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ALMOST CONTACT MANIFOLDS WITH SPECIFIED AFFINE CONNEXIONS III

BY

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In this paper we shall define affine connexions in an almost contact manifold and obtain some properties.

1. Introduction. Let us introduce in an odd dimensional ($n = (2m+1)$-dimensional) real differential manifold of differentiability class C^∞, a vector-valued C^∞ linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$A(T) = 1, \tag{1.1}$$

$$A(\bar{X}) = 0 \quad \text{for arbitrary vector field } X, \tag{1.2a}$$

where

$$\bar{X} \stackrel{\text{def}}{=} F(X), \tag{1.2b}$$

$$\bar{T} = 0, \tag{1.3}$$

$$\bar{\bar{X}} + X = TA(X) \quad \text{for arbitrary vector field } X. \tag{1.4}$$

Then (F, T, A) is called an *almost contact structure* and the manifold is called an *almost contact manifold.*

Agreement (1.1). *Equations containing $X, Y, Z, U \ldots$ will hold for arbitrary vector fields $X, Y, Z, U \ldots$*

2. The manifold V_n. Let us introduce in the almost contact manifold defined as above an affine connexion D satisfying

$$A(Y) D_X T + (D_X A)(Y) T = 0. \tag{2.1a}$$

Agreement (2.1). *We shall mean by V_n an almost contact manifold in which an affine connexion D satisfying* (2.1a) *has been introduced.*

THEOREM (2.1). *Equation* (2.1) *implies*

$$A(Y)D_{\overline{X}}T+(D_{\overline{X}}A)(Y)T = 0, \tag{2.1b}$$

$$\overline{D_X T} = 0, \tag{2.2a}$$

$$\overline{D_{\overline{X}} T} = 0, \tag{2.2b}$$

$$D_X T = A(D_X T)T, \tag{2.3a}$$

$$D_{\overline{X}}T = A(D_{\overline{X}}T)T, \tag{2.3b}$$

$$(D_X A)(\overline{Y}) = -A(D_X\overline{Y}) = -A\big((D_X F)(Y)\big) = 0. \tag{2.4a}$$

$$(D_{\overline{X}}A)(\overline{Y}) = -A(D_{\overline{X}}\overline{Y}) = 0. \tag{2.4b}$$

Proof. Barring (2.1a) and using (1.3), we obtain (2.2a). Barring (2.2a) and using (1.4), we obtain (2.3a). Barring Y in (2.1a) and using (1.2a), we obtain (2.4a). Barring X in (2.1a) we obtain (2.1b).

THEOREM (2.2). *We have in* V_n

$$A(Y)\operatorname{div}T+(D_T A)(Y) = 0, \tag{2.5a}$$

where

$$\operatorname{div}X \overset{\text{def}}{=} (C_1^1 \nabla X), \tag{2.5b}$$

$$(\nabla X)(Y) \overset{\text{def}}{=} D_Y X, \tag{2.5c}$$

$$(D_X F)(T) = 0, \tag{2.6a}$$

$$(\operatorname{div}F)(T) = 0, \tag{2.6b}$$

$$\operatorname{div}T = A(D_T T). \tag{2.7}$$

Proof. Contracting (2.1a), we get (2.5a). From (1.3) we have $(D_X F)(T)+\overline{D_X T} = 0$. Using (2.2a) in this equation, we get (2.6a). Contracting this equation, we get (2.6b). Putting T for Y in (2.5a) and using (1.1), we get (2.7).

THEOREM (2.3). *We also have in* V_n

$$(D_X A)(Z) = -A(Z)A(D_X T), \tag{2.8a}$$

$$(D_X A)(Z)A(D_Y T) = A(Z)(D_X A)(D_Y T). \tag{2.8b}$$

Proof. (2.8) follows from (2.1a) and (1.1).

