ANNALES
POLONICI MATHEMATICI
L (1990)

Lifting uniformly continuous maps with values in nuclear Fréchet spaces

by NGUYEN VAN KHUE (Warszawa)

Abstract. Let j be a continuous linear map from a Fréchet space E onto a nuclear Fréchet space F and let f be a uniformly continuous map from a metric space X into F. It is shown that if X is convex in the sense of Menger [7], then there exists a uniformly continuous map g: $X \to E$ such that $jg = f$. The result is not true without the assumption of the convexity of X.

Let j be a continuous linear map from a Fréchet space E onto a Fréchet space F. A well-known theorem of Bartle and Graves [1] says that for every continuous map from a topological space X into F there exists continuous map g: $X \to E$ such that $jg = f$. The aim of this note is to consider the following problem.

PROBLEM. Given a continuous linear map j from a Fréchet space E onto a Fréchet space F and a uniformly continuous map f from a metric space X into F. Does there exist a uniformly continuous map g: $X \to E$ such that $jg = f$?

In this note we give a negative answer to this problem and show that the answer to this problem is positive if F is a nuclear Fréchet space and X is convex in the sense of Menger [7].

1. Lifting a uniformly continuous map. A metric space (X, d) is said to be *convex* [7] iff every pair of points in X can be joined in X by an arc isometric to a segment of the real line $\boldsymbol{R}^1$.

1.1. THEOREM. *Let j be a continuous linear map from a Fréchet space E onto a nuclear Fréchet space F. Then for every uniformly continuous map f from a convex metric space X into F there exists a uniformly continuous map g: $X \to E$ such that $jg = f$.*

Theorem 1.1 is an immediate consequence of Lemmas 1.2, 1.3, 1.4.

1.2. LEMMA. *Let j: $E \to F$ be as in 1.1. Then for every continuous linear map f from a normed space X into F there exists a continuous linear map g: $X \to E$ such that $jg = f$.*

Proof. Let $S = \{x \in X: \|x\| \leqslant 1\}$. Since F is a nuclear Fréchet space there exists a compact absolutely convex set $K_1 \supset K = f(S)$ such that the canonical map i: $F(K) \to F(K_1)$ is nuclear, where $F(K)$ denotes the linear subspace of F spanned by K equipped with the norm induced by K. By a theorem of Bartle and Graves [1] there exists a compact absolutely convex set $L \subset E$ such that $j(L) = K_1$. Let $j' = j \mid E(L)$: $E(L) \to F(K_1)$. Since $h = if$: $X \to F(K_1)$ is a nuclear map there exist sequences $(a_n') \subset X'$ and $(y_n) \subset F(K_1)$ such that

$$\sum_{n=1}^{\infty} \|a_n\| < \infty, \quad \sup\{\|y_n\|, \ n \in N\} < \infty$$

and

$$h(x) = \sum_{n=1}^{\infty} a_n(x) y_n \quad \text{for every } x \in X.$$

Take a sequence $(z_n) \subset E(L)$ such that $\sup\{\|z_n\|, n \in N\} < \infty$ and $j(z_n) = y_n$ for every $n \in N$. Setting

$$f(x) = \sum_{n=1}^{\infty} a_n(x) z_n \quad \text{for every } x \in X$$

we get a continuous linear map g: $X \to E$ such that $jg = f$.

Adapting the terminology of Mankiewicz [4]–[6], let us say that a map f from a metric space (X, ϱ) into a Fréchet space F is a *Lipschitz map* iff for every continuous pseudonorm p on F there exists a constant $K_p > 0$ such that

$$p(f(x) - f(y)) \leqslant K_p \varrho(x, y) \quad \text{for every } x, y \in X.$$

1.3. LEMMA. *Let j: $E \to F$ be as in 1.1. Then for every Lipschitz map f from a metric space X into F there exists a Lipschitz map g: $X \to E$ such that $jg = f$.*

Proof. Let $L(X)$ denote the linear space spanned formally by X. Equip $L(X)$ with the norm

$$\Big\|\sum_{i=1}^{n} \lambda_i x_i\Big\| = \sup\Big\{\Big|\sum_{i=1}^{n} \lambda_i \sigma(x_i)\Big|: \ \sigma \in \mathrm{Lip}(X), \ \|\sigma\| \leqslant 1\Big\},$$

where $\mathrm{Lip}(X)$ denotes the Banach space of all Lipschitz functions on X vanishing at a fixed point x_0 equipped with the Lipschitz norm.

