.20) and (8.5) we see that

$$R_n'(t) \leqslant \frac{M+N}{1-L} \cdot R_n(t) + \frac{b_n}{1-L} \leqslant \frac{M+N}{1-L} \cdot k_n(t) + \frac{b_n}{1-L} = k_n'(t) \quad \text{for} \quad t \in \Delta_1(n),$$

which means that relations (8.10) hold.

This completes the proof of (8.8) in the interval $\Delta_1(q)$: $0 \leqslant t < \delta/q$.

Now we shall prove that in the interval $\delta/q \leqslant t < \tau$:

$$\begin{aligned} &\|y_p(t) - y_q(t)\| \leqslant k_n(t), \\ &\|y_p'(t) - y_q'(t)\| \leqslant k_n'(t) \quad \text{for} \quad \delta/q \leqslant t < \tau \ (p \geqslant q \geqslant n). \end{aligned} \tag{8.21}$$

In fact, if t is in the interval $\delta/q \leqslant t < \tau$, then by (8.2) we obtain:

$$\begin{aligned} &\|y_p'(t) - y_q'(t)\| \\ &= \left\| h\left(t, y_p(t), y_p\left(t - \frac{\delta}{p}\right), y_p'\left(t - \frac{\delta}{p}\right)\right) - h\left(t, y_q(t), y_q\left(t - \frac{\delta}{q}\right), y_q'\left(t - \frac{\delta}{q}\right)\right) \right\| \\ &\leqslant M \cdot \|y_p(t) - y_q(t)\| + N \cdot \left\| y_p\left(t - \frac{\delta}{p}\right) - y_q\left(t - \frac{\delta}{q}\right) \right\| + L \cdot \left\| y_p'\left(t - \frac{\delta}{p}\right) - y_q'\left(t - \frac{\delta}{q}\right) \right\|. \end{aligned} \tag{8.22}$$

But

$$\begin{aligned} &\left\| y_p\left(t - \frac{\delta}{p}\right) - y_q\left(t - \frac{\delta}{q}\right) \right\| \\ &\qquad \leqslant \left\| y_p\left(t - \frac{\delta}{p}\right) - y_p(t) \right\| + \left\| y_p(t) - y_q(t) \right\| + \left\| y_q(t) - y_q\left(t - \frac{\delta}{q}\right) \right\|, \\ &\left\| y_p'\left(t - \frac{\delta}{p}\right) - y_q'\left(t - \frac{\delta}{q}\right) \right\| \\ &\qquad \leqslant \left\| y_p'\left(t - \frac{\delta}{p}\right) - y_p'(t) \right\| + \left\| y_p'(t) - y_q'(t) \right\| + \left\| y_q'(t) - y_q'\left(t - \frac{\delta}{q}\right) \right\|, \end{aligned} \tag{8.23}$$

whence according to (8.22) and (8.23) we have

$$\begin{aligned} (1-L) \cdot \|y_p'(t) - y_q'(t)\| \leqslant (M+N) \cdot \|y_p(t) - y_q(t)\| + \\ + N \cdot \left[\left\| y_p(t) - y_p\left(t - \frac{\delta}{p}\right) \right\| + \left\| y_q(t) - y_q\left(t - \frac{\delta}{q}\right) \right\| \right] + \\ + L \cdot \left[\left\| y_p'(t) - y_p'\left(t - \frac{\delta}{p}\right) \right\| + \left\| y_q'(t) - y_q'\left(t - \frac{\delta}{q}\right) \right\| \right], \end{aligned} \tag{8.24}$$

for $\delta/q \leqslant t < \tau$ $(p \geqslant q \geqslant n)$.

Hence, (8.24), (7.20), the conditions of monotonicity (5.12) and definition (8.6) of the constant b_n imply

$$(1-L)\cdot\|y'_p(t)-y'_q(t)\| \leqslant (M+N)\cdot\|y_p(t)-y_q(t)\|+b_n\,, \tag{8.25}$$

for $\delta/q \leqslant t < \tau$ $(p \geqslant q \geqslant n)$.