We know that the curvature tensor K is the vector-valued tri-linear function given by

$$K(X, Y, Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z. \tag{2.9}$$

THEOREM (2.4). *We have in* V_n

$$K(X, Y, T) = A(K(X, Y, T))T. \tag{2.10}$$

Proof. Using (2.9), (2.3a), (2.1a) and (1.1), we get

$$\begin{aligned} K(X, Y, T) &= D_X D_Y T - D_Y D_X T - D_{[X,Y]} T \\ &= (D_X T)A(D_Y T) + (D_X A)(D_Y T)T + A(D_X D_Y T)T - \\ &\quad - (D_Y T)A(D_X T) - (D_Y A)(D_X T)T - A(D_Y D_X T)T - \\ &\quad - A(D_{[X,Y} \dot{T})T = TA(K(X, Y, T)). \end{aligned}$$

COROLLARY (2.1). *We have in* V_n

$$\overline{K(X, Y, T)} = 0, \tag{2.11}$$

$$K(X, Y, T)A(D_Z T) = (D_Z T)A(K(X, Y, T)). \tag{2.12}$$

Proof. Barring (2.10) and using (1.3), we get (2.11). (2.12) follows from (2.10) and (2.3a).

THEOREM (2.5). *Put*

$$\operatorname{Ric}(Y, Z) \overset{\text{def}}{=} (C_1^1 K)(Y, Z). \tag{2.13}$$

Then

$$\operatorname{Ric}(Y, T) = A(K(T, Y, T)). \tag{2.14}$$

Proof. (2.14) follows from (2.13) and (2.10).

COROLLARY (2.2). *We have in* V_n

$$\operatorname{Ric}(T, T) = 0. \tag{2.15}$$

Proof. (2.15) follows from (2.14) and (2.9).

THEOREM (2.6). *Bianchi's second identities yield in* V_n

$$\begin{aligned} &(D_X T)A(K(Y, Z, T)) + (D_Y T)A(K(Z, X, T)) + \\ &\quad + (D_Z T)A(K(X, Y, T)) + T\{(D_X A)(K(Y, Z, T)) + \\ &\quad + (D_Y A)(K(Z, X, T)) + (D_Z A)(K(X, Y, T))\} = 0, \end{aligned} \tag{2.16}$$

or

$$\begin{aligned} &A(D_X T)A(K(Y, Z, T)) + A(D_Y T)A(K(Z, X, T)) + \\ &\quad + A(D_Z T)A(K(X, Y, T)) + (D_X A)(K(Y, Z, T)) + \\ &\quad + (D_Y A)(K(Z, X, T)) + (D_Z A)(K(X, Y, T)) = 0, \end{aligned} \tag{2.17}$$

or

$$A(D_XT)K(Y,Z,T)+A(D_YT)K(Z,X,T)+ \\ +A(D_ZT)K(X,Y,T)+T\{(D_XA)(K(Y,Z,T))+ \\ +(D_YA)(K(Z,X,T))+(D_ZA)(K(X,Y,T))\} = 0. \tag{2.18}$$

Proof. From (2.10) we have

$$(D_ZK)(X,Y,T) = (D_ZT)A(K(X,Y,T))+T(D_ZA)(K(X,Y,T))+ \\ +TA((D_ZK)(X,Y,T)).$$

Writing two other equations by the cyclic permutation of X, Y, Z, adding the three equations and using Bianchi's second identities, viz.

$$(D_XK)(Y,Z,T)+(D_YK)(Z,X,T)+(D_ZK)(X,Y,T)+ \\ +K(S(X,Y),Z,T)+K(S(Y,Z),X,T)+K(S(Z,X),Y,T) = 0,$$

and (2.10), we obtain (2.16). Substituting from (2.3a) in (2.16), we obtain (2.17). Substituting from (2.10) in (2.17), we obtain (2.18).

THEOREM (2.7). *We have in* V_n

$$A(K(X,Y,Z)) = A(Z)A(K(X,Y,T)). \tag{2.19}$$

Proof. We have from (1.1) and (2.1a)

$$(D_YA)(Z) = -A(Z)A(D_YT).$$

Consequently,

$$(D_XD_YA)(Z) = -(D_XA)(Z)A(D_YT)-A(Z)(D_XA)(D_YT)- \\ -A(Z)A(D_XD_YT).$$

Using (2.8) in this equation, we get

$$(D_XD_YA)(Z) = 2A(Z)A(D_XT)A(D_YT)-A(Z)A(D_XD_YT).$$

Consequently,

$$-A(K(X,Y,Z)) = (D_XD_YA-D_YD_XA-D_{[X,Y]}A)(Z) \\ = -A(Z)A(K(X,Y,T)).$$

COROLLARY (2.3). *We also have in* V_n

$$A(K(X,Y,\bar{Z})) = 0. \tag{2.20}$$

Proof. Barring Z in (2.19) and using (1.2a), we obtain (2.20).