Given a Lipschitz map f: $X \to F$. Extending linearly f over $L(X)$ we get a continuous linear map f': $L(X) \to F$. By 1.2 there exists a continuous linear map g': $L(X) \to E$ such that $jg' = f'$. Whence the restriction $g = g'|X$ satisfies the required properties.

1.4. LEMMA. *Let f be a uniformly continuous map from a convex metric space (X, d) into a Fréchet space F. Then there exists a metric ϱ on X uniformly equivalent to d such that f: $(X, \varrho) \to F$ satisfies the Lipschitz condition.*

Proof. Let (p_n) be an increasing sequence of pseudonorms inducing the topology of F. Since f is uniformly continuous there is a sequence of positive numbers (ε_n) such that

$$\sup\{p_n(f(x)-f(y)):\ d(x, y) \leqslant \varepsilon_n\} \leqslant 1.$$

Let us put

$$\varrho(x, y) = d(x, y) + \sum_{n=1}^{\infty} 2^{-n}\min\{p_n(f(x)-f(y)), 1\} \quad \text{for } x, y \in X.$$

Obviously ϱ is uniformly equivalent to d on X. Let us check that f: $(X, \varrho) \to F$ satisfies the Lipschitz condition.

Take $n \in N$ and $x, y \in X$. If $d(x, y) \leqslant \varepsilon_n$, then

$$p_n(f(x)-f(y)) \leqslant 2^n \varrho(x, y).$$

If $d(x, y) > \varepsilon_n$ then by the convexity of X there exist points $x_0, x_1, \ldots, x_k \in X$ such that $x_0 = x$, $x_k = y$ and $d(x_{i+1}, x_i) \leqslant \frac{1}{k} d(x, y) \leqslant \varepsilon_n$ for $i = 0, 1, \ldots, k-1$, where $k = \left[\frac{1}{\varepsilon_n} d(x, y)\right] + 1$.

Then we get

$$p_n(f(x)-f(y)) \leqslant \sum_{n=0}^{k-1} p_n(f(x_{i+1})-f(x_i)) \leqslant k \leqslant \frac{1}{\varepsilon_n} d(x, y) + 1,$$

$$\frac{2}{\varepsilon_n} d(x, y) \leqslant \frac{2}{\varepsilon_n} \varrho(x, y).$$

Thus f satisfies the Lipschitz condition with the Lipschitz constant $K_n = \max\{2^n, 2/\varepsilon_n\}$.

This completes the proof of Lemma 1.4 and hence concludes the proof of Theorem 1.1.

In the following we will show that Theorem 1.1 is not true if the convex metric space X is replaced by $s = \boldsymbol{R}^{\infty}$. We need the following

1.5. Definition. Let $\mathscr{D}$ denote a class of Fréchet spaces. A Fréchet space $G \in \mathscr{D}$ is said to have the LP($\mathscr{S}$) (resp. LLP($\mathscr{D}$), LUP($\mathscr{D}$)) iff for every continuous linear map j from $E \in \mathscr{S}$ onto $F \in \mathscr{D}$ and for every continuous linear map (resp. Lipschitz map, uniformly continuous map) f: $G \to F$ there exists a continuous linear map (resp. Lipschitz map, uniformly continuous map) g: $G \to E$ such that $jg = f$.

