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Method of difference inequalities for parabolic equations with mixed derivatives

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Abstract. We shall consider here a difference inequality with difference quotients corresponding to mixed partial derivatives of the second order. This inequality will be used to estimate the convergence of a difference scheme for a parabolic differential equation with mixed derivatives. In this difference scheme a weighted average of symmetric difference quotients will replace the pure partial derivatives of the second order.

The result obtained for one equation can be generalized, in the same way, for a system of partial equations of the parabolic type.

1. The present paper can be regarded as an immediate continuation of works [7], [4] and [5] dealing with theorems on difference inequalities.

In [7] A. Pliś proved a theorem on a difference inequality corresponding to a differential inequality of the first order and suggested the possibility of using it for estimating the convergence of a difference scheme for an equation of the first order.

In [1] the author developed the idea, obtaining in a different way the result of Z. Kowalski [6]. The method was further developed in [2], [3], [4] and [5]. In particular, the paper [5] concerns a convergence of a difference scheme for a system of parabolic differential equations of the form:

$$\frac{\partial y_i}{\partial x_0} = f_i\left(x, y, \frac{\partial y_i}{\partial x_1}, \ldots, \frac{\partial y_i}{\partial x_n}, \ldots, \frac{\partial^2 y_i}{\partial x_1^2}, \ldots, \frac{\partial^2 y_i}{\partial x_n^2}\right), \quad i = 1, \ldots, m.$$

A difference scheme for this system was obtained in the following manner: the derivatives $\dfrac{\partial y_i}{\partial x_k}, \dfrac{\partial^2 y_i}{\partial x_k^2}$, $k \geqslant 1$, were replaced by symmetric difference quotients.

Application of the same method of proof when the equation is of the form:

$$u_t = f(t, x, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_nx_n}, u_{x_1x_2}, \ldots, u_{x_{n-1}x_n})$$

requires another difference scheme. The present paper is devoted to this problem. In Section 2 we shall give a theorem on a difference inequality and as its consequence, in Section 3 we shall demonstrate a convergence of a difference scheme and we shall give an estimation of this convergence.

Moreover, in Section 4 we try to discuss the applied method.

2. Notations. We shall consider the nodal points x^M of the euclidean space R^{n+1}, $0 \leqslant M \leqslant P$, where $M = (m_0, m_1, \ldots, m_n)$, $P = (p_0, p_1, \ldots, p_n)$ are given systems of integers, and $0 \leqslant M \leqslant P$ denotes $0 \leqslant m_i \leqslant p_i$, $i = 0, 1, \ldots, n$;

$$x^M = (x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}), \quad x_0^{m_0} = m_0 k, \quad x_i^{m_i} = m_i h, \quad i = 1, \ldots, n,$$

where $h = \tau/N$, $k = \tau/N_1$ are positive numbers. To each nodal point x^M there correspond real numbers u^M and v^M. Moreover, we shall introduce a notation for an operation on the multi-indices M:

$$\begin{aligned} +M &= (m_0+1, m_1, \ldots, m_n), \\ +jM &= (m_0, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_n), \quad j = 1, \ldots, n. \\ -jM &= (m_0, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_n), \quad j = 1, \ldots, n. \end{aligned}$$

We need also special notations for difference quotients of the first and second order:

$$\begin{aligned} \delta_0 u^M &= \frac{1}{k}(u^{+M} - u^M), \\ \delta_i u^M &= \frac{1}{2h}(u^{+iM} - u^{-iM}), \quad i = 1, 2, \ldots, n, \\ \delta_{ii} u^M &= \frac{1}{h^2}(u^{+iM} - 2u^M + u^{-iM}), \quad i = 1, 2, \ldots, n, \\ \sigma_{ii} u^M &= \frac{1}{n-1}\delta_{ii}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} (\alpha u^{+jM} + \beta u^M + \alpha u^{-jM})\Big), \quad i = 1, \ldots, n, \\ \delta_{ij} u^M &= \frac{1}{4h^2}(u^{+i(+jM)} + u^{-i(-jM)} - u^{+i(-jM)} - u^{-i(+jM)}), \\ &\qquad\qquad i = 1, \ldots, n, \ j = 2, \ldots, n, \ i < j. \end{aligned}$$