THEOREM (2.8). *Let us put*

$$M(X,\ Y) \overset{\text{def}}{=} D_{\overline{X}}\overline{Y}+\overline{\overline{D_X Y}}-\overline{D_X\overline{Y}}-\overline{D_{\overline{X}}Y}. \tag{2.21}$$

Then

$$M(X, T) = 0. \tag{2.22}$$

Proof. Putting T for Y in (2.21) and using (2.2) and (1.3), we get (2.22).

We will not consider Nijenhuis tensor in this section. But related to Nijenhuis tensor of an almost complex manifold there are three other tensors in an almost contact manifold:

$$P(X,\ Y) \overset{\text{def}}{=} (D_Y A)(\overline{X})-(D_X A)(\overline{Y})+(D_{\overline{Y}}A)(X)-(D_{\overline{X}}A)(Y), \tag{2.23}$$

$$Q(X) \overset{\text{def}}{=} (D_T F)(X)-(D_X F)(T)-D_{\overline{X}}T, \tag{2.24}$$

$$R(X) \overset{\text{def}}{=} (D_X A)(T)-(D_T A)(X). \tag{2.25}$$

Here P is a scalar-valued bilinear function, Q is a vector-valued linear function, and R is a 1-form.

THEOREM (2.9). *In V_n, P, Q and R assume the forms*

$$P(X,\ Y) = (D_{\overline{Y}}A)(X)-(D_{\overline{X}}A)(Y), \tag{2.26}$$

$$Q(X) = (D_T F)(X)-D_{\overline{X}}T, \tag{2.27}$$

$$R(X) = -A(D_X T)+A(X)\operatorname{div}T. \tag{2.28}$$

Proof. Substituting from (2.4a), (2.6a), (1.1) and (2.5a) in (2.23) through (2.25), we obtain (2.26) through (2.28).

THEOREM (2.10). *We have in V_n*

$$P(X, T) = A(D_{\overline{X}}T) = A\big((D_T F)(X)\big)-A\big(Q(X)\big), \tag{2.29}$$

$$P(\overline{X}, T) = -A(D_X T)+A(X)\operatorname{div}T = R(X), \tag{2.30}$$

$$P(\overline{X}, \overline{Y}) = 0, \tag{2.31}$$

$$P(\overline{X},\ Y) = (D_X A)(Y)+A(X)A(Y)\operatorname{div}T, \tag{2.32a}$$

$$P(\overline{X},\ Y)+P(X, \overline{Y}) = (D_X A)(Y)-(D_Y A)(X), \tag{2.32b}$$

$$Q(T) = R(T) = 0. \tag{2.33}$$

Proof. Putting T for Y in (2.26) and using (1.3) and (2.27), we obtain (2.29). Barring X in (2.29) and using (2.7) and (2.28), we get (2.30). Barring X in (2.26) and using (2.4) and (2.5a), we get (2.32a). (2.31) and (2.32b) follow from (2.32a). Putting T for X in (2.27) and (2.28), we get (2.33).

3. Manifold W_n. Let us now introduce in the almost contact manifold an affine connexion D satisfying (2.1a), viz.

$$A(Y)D_X T + T(D_X A)(Y) = 0, \tag{3.1}$$

and

$$(D_X F)(Y) + (D_Y F)(X) = 0. \tag{3.2}$$

Agreement (3.1). *An almost contact manifold in which an affine connexion D satisfying* (3.1) *and* (3.2) *has been introduced will be denoted by W_n.*

In consequence of (1.2b), equation (3.2) is equivalent to

$$D_X \bar{Y} + D_Y \bar{X} = \overline{D_X Y} + \overline{D_Y X}. \tag{3.3}$$

Note (3.1). It may be noted that all the results of section 2 hold also in W_n. In addition, we have the following results:

Theorem (3.1). *Equations* (3.1) *and* (3.2) *imply*

$$\overline{D_X \bar{\bar{Y}}} + \overline{D_Y \bar{\bar{X}}} + D_X Y + D_Y X = T\{A(D_X Y) + A(D_Y X)\}, \tag{3.4}$$

$$D_{\bar{Y}} \bar{\bar{X}} + D_Y X = TA(D_Y X) + \overline{D_{\bar{Y}} X} - \overline{D_Y \bar{\bar{X}}}, \tag{3.5}$$

$$\overline{D_{\bar{Y}} \bar{\bar{X}}} + \overline{D_Y X} + D_{\bar{Y}} X - D_Y \bar{\bar{X}} = TA(D_{\bar{Y}} X), \tag{3.6}$$

$$D_T \bar{\bar{X}} = \overline{D_T X}, \tag{3.7}$$

$$\overline{D_T \bar{\bar{X}}} + D_T X = TA(D_T X). \tag{3.8}$$

Proof. Barring (3.3) and using (1.4), we obtain (3.4). Barring Y in (3.3) and using (1.4) and (3.4), we obtain (3.5). Barring (3.5) throughout and using (1.4) and (2.4), we get (3.6). Putting T for X in (3.3) and using (1.3) and (2.2a), we get (3.7). Similarly, putting T for X in (3.4) and using (1.4) and (2.3a), we obtain (3.8).

Theorem (3.2). *We have in W_n*

$$M(X, Y) = 2(\overline{\overline{D_X Y}} - \overline{D_X \bar{\bar{Y}}}) = 2(D_{\bar{X}} \bar{Y} - \overline{D_{\bar{X}} Y}), \tag{3.9a}$$

$$M(X, Y) = -2\overline{(D_X F)(Y)} = 2(D_{\bar{X}} F)(Y). \tag{3.9b}$$

Proof. Equation (3.5) can be written as

$$D_{\bar{X}} \bar{Y} - \overline{\overline{D_X Y}} - \overline{D_{\bar{X}} Y} + \overline{D_X \bar{\bar{Y}}} = 0.$$

Using this equation in (2.21), we obtain (3.9a). Using (1.2b) in (3.9a), we get (3.9b).

Corollary (3.1). *M is skew-symmetric in W_n.*

Proof. The statement follows from (3.9a) and (3.2).

We consider the vector-valued bilinear function N defined by

$$N(X, Y) \stackrel{\text{def}}{=} M(X, Y) - M(Y, X). \tag{3.10a}$$

We, therefore, have the following

THEOREM (3.3). *We have in W_n*

$$N(X, Y) = -4\overline{(D_X F)(Y)} = 4(D_{\bar X} F)(Y) = 2M(X, Y), \tag{3.10b}$$

$$A\big(N(X, Y)\big) = A\big(M(X, Y)\big) = 0. \tag{3.11}$$

Proof. (3.10b) follows from (3.10a) and (3.9b). Using (1.2a) in (3.10b), we get (3.11).

THEOREM (3.4). *We also have in W_n*

$$M(T, Y) = 0, \tag{3.12}$$

$$M(\bar X, \bar Y) + M(X, Y) = 0, \tag{3.13a}$$

$$M(X, \bar Y) = M(\bar X, Y), \tag{3.13b}$$

$$N(X, T) = 0, \tag{3.14}$$

$$N(\bar X, \bar Y) + N(X, Y) = 0, \tag{3.15a}$$

$$N(X, \bar Y) = N(\bar X, Y). \tag{3.15b}$$

Proof. Putting T for X in (3.9b) and (3.10b), we get (3.12) and (3.14). From (3.9a), we have $M(\bar X, \bar Y) = 2(\overline{D_{\bar X} Y} - \overline{D_{\bar X} \bar{\bar Y}})$. But from (3.6) and (1.4) we have

$$\overline{\overline{D_{\bar X} \bar Y}} + \overline{D_{\bar X} Y} = \overline{D_X \bar Y} - \overline{\overline{D_X Y}} \quad \text{or} \quad \overline{\overline{D_{\bar X} \bar Y}} - \overline{D_{\bar X} \bar{\bar Y}} = \overline{D_X \bar Y} - \overline{\overline{D_X Y}}.$$

Hence, using (3.9a) again, we get $M(\bar X, \bar Y) = 2(\overline{D_X \bar Y} - \overline{\overline{D_X Y}})$ $= -M(X, Y)$.