1.6. Theorem. *Let $\mathscr{N}$ denote the class of all nuclear Fréchet spaces. Then the following conditions are equivalent*:

(i) $\dim G < \infty$,

(ii) *G has the* LP($\mathscr{N}$),

(iii) *G has the* LLP($\mathscr{N}$),

(iv) *G has the* LUP($\mathscr{N}$).

Proof. The equivalence between (i) and (ii) has been proved by Gejler [3]. The implications (i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv) follow immediately from 1.3 and 1.1 respectively.

(iii)$\Rightarrow$(ii) Let $f\colon G \to F$ be a continuous linear map. Take a Lipschitz map $g_1\colon G \to E$ such that $jg_1 = f$. By a theorem of Mankiewicz [5] the differential $g = (Dg_1)_{x_0}$ exists for some $x_0 \in G$. Thus we have

$$jg = j(Dg_1)_{x_0} = D(jg_1)_{x_0} = (Df)_{x_0} = f.$$

(iv)$\Rightarrow$(ii) Let $f\colon G \to F$ be a continuous linear map and $g_1\colon G \to E$ be a uniformly continuous map such that $jg_1 = f$. Using the argument of Mankiewicz [4] one can take a Lipschitz map $g_2\colon G \to E$ such that $jg_2 = f$. By (iii)$\Rightarrow$(ii) there exists a continuous linear map $g\colon G \to E$ satisfying the condition $jg = f$.

Theorem 1.6 has a useful corollary

1.7. COROLLARY. *Theorem* 1.1 *is not true without the assumption of the convexity of* X.

Proof. Take $s = \mathbf{R}^\infty$. Since $\dim s = \infty$, by 1.6 there exists a continuous linear map j from a nuclear Fréchet space E onto a nuclear Fréchet space F and a continuous linear map $f\colon s \to F$ such that there is no continuous linear map $g\colon s \to E$ with $jg = f$. By the implication (iv)$\Rightarrow$(ii) it now follows that there is no uniformly continuous map $g\colon G \to E$ such that $jg = f$.

2. Lifting locally uniformly continuous maps. A map f from a uniform space X into a uniform space Y is said to be *locally uniformly continuous* iff for every $x \in X$ there exists a neighbourhood $U(x) \subset X$ such that $f|U(x)$ is uniformly continuous. The following proposition shows that the class of locally uniformly continuous maps is much more narow than the class of continuous maps.

2.1. PROPOSITION. *A metric space X containing no isolated points is locally compact if and only if every continuous function on X is locally uniformly continuous.*

Proof. Assume that X is not locally compact. Take $x_0 \in X$ and a basis of neighbourhoods $U_n(x_0) = \{x \in X\colon d(x, x_0) \leqslant \varepsilon_n\}$ such that

(a) $U_n(x_0)$ is not compact for every $n \in \mathbf{N}$, and

(b) There exists a sequence $(x_n^k)_{k=1}^{\infty} \subset U_n(x_0) \setminus U_{n-1}(x_0)$ which contains no converging subsequences.

Since X contains no isolated points there exists a sequence $(y_n^k) \subset U_n(x_0) \setminus U_{n-1}(x_0)$ such that

$$\lim_{k \to \infty} \varrho(x_n^k, y_n^k) = 0 \quad \text{for every } n \in \mathbf{N}, \tag{2.2}$$

$$x_n^k \neq y_n^k \quad \text{for every } k \in \mathbf{N}, \tag{2.3}$$

It is easy to see that the set

$$A = (x_n^k)_{k,n=1}^{\infty} \cup (y_n^k)_{k,n=1}^{\infty} \cup (x_0)$$

is closed in X. Define a map f: $A \to \boldsymbol{R}^1$ by

$$(2.4)\qquad f(x) = \begin{cases} 1/n & \text{if } x = x_n^k \text{ for every } k \in \boldsymbol{N}, \\ 2/n & \text{if } x = y_n^k \text{ for every } k \in \boldsymbol{N}, \\ 0 & \text{if } x = x_0. \end{cases}$$

It is easy to see that f is continuous. By the classical Tietze theorem there exists a continuous map f': $X \to \boldsymbol{R}^1$ such that $f'|A = f$.