In order to abbreviate the text let us introduce notations for certain multidimensional vectors:

$$\begin{aligned} \delta u^M &= (\delta_1 u^M, \delta_2 u^M, \ldots, \delta_n u^M), \\ \sigma u^M &= (\sigma_{11} u^M, \sigma_{22} u^M, \ldots, \sigma_{nn} u^M), \\ \varrho u^M &= (\delta_{12} u^M, \delta_{13} u^M, \ldots, \delta_{rs} u^M, \ldots, \delta_{n-1n} u^M). \end{aligned}$$

THEOREM 1. *Let us suppose that the scalar function* $f(A)$, *where* $A = (x_0, x_1, \ldots, x_n, u, p_1, \ldots, p_n, q_1, \ldots, q_n, w_{12}, w_{13}, \ldots, w_{rs}, \ldots, w_{n-1n})$, $r = 1, \ldots, n-1$, $s = 2, \ldots, n$, $r < s$, *is of the class* C^1 *in the set* $D = [0, \tau]^{n+1} \times R^{2n+1+n(n-1)/2}$, *and satisfies the following conditions:*

$$(1.1) \qquad |f_{p_i}(A)| < \frac{2}{h}\Big(\beta f_{q_i}(A) - \frac{2a}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} f_{q_j}(A)\Big), \qquad i = 1, \ldots, n,$$

$$(1.2) \qquad |f_{w_{ij}}(A)| < \frac{4a}{n-1}\big(f_{q_i}(A) + f_{q_j}(A)\big),$$

$$i = 1, \ldots, n-1,\ j = 2, \ldots, n,\ i < j,$$

where a *and* β *are positive numbers such that* $2a + \beta = 1$.

Let the mesh sizes k *and* h *be chosen such that*

$$(1.3) \qquad 1 + kf_u(A) - \frac{2\beta_k}{h^2}\sum_{i=1}^{n} f_{q_i}(A) > 0.$$

Moreover, we suppose that for $0 \leqslant m_0 \leqslant p_0 - 1$, $1 \leqslant m_i \leqslant p_i$, $i = 1, \ldots, n$, *the numbers* u^M *and* v^M *satisfy the following difference inequalities:*

$$(1.4) \qquad \delta_0 u^M \leqslant f(x^M, u^M, \delta u^M, \sigma u^M, \varrho u^M),$$

$$(1.5) \qquad \delta_0 v^M \geqslant f(x^M, v^M, \delta v^M, \sigma v^M, \varrho v^M),$$

and on the parabolic boundary of the mesh, i.e. when $m_0 = 0$, *or* $m_1 = 0, \ldots,$ *or* $m_n = 0$, *or* $m_1 = N$, *or* $m_2 = N, \ldots,$ *or* $m_n = N$, *we have*

$$(1.6) \qquad u^M \leqslant v^M.$$

Then the inequalities

$$(1.7) \qquad u^M \leqslant v^M$$

hold for $0 \leqslant M \leqslant P$.

We shall prove our theorem by induction on m_0. By virtue of (1.6) the theorem is true for $m_0 = 0$.

Let $r^M = u^M - v^M$. It is sufficient to prove that the inequality $r^M \leqslant 0$ for $m_0 = m_0'$ implies $r^{+M} \leqslant 0$. A substraction of inequality (1.4) from (1.5) yields

$$\delta_0 r^M = \delta_0 u^M - \delta_0 v^M \leqslant f(x^M, u^M, \delta u^M, \sigma u^M, \varrho u^M) - \\ - f(x, v^M, \delta v^M, \sigma v^M, \varrho v^M).$$

Hence by the mean value theorem we have

$$\delta_0 r^M \leqslant f_u(\sim) r^M + \sum_{i=1}^{n} f_{p_i}(\sim)\,\delta_i r^M + \sum_{i=1}^{n} f_{q_i}(\sim)\,\sigma_{ii} r^M + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}(\sim)\,\delta_{ij} r^M$$

$$= f_u(\sim) r^M + \sum_{i=1}^{n} f_{p_i}(\sim)\frac{1}{2h}\left(r^{+iM} - r^{-iM}\right) +$$

$$+ \frac{1}{\overline{n-1}h^2}\sum_{i=1}^{n} f_{q_i}(\sim)\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha r^{+i(+jM)} - 2\alpha r^{+jM} + r^{-i(+jM)} +$$

$$+ \beta r^{+iM} - 2\beta r^M + \beta r^{-iM} + \alpha r^{+i(-jM)} - 2\alpha r^{-jM} + \alpha r^{-i(-jM)}\Bigg) +$$

$$+ \frac{1}{4h^2}\sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}(\sim)\left(r^{+i(+jM)} + r^{-i(-jM)} - r^{+i(-jM)} - r^{-i(+jM)}\right).$$