Barring X in (3.13a) and using (1.4) and (3.12), we get (3.13b). (3.15) follows from (3.10a) and (3.13).

THEOREM (3.5). *The following equations hold also in W_n:*

$$\overline{M(\bar X, Y)} = \overline{M(X, \bar Y)} = M(\bar X, \bar Y) = -M(X, Y), \tag{3.16}$$

$$\overline{N(\bar X, Y)} = \overline{N(X, \bar Y)} = N(\bar X, \bar Y) = -N(X, Y). \tag{3.17}$$

Proof. In consequence of (3.9b), (2.2a), (1.4), (1.3), (2.4a) and (1.2a), we have

$$\begin{aligned} M(X, \bar Y) = -2\overline{(D_X F)(\bar Y)} = -2\overline{(D_X \bar{\bar Y} - \overline{D_X \bar Y})} &= 2\overline{(D_X Y - D_X \bar Y)} \\ &= -2(D_X F)(Y). \end{aligned}$$

Hence, in consequence of (3.9b) and (3.13),

$$\overline{M(X, \overline{Y})} = -2\overline{(D_X F)(Y)} = \overline{M(\overline{X}, Y)} = M(X, Y) = -M(\overline{X}, \overline{Y}).$$

(3.17) follows from (3.16) and (3.10b).

Note (3.2). Since in W_n there is

$$(D_T F)(X) = 0, \tag{3.18}$$

equations (2.27) and (2.29) assume the forms

$$Q(X) = -D_{\overline{X}} T, \tag{3.19}$$

$$P(X, T) = A(D_{\overline{X}} T) = -A(Q(X)). \tag{3.20}$$

4. Manifold P_n. In this section we shall introduce in an almost contact manifold an affine connexion D satisfying (2.1a), viz.

$$A(Y) D_X T + T(D_X A)(Y) = 0 \tag{4.1}$$

and

$$(D_X F)(Y) + (D_{\overline{X}} F)(\overline{Y}) = 0. \tag{4.2}$$

AGREEMENT (4.1). *An almost contact manifold in which an affine connexion D satisfying* (4.1) *and* (4.2) *have been introduced will be denoted by P_n.*

Note (4.1). All the results of section 2 hold also in P_n. In addition, we have the following results:

THEOREM (4.1). *Equations* (4.1) *and* (4.2) *imply*

$$D_X \overline{\overline{Y}} + D_{\overline{X}} \overline{\overline{\overline{Y}}} = \overline{D_X Y} + \overline{D_{\overline{X}} \overline{\overline{Y}}}, \tag{4.3a}$$

which is equivalent to

$$D_X \overline{\overline{Y}} + TA(D_{\overline{X}} Y) = D_{\overline{X}} Y + \overline{D_X Y} + \overline{D_{\overline{X}} \overline{\overline{Y}}}, \tag{4.3b}$$

or to

$$(D_X F)(Y) = \overline{(D_{\overline{X}} F)(Y)}, \tag{4.3c}$$

or to

$$\overline{(D_X F)(Y)} = -(D_{\overline{X}} F)(Y). \tag{4.3d}$$

Proof. Using (1.2b) in (4.2), we get (4.3a). In consequence of (1.4), equation (4.3a) assumes form (4.3b). In consequence of (1.2a), equation (4.3b) assumes form (4.3c). Barring (4.3c) and using (2.4), we obtain (4.3d).

THEOREM (4.2). *Equations* (4.1) *and* (4.2) *imply*

$$D_{\bar{X}}\bar{Y}+\overline{D_X\bar{Y}}+D_X Y=\overline{D_{\bar{X}}Y}+TA(D_X Y), \tag{4.4}$$

$$D_T\bar{X}=\overline{D_T X}, \tag{4.5}$$

$$\overline{D_T\bar{X}}+D_T X=TA(D_T X). \tag{4.6}$$

Proof. Putting T for X in (4.3a) and using (1.3), we get (4.5) Barring (4.5) and using (1.4), we get (4.6). Barring X in (4.3b) and using (1.4) and (4.6), we get (4.4).