So the function $\|y_p(t)-y_q(t)\|$ satisfies the differential inequality (cf. Theorem B):

$$\bar{D}_+\|y_p(t)-y_q(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_p(t)-y_q(t)\|+\frac{b_n}{1-L}, \tag{8.26}$$

for $\delta/q \leqslant t < \tau$ $(p \geqslant q \geqslant n)$, the function $k_n(t)$ satisfies the differential equation (8.5) for $t \in \theta$, and initial values for $t = \delta/q$ fulfil condition

$$\left\|y_p\left(\frac{\delta}{q}\right)-y_q\left(\frac{\delta}{q}\right)\right\| \leqslant k_n\left(\frac{\delta}{q}\right), \tag{8.27}$$

because of (8.8). Hence from Theorem A it follows

$$\|y_p(t)-y_q(t)\| \leqslant k_n(t) \quad \text{for} \quad \frac{\delta}{q} \leqslant t < \tau \ (p \geqslant q \geqslant n)\,. \tag{8.28}$$

In addition, from (8.25), (8.28) and (8.5) we obtain

$$\begin{aligned}\|y'_p(t)-y'_q(t)\| &\leqslant \frac{M+N}{1-L}\cdot\|y_p(t)-y_q(t)\|+\frac{b_n}{1-L}\\ &\leqslant \frac{M+N}{1-L}\cdot k_n(t)+\frac{b_n}{1-L}=k'_n(t) \quad \text{for} \quad \frac{\delta}{q}\leqslant t<\tau \ (p\geqslant q\geqslant n)\,,\end{aligned}$$

which completes the proof of (8.21).

Thus, according to (8.8) and (8.21):

$$\begin{aligned}&\|y_p(t)-y_q(t)\| \leqslant k_n(t)\,,\\ &\|y'_p(t)-y'_q(t)\| \leqslant k'_n(t) \quad \text{for} \quad t \in \theta \ (p \geqslant q \geqslant n)\,,\end{aligned} \tag{8.29}$$

in the whole interval θ.

But $k_n(t)$ is the solution of the non-homogeneous linear equation (8.5) with constant coefficients, $k_n(0) = 0$, and the term b_n tends to zero, as $n \to +\infty$ because of (8.7), whence

$$k_n(t) \rightrightarrows 0, \quad k'_n(t) \rightrightarrows 0, \quad \text{as} \quad n \to +\infty,\ t \in \theta\,, \tag{8.30}$$

(cf. for example Kamke [1], p. 145).

From (8.29) and (8.30) it follows that the sequence $y_n(t)$ and the sequence of derivatives $y'_n(t)$ fulfil the Cauchy criterion of uniform convergence in the interval θ, consequently they are almost uniformly convergent in the interval I'.

b) Let us denote by $\varphi(t)$, $t \in I'$, the limit function of the sequence $y_n(t)$. Since the sequence of (right) derivatives $y_n'(t)$ is also almost uniformly convergent, the theorem of T. Ważewski ([6], théorème 3) implies that

$$(8.31) \qquad y_n(t) \Rightarrow \varphi(t), \quad y_n'(t) \Rightarrow \varphi'(t), \quad \text{as} \quad n \to +\infty, \ t \in I'.$$

We shall prove now that $\varphi(t)$, $t \in I'$, is the solution of the equation (8.1) and $\varphi(0) = 0$.

To this end we show that

$$(8.32) \qquad \left\| y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big) \right\| \leqslant b_n \quad \text{for} \quad t \in \theta,$$

since from (8.32), (8.31) and (8.7) follows, as $n \to +\infty$, the identity

$$(8.33) \qquad \varphi'(t) \equiv h\big(t, \varphi(t), \varphi(t), \varphi'(t)\big) \quad \text{for} \quad t \in \theta,$$

in an arbitrary prescribed interval θ, bounded and contained in the interval I'.