We now group the terms containing the numbers r in the same nodes of the mesh:

$$r^{+M} \leqslant \left(1 + kf_u - \frac{2\beta_k}{h^2}\sum_{i=1}^{n} f_{q_i}\right) r^M +$$

$$+ \sum_{i=1}^{n}\left(\frac{1}{2h}f_{p_i} + \frac{\beta}{h^2}f_{q_i} - \frac{2\alpha}{(n-1)h^2}\sum_{\substack{j=1\\ j\neq i}}^{n} f_{q_j}\right) r^{+iM} +$$

$$+ \sum_{i=1}^{n}\left(-\frac{1}{2h}f_{p_i} + \frac{\beta}{h^2}f_{q_i} - \frac{2}{(n-1)h^2}\sum_{\substack{j=1\\ j\neq i}}^{n} f_{q_j}\right) r^{-iM} +$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{n}\left[\frac{\alpha}{(n-1)h^2}\left(f_{q_i} + f_{q_j}\right) + \frac{1}{4h^2}f_{w_{ij}}\right]\left(r^{+i(+jM)} + r^{-i(-jM)}\right) +$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{n}\left[\frac{\alpha}{(n-1)h^2}\left(f_{q_i} + f_{q_j}\right) - \frac{1}{4h^2}f_{w_{ij}}\right]\left(r^{-i(+jM)} + r^{+i(-jM)}\right).$$

By virtue of Assumptions (1.1)–(1.3), the coefficients of numbers r are positive. It follows from the induction assumption that r^M, r^{+iM}, r^{-iM}, $r^{+i(+jM)}$, $r^{-i(+jM)}$, $r^{-i(-jM)}$, $r^{+i(-jM)}$ are non-positive for $M = (m_0', m_1, \ldots, m_n)$. Thus we have proved the non-positivity of r^{+M} for m_i, $i \geqslant 1$, satisfying the inequalities $1 \leqslant m_i \leqslant p_i - 1$, which together with Assumption (1.6) gives the non-positivity of r^{+M} for all $M = (m_0', m_1, \ldots, m_n)$. This completes the proof.

3. We now apply Theorem 1 in proving the convergence of a difference scheme for a partial differential equation of the parabolic type with mixed derivatives

$$y_{x_0} = f(x, y, y_{x_1}, \ldots, y_{x_n}, y_{x_1x_1}, \ldots, y_{x_nx_n}, y_{x_1x_2}, \ldots, y_{x_{n-1}x_n}), \tag{2.1}$$

with boundary conditions

$$\begin{aligned} y(x) &= \varphi_j(x^M) \quad \text{for } x_j = 0,\ j = 0, 1, \ldots, n, \\ y(x) &= \psi_j(x^M) \quad \text{for } x_j = \tau,\ j = 1, \ldots, n. \end{aligned} \tag{2.2}$$

We obtain this scheme by substituting symmetric difference quotients for derivatives of the first order, and the weighted averages of symmetric difference quotients for derivatives of the second order.

Let us retain the notations used in the preceding part. To each nodal point x^M of the mesh we attach the numbers z^M. Let us consider the following scheme:

$$\delta_0 z^M = f(x^M, z^M, \delta z^M, \sigma z^M, \varrho z^M) \tag{2.3}$$

with boundary conditions

$$\begin{aligned} z^M &= \varphi_j(x^M) \quad \text{for } x_j = 0,\ j = 0, 1, \ldots, n, \\ z^M &= \psi_j(x^M) \quad \text{for } x_j = \tau,\ j = 1, 2, \ldots, n. \end{aligned} \tag{2.4}$$