Note (4.2). It may be noted that the result of a repeated operation of barring different vectors yields a closed cycle of only two equations (4.3) and (4.4).

Note (4.3). Equations (4.3) and (4.4) of P_n are the same as equations (3.5) and (3.6) of W_n.

THEOREM (4.3). *The following equations hold in* P_n:

$$M(X, Y)=2(D_{\bar{X}}F)(Y)=-2\overline{(D_X F)(Y)}, \tag{4.7a}$$

$$M(X, Y)=2(D_{\bar{X}}\bar{Y}-\overline{D_{\bar{X}}Y})=2(\overline{\overline{D_X Y}}-\overline{D_X\bar{Y}}), \tag{4.7b}$$

$$\overline{M(\bar{X}, Y)}=\overline{M(X, \bar{Y})}=M(X, Y)=-M(\bar{X}, \bar{Y}), \tag{4.8a}$$

$$M(X, \bar{Y})=M(\bar{X}, Y), \tag{4.8b}$$

$$A\big(M(X, Y)\big)=0, \tag{4.9}$$

$$M(X, T)=M(T, Y)=0. \tag{4.10}$$

Proof. Using (4.3a) in (2.21), we obtain (4.7b). In consequence of (1.2b), (4.7b) assumes form (4.7a). Putting T for X and Y in (4.7a) and using (1.3) and (2.2a), we get (4.10). Equation (4.9) follows from (4.7a) and (1.2a).

Barring X and Y in (4.7b) and using (1.4), (4.5), (2.2a) and (1.3), we get

$$M(X, \bar{Y})=M(\bar{X}, Y)=2(\overline{D_X Y}-D_X\bar{Y}).$$

Hence

$$M(\bar{X}, \bar{Y})=2(\overline{D_{\bar{X}}Y}-D_X\bar{Y})=-M(X, Y)=2(\overline{D_X\bar{Y}}-\overline{\overline{D_X Y}})$$
$$=-\overline{M(X, \bar{Y})}=-\overline{M(\bar{X}, Y)}.$$

Hence we have (4.8a) and (4.8b).

COROLLARY (4.1). *We have in* P_n

$$N(X, Y) = 2\{(D_{\overline{X}}F)(Y) - (D_{\overline{Y}}F)(X)\} \tag{4.11}$$

$$= 2\{\overline{(D_Y F)(X)} - \overline{(D_X F)(Y)}\},$$

$$\overline{N(X, \overline{Y})} = \overline{N(\overline{X}, Y)} = N(X, Y) = -N(\overline{X}, \overline{Y}), \tag{4.12a}$$

$$N(X, \overline{Y}) = N(\overline{X}, Y), \tag{4.12b}$$

$$A\big(N(X, Y)\big) = 0, \tag{4.13}$$

$$N(X, T) = N(T, Y) = 0. \tag{4.14}$$

Proof. Using (3.10a) in (4.7) through (4.10), we get (4.11) through (4.14).

Note (4.4). It may be noted that (4.11) of P_n is not the same as equation (3.10b) of W_n, whereas other equations of theorem (4.3) and corollary (4.1) hold also in P_n and W_n.

Note (4.5). Since in P_n there is

$$(D_T F)(X) = 0, \tag{4.15}$$

equations (2.27) and (2.29) assume the forms

$$Q(X) = -D_{\overline{X}}T, \tag{4.16}$$

$$P(X, T) = A(D_{\overline{X}}T) = -A\big(Q(X)\big), \tag{4.17}$$

respectively.