We begin with the proof of inequality

$$(8.34) \qquad \left\| y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big) \right\| \leqslant b_n \quad \text{for} \quad t \in \Delta_1(n).$$

In fact, using successively (8.2), (7.18), (4.2) and the definition (5.7) of the constant a_n we obtain

$$(8.35) \qquad \left\| y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big) \right\|$$
$$= \left\| h\big(t, y_n(t), 0, d_n\big) - h\big(t, y_n(t), y_n(t), y_n'(t)\big) \right\|$$
$$\leqslant N \cdot \|y_n(t) - y_n(0)\| + L \cdot \|y_n'(t) - d_{n+1}\| + L \cdot \|d_{n+1} - d_n\|$$
$$\leqslant N \cdot \left[R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n} \cdot s'(\tau) \right] + L \cdot R_n'\left(\frac{\delta}{n}\right) + 2 \cdot L^n \cdot \frac{P}{1-L} \leqslant a_n \quad \text{for} \quad t \in \Delta_1(n).$$

So (8.34) will be proved, if $a_n \leqslant b_n$. But from the equation (5.6) if follows that $S_n(t)$ is an increasing function in the interval $\delta/n \leqslant t < \tau$, and $S_n\left(\frac{\delta}{n}\right) = R_n\left(\frac{\delta}{n}\right) + \frac{\delta}{n} \cdot s'(\tau)$ because of (5.8), therefore $R_n\left(\frac{\delta}{n}\right) \leqslant S_n\left(2 \cdot \frac{\delta}{n}\right)$. So from the equation (5.1) and (5.6) we obtain

$$R_n'\left(\frac{\delta}{n}\right) = \frac{M+N}{1-L} \cdot R_n\left(\frac{\delta}{n}\right) + \frac{C_n(\delta/n)}{1-L} \leqslant \frac{M+N}{1-L} \cdot S_n\left(2 \cdot \frac{\delta}{n}\right) + \frac{a_n}{1-L} = S_n'\left(2 \cdot \frac{\delta}{n}\right),$$

which means, that the constants a_n and b_n defined by (5.7) and (8.6) satisfy the desired inequality $a_n \leqslant b_n$.

This concludes the proof of (8.34).

Now the inequality

$$(8.36) \qquad \left\| y_n'(t) - h\big(t, y_n(t), y_n(t), y_n'(t)\big) \right\| \leqslant b_n \quad \text{for} \quad \frac{\delta}{n} \leqslant t < \tau,$$

will be proved. In fact, from (8.2), (7.20) and (8.6) we obtain

$$(8.37)\qquad \left\|y_n'(t)-h\big(t,y_n(t),y_n(t),y_n'(t)\big)\right\|$$

$$=\left\|h\left(t,y_n(t),y_n\left(t-\frac{\delta}{n}\right),y_n'\left(t-\frac{\delta}{n}\right)\right)-h\big(t,y_n(t),y_n(t),y_n'(t)\big)\right\|$$

$$\leqslant N\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\|+L\cdot\left\|y_n'(t)-y_n'\left(t-\frac{\delta}{n}\right)\right\|$$

$$\leqslant N\cdot S_n\left(2\cdot\frac{\delta}{n}\right)+L\cdot S_n'\left(2\cdot\frac{\delta}{n}\right)\leqslant b_n\quad\text{for}\quad\frac{\delta}{n}\leqslant t<\tau,$$

which completes the proof of (8.36).

According to (8.36) and (8.34), the inequality (8.32) and the identity (8.33) hold in the whole interval θ.

In addition $\varphi(0)=0$, since $y_n(0)=0$ $(n=1,2,3,\ldots)$, which completes the proof of existence.

c) We shall prove now the uniqueness of the solution. To this purpose let us suppose that $\psi(t)$, $t\in I'$, is the solution of (8.1) satisfying the initial condition $\psi(0)=0$. Then we obtain

$$\|\varphi'(t)-\psi'(t)\|=\left\|h\big(t,\varphi(t),\varphi(t),\varphi'(t)\big)-h\big(t,\psi(t),\psi(t),\psi'(t)\big)\right\|$$

$$\leqslant(M+N)\cdot\|\varphi(t)-\psi(t)\|+L\cdot\|\varphi'(t)-\psi'(t)\|\quad\text{for}\quad t\in I',$$

and

$$(8.38)\qquad \|\varphi'(t)-\psi'(t)\|\leqslant\frac{M+N}{1-L}\cdot\|\varphi(t)-\psi(t)\|\quad\text{for}\quad t\in I',$$

which means that the function $\|\varphi(t)-\psi(t)\|$ satisfies the differential inequality (cf. Theorem B):

$$(8.39)\qquad \bar{D}_+\|\varphi(t)-\psi(t)\|\leqslant\frac{M+N}{1-L}\cdot\|\varphi(t)-\psi(t)\|\quad\text{for}\quad t\in I'.$$