THEOREM 2. *Let us suppose that*

1^0 *the function* $f(x, u, p, q, w)$,

$$x \in R^{n+1}, \quad p = (p_1, \ldots, p_n) \in R^n,$$

$$q = (q_1, q_2, \ldots, q_n) \in R^n, \quad w = (w_{12}, w_{13}, \ldots, w_{n-1n}) \in R^{n(n-1)/2},$$

is of the class C^1 *in the domain* $D = [0, \tau]^{n+1} \times R^{2n+1+n(n-1)/2}$,

2^0 *the derivatives of this function satisfy in the domain* D *the following conditions*

$$0 \leqslant f_u \leqslant L,$$

$$|f_{p_i}| < \frac{2}{h}\Big(\beta f_{q_i} - \frac{2a}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} f_{q_j}\Big), \quad i = 1, \ldots, n,$$

$$|f_{w_{ij}}| < \frac{4a}{n-1}(f_{q_i} + f_{q_j}), \quad i, j = 1, \ldots, n,\ i < j,$$

where a *and* β *are positive numbers,* $2a + \beta = 1$, *and the numbers* k *and* h *are chosen such that*

$$1 + kf_u - \frac{2\beta_k}{h^2}\sum_{i=1}^{n} f_{q_i} > 0,$$

3^0 *the function* $y(x)$ *is of the class* C^2 *in the domain* $E = [0, \tau]^{n+1}$ *and satisfies* (2.1) *with boundary conditions* (2.2), *and* z^M *satisfies* (2.3) *with conditions* (2.4).

Let us denote $u^M = y^M - z^M$.

Then the inequalities

$$|u^M| < \frac{\varepsilon(h)}{L}(e^{Lkm_0} - 1) \tag{2.5}$$

hold for $0 \leqslant M \leqslant P$, *and*

$$u^M \to 0 \quad \textit{when } h \to 0. \tag{2.6}$$

Proof. In the nodes x^M of the mesh the function $y(x)$ satisfies the following difference equation:

$$\delta_0 y^M = f(x^M, y^M, \delta y^M, \sigma y^M, \varrho y^M) + \varepsilon^M(h)$$

with boundary conditions

$$y^M = \varphi_j(x^M) \quad \text{for } x_j = 0,\ j = 0, 1, \dots, n,$$
$$y^M = \psi_j(x^M) \quad \text{for } x_j = \tau,\ j = 1, 2, \dots, n,$$

and

$$\varepsilon(h) = \max_M \varepsilon^M(h) \to 0, \qquad \text{when } h \to 0. \tag{2.7}$$

Hence it follows easily that

$$\delta_0 u^M = \delta_0 y^M - \delta_0 z^M = f(x^M, y^M, \delta y^M, \sigma y^M, \varrho y^M) - \\ - f(x^M, z^M, \delta z^M, \sigma z^M, \varrho z^M) + \varepsilon^M(h).$$

The mean value theorem yields

$$\delta_0 u^M = f_u(\sim) u^M + \sum_{i=1}^n f_{p_i}(\sim)\delta_i u^M + \sum_{i=1}^n f_{q_i}(\sim)\sigma_{ii} u^M + \\ + \sum_{\substack{i,j=1\\ i<j}}^n f_{w_{ij}}(\sim)\delta_{ij} u^M + \varepsilon^M(h). \tag{2.8}$$

Let us define the values v^M in the following way:

$$v^M = \frac{\varepsilon(h)}{L}(e^{Lkm_0} - 1), \qquad 0 \leqslant M \leqslant P. \tag{2.9}$$

It is easily seen that v^M satisfy the inequalities

$$\delta_0 v^M \geqslant L v^M + \varepsilon(h), \tag{2.10}$$

and that $\delta v^M = 0$, $\sigma v^M = 0$, $\varrho v^M = 0$. Thus we can write

$$(2.11)\qquad \delta_0 v^M \geqslant L v^M + \varepsilon(h) \geqslant f_u v^M + \sum_{i=1}^{n} f_{p_i}\delta_i v^M + \sum_{i=1}^{n} f_{q_i}\sigma_{ii} v^M + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}\delta_{ij} v^M + \varepsilon(h).$$