5. Appendix. We prove the following

THEOREM (5.1). *Let* D *be an arbitrary connexion satisfying*

$$A(X)D_Y T + (D_Y A)(X)T = 0. \tag{5.1}$$

We can always find a connexion B *such that* $B_X Y$ *is a linear combination of* $D_X Y$ *and different vectors obtained by barring* X, Y *and* D_X, *and that* $B_X Y$ *satisfies*

$$(B_X F)(Y) + (D_Y F)(X) = 0, \tag{5.2}$$

$$\overline{B_X T} = 0. \tag{5.3}$$

Then $B_X Y$ *is given by*

$$B_X Y = \alpha(D_X Y - \overline{D_X \overline{Y}}) + \beta(D_{\overline{X}} Y - \overline{D_{\overline{X}} \overline{Y}}) + \gamma(D_X \overline{Y} + \overline{D_X Y}) + \delta(D_{\overline{X}} \overline{Y} + \overline{D_{\overline{X}} Y}). \tag{5.4}$$

Proof. Equation (5.2) is equivalent to

$$B_X \overline{Y} + B_Y \overline{X} = \overline{B_X Y + B_Y X}. \tag{5.5}$$

Let us put

$$(5.6)\qquad B_X Y = \alpha D_X Y + \beta D_{\bar X} Y + \gamma D_X \bar Y + \delta D_{\bar X}\bar Y + \theta \overline{D_X Y} + \varphi \overline{D_{\bar X} Y} + \\ + \varrho \overline{D_X \bar Y} + \sigma \overline{D_{\bar X} \bar Y}.$$

Then, using (5.1), (1.3) and (2.2a), we have

$$(5.7)\qquad B_X \bar Y = \alpha D_X \bar Y + \beta D_{\bar X} \bar Y - \gamma D_X Y + \gamma A(D_X Y)T - \delta D_{\bar X} Y + \\ + TA(D_{\bar X} Y)\delta + \theta \overline{D_X \bar Y} + \varphi \overline{D_{\bar X}\bar Y} - \varrho \overline{D_X Y} - \sigma \overline{D_{\bar X} Y},$$

$$(5.8)\qquad \overline{B_X Y} = \alpha \overline{D_X Y} + \beta \overline{D_{\bar X} Y} + \gamma \overline{D_X \bar Y} + \delta \overline{D_{\bar X}\bar Y} - \theta D_X Y + \theta A(D_X Y)T - \\ - \varphi D_{\bar X} Y + \varphi A(D_{\bar X} Y)T - \varrho D_X \bar Y - \sigma D_{\bar X}\bar Y.$$

Using (5.7) and (5.8) in (5.5), we get

$$\alpha(D_X\bar Y + D_Y\bar X) + \beta(D_{\bar X}\bar Y + D_{\bar Y}\bar X) - \gamma(D_X Y + D_Y X) + \\ + \gamma TA(D_X Y + D_Y X) - \delta(D_{\bar X} Y + D_{\bar Y} X) + \delta TA(D_{\bar X} Y + D_{\bar Y} X) + \\ + \theta(\overline{D_X \bar Y} + \overline{D_Y \bar X}) + \varphi(\overline{D_{\bar X}\bar Y} + \overline{D_{\bar Y}\bar X}) - \varrho(\overline{D_X Y} + \overline{D_Y X}) - \sigma(\overline{D_{\bar X} Y} + \overline{D_{\bar Y} X}) \\ = \alpha(\overline{D_X Y} + \overline{D_Y X}) + \beta(\overline{D_{\bar X} Y} + \overline{D_{\bar Y} X}) + \gamma(\overline{D_X \bar Y} + \overline{D_Y \bar X}) + \delta(\overline{D_{\bar X}\bar Y} + \\ + \overline{D_{\bar Y}\bar X}) - \theta(D_X Y + D_Y X) + \theta TA(D_X Y + D_Y X) - \varphi(D_{\bar X} Y + D_{\bar Y} X) + \\ + \varphi TA(D_{\bar X} Y + D_{\bar Y} X) - \varrho(D_X\bar Y + D_Y\bar X) - \sigma(D_{\bar X}\bar Y + D_{\bar Y}\bar X).$$

By comparison, we get $\varrho = -\alpha$, $\sigma = -\beta$, $\theta = \gamma$ and $\varphi = \delta$. Substituting these equations in (5.6), we get (5.4).

From (5.4) and (2.2) we get

$$\overline{B_X T} = \alpha \overline{D_X T} + \beta \overline{D_{\bar X} T} + \gamma \overline{\overline{D_X T}} + \delta \overline{D_{\bar X} T} = 0.$$

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