The function $\xi(t)\equiv 0$, $t\in I'$, is the unique solution (also the greatest solution) of the equation

$$(8.40)\qquad \xi'(t)=\frac{M+N}{1-L}\cdot\xi(t)\quad\text{for}\quad t\in I',$$

satisfying the condition $\xi(0)=0$, therefore (8.39), (8.40) and Theorem A imply

$$\|\varphi(t)-\psi(t)\|\leqslant\xi(t)\equiv 0\quad\text{for}\quad t\in I',$$

whence $\varphi(t)\equiv\psi(t)$ for $t\in I'$, which completes the proof of uniqueness.

d) The error estimates (8.3) follow from (8.29), as $p\to+\infty$.

This completes the proof of Theorem 2.

§ 9. Now we shall give another error estimate for $y_n(t)$ in the interval I'. It will be derived with the aid of differences $y_n(t)-y_n\left(t-\frac{\delta}{n}\right)$ and $y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)$, only one function $y_n(t)$ being involved.

THEOREM 3. *Let us suppose that the right-hand member $h(t, u, v, w)$ of the equation* (8.1) *satisfies assumptions* H, *and that the function $y_n(t)$ is constructed for some n with the aid of equations* (8.2) *in the interval I'.*

Assume in addition that

$$\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant \varepsilon(t), \qquad \left\|y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)\right\| \leqslant \eta(t) \quad \text{for} \quad \frac{\delta}{n} \leqslant t < a, \tag{9.1}$$

$\varepsilon(t)$ and $\eta(t)$ being given.

Under these assumptions the error estimates for $y_n(t)$ are provided by

$$\|y_n(t)-\varphi(t)\| \leqslant x(t), \qquad \|y'_n(t)-\varphi'(t)\| \leqslant x'(t) \quad \text{for} \quad 0 \leqslant t < a. \tag{9.2}$$

Here $x(t)$ is the solution of the linear equation

$$x'(t) = \frac{M+N}{1-L}\cdot x(t)+\frac{N\cdot\varepsilon(t)+L\cdot\eta(t)}{1-L} \quad \text{for} \quad \frac{\delta}{n} \leqslant t < a, \tag{9.3}$$

satisfying the initial condition $x\left(\frac{\delta}{n}\right) = l_n\left(\frac{\delta}{n}\right)$, and $x(t) = l_n(t)$ for $t \in \Delta_1(n)$. The function $l_n(t)$ satisfies the linear non-homogeneous equation

$$l'_n(t) = \frac{M+N}{1-L}\cdot l_n(t)+\frac{D_n(t)}{1-L} \quad \text{for} \quad t \in I', \tag{9.4}$$

and the initial condition $l_n(0) = 0$, the function $D_n(t)$ being defined by

$$D_n(t) = \left(\frac{P\cdot(M+N)}{1-L}+K\right)\cdot t+2\cdot L^n\cdot\frac{P}{1-L} \quad \text{for} \quad t \in I'. \tag{9.5}$$

Proof. Let us observe first that

$$\|y_p(t)-t\cdot y'_p(0)\| \leqslant l_n(t), \qquad \|y'_p(t)-y'_p(0)\| \leqslant l'_n(t) \quad \text{for} \quad t \in I' \ (p \geqslant n). \tag{9.6}$$

The proof of (9.6) in the interval I' is similar to that of (7.1); it is sufficient to place $l_n(t)$ and $D_n(t)$ in the formula (7.3)-(7.11) instead of $R_n(t)$ and $C_n(t)$ respectively.