· From (2.2)–(2.4) it follows that f' is not uniformly continuous on $U_n(x_0)$ for any $n \in \boldsymbol{N}$. This proves the proposition.

Let us note that 2.1 may not be true if X contains isolated points.

2.5. Example. Let B be an infinite dimensional Banach space. For every $n \in \boldsymbol{N}$, take a sequence $(x_n^k) \subset B$ satisfying the conditions

$$(2.6)\qquad \|x_n^k\| = 1/n \quad \text{for every } k \in \boldsymbol{N}.$$

$$(2.7)\qquad \|x_n^k - x_n^l\| \geqslant \varepsilon_n > 0 \quad \text{for every } k, l \in \boldsymbol{N}.$$

Let $X = (x_n^k)_{k,n=1}^{\infty} \cup (0)$.

It is easy to see that every continuous function on X is uniformly continuous, however, X is not locally compact.

2.8. Definition. A Fréchet space G is said to have the LLUP iff for every continuous linear map j from a Fréchet space E onto a Fréchet space F and, for every locally uniformly, uniformly continuous map f: $G \to F$ there exists a locally uniformly continuous map g: $G \to E$ such that $jg = f$.

Let us prove the following

2.9. Theorem. *If G has the* LLUP, *then G is a Montel–Fréchet space admiting a continuous norm.*

Proof. Let $G_0 = G \setminus (0)$. By $\mathscr{L}(G_0)$ we denote the Fréchet space of all functions λ: $G_0 \to \boldsymbol{R}^1$ satisfying the condition

$$p'_n(\lambda) = \sum_{x \in G_0} \lambda(x) p_n(x) < \infty \quad \text{for every } n \in \boldsymbol{N},$$

where (p_n) denotes an increasing sequence of pseudonorms inducing the topology of G. Define a map j: $\mathscr{L}(G_0) \to G$ by

$$j(\lambda) = \sum_{x \in G_0} \lambda(x) x.$$

Since G has the LLUP, there exists a locally uniformly continuous map r: $G \to \mathscr{L}(G_0)$ such that

$$(2.10)\qquad jr = \mathrm{id}_G.$$

Assume to the contrary that G is not a Montel–Fréchet space. Take a bounded sequence $(x_k) \subset G$ which contains no converging subsequences and a countable set $S \subset G_0$ such that

$$r(x_k) \subset \mathscr{L}(s) \subset \mathscr{L}(G_0).$$

Since $\mathscr{L}(S)$ is separable, by a theorem of Gejler [3] there exists a Montel–Fréchet space F and a continuous linear map g from F onto $\mathscr{L}(S)$. Take a locally uniformly continuous map r_1: $G \to F$ such that $gr_1 = \pi r$, where π: $\mathscr{L}(G_0) \to \mathscr{L}(S)$ is the restriction map. Let U be a neighbourhood of zero in G such that $r_1|U$ is uniformly continuous.

Let (P_n), (Q_n) be increasing sequences of pseudonorms inducing the topologies of G and F respectively such that

$$U_n = \{x \in G\colon\ P_n(x) \leqslant 1\} \subset U \qquad \text{for every } n \in N$$

and

$$Q_n(r_1(x) - r_1(y)) \leqslant 1 \qquad \text{whenever } x, y \in U \text{ and } P_n(x-y) \leqslant 1.$$