Let us compare the inequalities

$$(2.12)\qquad \delta_0 v^M \geqslant f_u v^M + \sum_{i=1}^{n} f_{p_i}\delta_i v^M + \sum_{i=1}^{n} f_{q_i}\sigma_{ii} v^M + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}\delta_{ij} v^M + \varepsilon(h)$$

and

$$(2.13)\qquad \delta_0 u^M \leqslant f_u u^M + \sum_{i=1}^{n} f_{p_i}\delta_i u^M + \sum_{i=1}^{n} f_{q_i}\sigma_{ii} u^M + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}\delta_{ij} u^M + \varepsilon(h).$$

The boundary values for y and z are equal and $v^M \geqslant 0$. Hence

$$(2.14)\qquad u^M = 0,\quad u^M \leqslant v^M \quad \text{for } m_j = 0,\ j = 0, 1, 2, \ldots, n, \text{ or } m_j = N,\ j = 1, 2, \ldots, n.$$

From conditions (2.12), (2.13), (2.14) and Theorem 1 it follows that

$$(2.15)\qquad u^M \leqslant v^M \quad \text{for } 0 \leqslant M \leqslant P.$$

In a similar way we obtain

$$(2.16)\qquad \delta_0(-v^M) \leqslant f_u(-v^M) + \sum_{i=1}^{n} f_{p_i}\delta_i(-v^M) + \sum_{i=1}^{n} f_{q_i}\sigma_{ii}(-v^M) + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}\delta_{ij}(-v^M) + \varepsilon^M(h)$$

and

$$(2.17)\qquad \delta_0 u^M \geqslant f_u u^M + \sum_{i=1}^{n} f_{p_i}\delta_i u^M + \sum_{i=1}^{n} f_{q_i}\sigma_{ii} u^M + \sum_{\substack{i,j=1\\ i<j}}^{n} f_{w_{ij}}\delta_{ij} u^M + \varepsilon^M(h).$$

We have also

$$(2.18)\qquad u^M = 0,\quad u^M \geqslant -v^M \quad \text{for } m_j = 0,\ j = 0, 1, \ldots, n, \text{ or } m_j = N,\ j = 1, 2, \ldots, n.$$

Conditions (2.16), (2.17), (2.18) and Theorem 1 yield

$$(2.19)\qquad u^M \geqslant -v^M \quad \text{for } 0 \leqslant M \leqslant P.$$

Comparing conditions (2.15) and (2.19) we have

$$|u^M| \leqslant v^M.$$

This is the first part of the thesis. The second part is an immediate consequence of (2.7).

4. The following example:

$$u_t = u_{xx} + a_{12} u_{xy} + u_{yy},$$

shows that our method is not applicable for the classical difference scheme with symmetric quotients in place of derivatives of the second order.

This classical scheme is of the form

$$\delta_0 u^M = \delta_{11} u^M + a_{12} \delta_{12} u^M + \delta_{22} u^M,$$

and numbers r^{+M} must satisfy the inequalities

$$r^{+M} \leqslant \left(1 - \frac{4k}{h^2}\right) r^M + \frac{1}{h^2} (r^{+1M} + r^{-1M} + r^{+2M} + r^{-2M}) +$$
$$+ \frac{a_{12}}{h^2} (r^{+1+2M} + r^{-1-2M}) - \frac{a_{12}}{h^2} (r^{-1+2M} + r^{+1-2M}),$$

which does not guarantee their non-positivity.

In the considered example one can use a modification of the preceding scheme, taking the quotient

$$\frac{1}{2h^2} (u^{+1+2M} + u^{+1-2M} + u^{-1+2M} + u^{-1-2M} - 4u^M)$$

to be the Laplacian. This is possible in the case of any linear parabolic equation.

In the case when the equation is essentially non-linear this modification is impossible.

Using the same technique of proof one can obtain an analogous result for a scheme of difference inequalities and for the convergence of special difference scheme for the equations

$$y^i_{x_0} = f^i(x, y, y^i_{x_1}, \ldots, y^i_{x_n}, y^i_{x_1x_1}, \ldots, y^i_{x_nx_n}, y^i_{x_1x_2}, \ldots, y_{x_{n-1}x_n}),$$
$$i = 1, 2, \ldots, m.$$

We have here presented a method for one equation in order to obtain a simpler notation and clearer idea of proof.

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