With the aid of (9.6) the following inequalities can be obtained in the first interval $\Delta_1(n)$:

$$\begin{aligned} &\|y_p(t)-y_q(t)\| \leqslant l_n(t)\,, \\ &\|y'_p(t)-y'_q(t)\| \leqslant l'_n(t) \quad \text{for} \quad t\in\Delta_1(q)\ (p\geqslant q\geqslant n)\,. \end{aligned} \tag{9.7}$$

The proof of (9.7) for $t\in\Delta_1(q)$ is similar to that of (8.8); it is sufficient to place $l_n(t)$ and $D_n(t)$ in the formula (8.11)-(8.17) instead of $R_n(t)$ and $C_n(t)$ respectively.

From (9.7) we obtain the following inequalities, as $p\to+\infty$ and $q=n$,

$$\begin{aligned} &\|y_n(t)-\varphi(t)\| \leqslant l_n(t)\,, \\ &\|y'_n(t)-\varphi'(t)\| \leqslant l'_n(t) \quad \text{for} \quad t\in\Delta_1(n)\,, \end{aligned} \tag{9.8}$$

whence relations (9.2) hold for $t\in\Delta_1(n)$.

We shall now prove that (9.2) hold also in the interval $\delta/n\leqslant t<a$. In fact, if t is in the interval $\delta/n\leqslant t<a$, then by (8.1) and (9.1) we obtain

$$\begin{aligned} \|y'_n(t)-\varphi'(t)\| &\leqslant \left\|h\left(t,\, y_n(t),\, y_n\left(t-\frac{\delta}{n}\right),\, y'_n\left(t-\frac{\delta}{n}\right)\right) - h\big(t,\, y_n(t),\, y_n(t),\, y'_n(t)\big)\right\| + \\ &\quad + \big\|h\big(t,\, y_n(t),\, y_n(t),\, y'_n(t)\big) - h\big(t,\, \varphi(t),\, \varphi(t),\, \varphi'(t)\big)\big\| \\ &\leqslant N\cdot\left\|y_n(t)-y_n\left(t-\frac{\delta}{n}\right)\right\| + L\cdot\left\|y'_n(t)-y'_n\left(t-\frac{\delta}{n}\right)\right\| + \\ &\quad + (M+N)\cdot\|y_n(t)-\varphi(t)\| + L\cdot\|y'_n(t)-\varphi'(t)\|\,, \end{aligned}$$

and

$$\|y'_n(t)-\varphi'(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_n(t)-\varphi(t)\| + \frac{N\cdot\varepsilon(t)+L\cdot\eta(t)}{1-L}\,, \tag{9.9}$$

for $\delta/n\leqslant t<a$. Thus, the function $\|y_n(t)-\varphi(t)\|$ satisfies the differential inequality (cf. Theorem B):

$$\bar{D}_+\|y_n(t)-\varphi(t)\| \leqslant \frac{M+N}{1-L}\cdot\|y_n(t)-\varphi(t)\| + \frac{N\cdot\varepsilon(t)+L\cdot\eta(t)}{1-L}\,, \tag{9.10}$$

for $\delta/n\leqslant t<a$, the function $x(t)$ satisfies equation (9.3), and initial values for $t=\delta/n$ fulfil condition

$$\left\|y_n\left(\frac{\delta}{n}\right)-\varphi\left(\frac{\delta}{n}\right)\right\| \leqslant x\left(\frac{\delta}{n}\right)$$

because of (9.8). Hence Theorem A implies

$$\|y_n(t)-\varphi(t)\| \leqslant x(t) \quad \text{for} \quad \frac{\delta}{n}\leqslant t<a\,. \tag{9.11}$$

In addition, from (9.9), (9.11) and (9.3) we obtain

$$\begin{aligned} \|y'_n(t)-\varphi'(t)\| &\leqslant \frac{M+N}{1-L}\cdot\|y_n(t)-\varphi(t)\| + \frac{N\cdot\varepsilon(t)+L\cdot\eta(t)}{1-L} \\ &\leqslant \frac{M+N}{1-L}\cdot x(t) + \frac{N\cdot\varepsilon(t)+L\cdot\eta(t)}{1-L} = x'(t)\,, \end{aligned}$$

for $\delta/n\leqslant t<a$, which completes the proof of Theorem 3.

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