If $x, y \in U$ and $P_n(x-y) \geqslant 1$ then there are points $a_0 = x, a_1, \ldots, a_k = y$ such that $P_n(a_{i+1} - a_i) \leqslant 1$ for $i = 0, \ldots, k-1$, where $k = [P_n(x-y)] + 1$. Whence we have

$$\begin{aligned} Q_n(r_1(x) - r_1(y)) &\leqslant \sum_{i=0}^{k-1} Q_n(r_1(a_{i+1}) - r_1(a_i)) \leqslant k \\ &\leqslant P_n(x-y) + 1 \leqslant 2P_n(x-y). \end{aligned} \tag{2.11}$$

Since (x_k) contains no converging subsequences, there exists $\varepsilon > 0$ such that $(\varepsilon x_k) \subset U \backslash U_{n_0}$ for some $n_0 \in N$. Whence from (2.11) we get

$$Q_n(r_1(\varepsilon x_k) - r_1(0)) \leqslant 2P_n(\varepsilon x_k) = 2\varepsilon P_n(x_k) \qquad \text{for every } k \in N$$

and for every $n \geqslant n_0$.

Since (x_k) is bounded in G we infer that $(r_1(\varepsilon x_k))$ is bounded, and hence precompact, in F. Thus

$$\left(\frac{1}{\varepsilon} jgr_1(\varepsilon x_k)\right) = \left(\frac{1}{\varepsilon} j\pi r(\varepsilon x_k)\right) = \left(\frac{1}{\varepsilon} jr(\varepsilon x_k)\right) = (x_k)$$

is precompact in F. This contradiction shows that G is a Montel–Fréchet space.

Now let us assume that G does not admit a continuous norm. By a theorem of Bessaga and Pełczyński [2], G has a subspace isomorphic to $s = \boldsymbol{R}^{\infty}$. Since G has the LLUP we infer that s does so.

Let j be a continuous linear map from a nuclear Fréchet space E onto a nuclear Fréchet space F and f: $s \to F$ be a continuous linear map. Take a locally uniformly continuous map g: $s \to E$ such that $jg = f$. Select an $n_0 \in N$ such that $g_1 = g|s'$: $s' \to E$ is uniformly continuous, where $s' = 0 \times \ldots \times 0 \times \prod_{n=n_0+1}^{\infty} R$.

By the implication (iv)$\Rightarrow$(ii) of Theorem 1.6 there exists a continuous linear map $h_1\colon s' \to F$ such that $jh_1 = f|s'$. Let $h_2\colon \boldsymbol{R}^{n_0} \times 0 \to E$ be a continuous linear map such that $jh_2 = f|\boldsymbol{R}^{n_0} \times 0$. Whence $h = h_1 \oplus h_2\colon s \to E$ is a continuous linear map satisfying the condition $jh = f$. By Theorem 1.6 we get $\dim s < \infty$, a contradiction.

Thus G admits a continuous norm and the theorem is proved.

Remark. The proof of Theorem 2.9 is similar to the proof of Theorem 2.1 of Gejler [3].

The following problem is still open:

PROBLEM. Let G be a Fréchet space which has the LLUP. Is it true that $\dim G < \infty$?

References

[1] R. G. Bartle and L. M. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. 72 (1952), 400–413.

[2] C. Bessaga and A. Pełczyński, *On a class of B_0-spaces*, Bull. Acad. Pol. Sci. 5 (1957), 375–379.

[3] W. Gejler, *Extending and lifting continuous linear maps*, Studia Math. 62 (1978), 295–303.

[4] P. Mankiewicz, *On Lipschitz mappings between Fréchet spaces*, ibidem 41 (1972), 225–241.

[5] —, *On differentiability of Lipschitz mappings in Fréchet spaces*, ibidem 45 (1973), 15–29.

[6] —, *On topological, Lipschitz and uniform classifications of LF-spaces*, ibidem 52 (1974), 109–142.

[7] K. Menger, *Untersuchungen über allegemeine Metrik*, Math. Ann. 100 (1928), 75–63.

[8] A. Pietsch, *Nuclear locally convex spaces*, Springer, Berlin 1972.

Reçu par la Rédaction le 19.11.1979
et en la forme revisée le 20.10.1988