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Remark on a mean ergodic theorem

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Let H denote a real, separable Hilbert space and let p be a probability distribution in H (i.e. normed, regular measure defined on a σ-field $\mathfrak{B}$ of Borelian subsets of H) such that

$$\int \|h\|^2 p(dh) < \infty \ (^1) . \tag{1}$$

The mathematical expectation M_p and the dispersion operator D_p of probability measure p are defined, as it is known, by the formulae

$$(M_p, h) = \int (g, h) p(dg) , \quad h \in H , \tag{2}$$

$$(D_p g, h) = \int (u - M_p, g)(u - M_p, h) p(du) , \quad g, h \in H \tag{3}$$

(see for example [2], [4]).

The probability distribution p is uniquely determined by its characteristic functional ([1], [2], [4]):

$$\hat{p}(h) = \int e^{i(g,h)} p(dg) . \tag{4}$$

The distribution

$$(p * q)(Z) = \int p(Z-h) q(dh) \quad \text{for} \quad Z \in \mathfrak{B} \tag{5}$$

is called the *convolution* of distributions p and q. The convolution of the distributions $p_1, p_2, \ldots, p_n$ will be denoted by $\prod\limits_{k=1}^{n}{}^{*} p_k$. For a linear, bounded operator A in H and for a probability distribution p we put by definition

$$(Ap)(Z) = p(A^{-1}Z) \quad \text{for} \quad Z \in \mathfrak{B} , \tag{6}$$

where $A^{-1}Z = \{h:\ Ah \in Z\}$.

A sequence of probability measures $\{p_n\}$ is said to be *weakly convergent* to p ($p_n \to p$) if

$$\int f(h) p_n(dh) \to \int f(h) p(dh) \quad (n \to \infty) \tag{7}$$

for any continuous function f bounded in H.

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(1) $\int \ldots$ means an integral over the whole space H.

The purpose of this paper is to prove the following

THEOREM. *Let U be the unitary operator in H, p a probability distribution in H satisfying condition* (1), *M let denote the mathematical expectation of p. Then the sequence of distributions*

$$p_n = \prod_{k=1}^{n}{}^{*} \frac{1}{n} U^k p \quad (n = 1, 2, \ldots) \tag{8}$$

converges weakly to a one-point distribution δ_{x_0}, where

$$x_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U^k M . \tag{9}$$

Limit (9) exists by the ergodic theorem of J. v. Neumann (see [3] or [5], § X, 3). Our theorem can be regarded as a generalization of the mentioned theorem of J. v. Neumann, if we observe that in case of a one-point distribution $p = \delta_M$ formula (8) is reduced to formula (9).

Proof. We have $p = p*\delta_{(-M)}*\delta_M$. The mathematical expectation of the distribution $p*\delta_{(-M)}$ is equal to 0. Moreover, we have

$$\prod_{k=1}^{n}{}^{*} \frac{1}{n} U^k p = \left(\prod_{k=1}^{n}{}^{*} \frac{1}{n} U^k (p*\delta_{(-M)})\right) * \left(\prod_{k=1}^{n}{}^{*} \frac{1}{n} U^k \delta_M\right)$$

$$= \left(\prod_{k=1}^{n}{}^{*} \frac{1}{n} U^k (p*\delta_{(-M)})\right) * \delta_{\frac{1}{n}\sum_{k=1}^{n} U^k M} .$$

Thus, in view of the continuity of a convolution with respect to a weak convergence of measures, it suffices to show that for $M = \Theta$ the sequence of distributions (8) converges to a one-point distribution δ_Θ. Let D denote the dispersion operator of p. We have

$$\hat{p}(h) = 1 - \tfrac{1}{2}(Dh, h) + w(h)\|h\|^2 , \tag{10}$$

where $w(h) \to 0$ as $h \to \Theta$.

$$\hat{p}_n(h) = \prod_{k=1}^{n} \hat{p}\left(\frac{1}{n} U^{-k} h\right), \tag{11}$$

$$\log \hat{p}_n(h) = \sum_{k=1}^{n} \left[1 - \frac{1}{2n^2}(DU^{-k}h, U^{-k}h) + \frac{1}{n^2} w\left(\frac{1}{n} U^{-k} h\right) \|h\|^2\right]. \tag{12}$$

The operator D is self-adjoint, non-negative, compact, with a finite trace.

Let $\{e_i\}$ be a basis in H, diagonal with respect to the operator D, i.e. $De_i = \lambda_i e_i$ $(i = 1, 2, \ldots)$. Obviously

$$\lambda_i \geqslant 0 \quad \text{and} \quad \sum_{k=1}^{n} \lambda_i = \operatorname{Tr} D < \infty .$$

Then we have

$$\frac{1}{n^2}\sum_{k=1}^{n}(DU^{-k}h,\, U^{-k}h) = \frac{1}{n^2}\sum_{k=1}^{n}\sum_{i=1}^{\infty}(U^{-k}h,\, e_i)^2\lambda_i$$
$$\leqslant \frac{1}{n^2}\sum_{k=1}^{n}\sum_{i=1}^{\infty}\|h\|^2\lambda_i = \frac{\|h\|^2}{n}\operatorname{Tr} D \to 0 \quad \text{as} \quad n\to\infty .$$

It can be likewise easily seen that

$$\frac{1}{n^2}\sum_{k=1}^{n} w\left(U^{-k}\frac{1}{n}h\right)\|h\|^2 \to 0 \qquad (n\to\infty) .$$

Now it is seen that

$$\lim_{n\to\infty}\hat{p}_n(h) = 1$$

for any $h \in H$. Thus the sequence of characteristic functionals of the distributions p_n converges to the characteristic functional of the one-point distribution δ_Θ. To prove that $p_n \to \delta_\Theta$ it suffices to verify that the sequence $\{B_n\}$ of dispersion operators of the distributions p_n satisfies the conditions

1° $\sup_n \operatorname{Tr} B_n < \infty$,

2° $\lim_{m\to\infty}\sup_n \sum_{k=m}^{\infty}(B_n\varphi_k,\, \varphi_k) = 0$,

where $\{\varphi_k\}$ is a basis in H. We have

$$(B_n h,\, h) = \frac{1}{n^2}\sum_{k=1}^{n}(DU^{-k}h,\, U^{-k}h) .$$

Thus

$$\operatorname{Tr} B_n = \frac{1}{n^2}\sum_{k=1}^{n}\sum_{i=1}^{\infty}(DU^{-k}\varphi_i,\, U^{-k}\varphi_i) = \frac{1}{n}\operatorname{Tr} D \to 0,$$

which completes the proof of the theorem.

References

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