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STOCHASTIC BEHAVIOR ABOVE FEIGENBAUM'S PARAMETER VALUE

M. V. JAKOBSON

Institute of Oceanology
Moscow, U.S.S.R.

We use the renormalization approach to study one parameter families of maps $f_\mu\colon I \to I$, $\mu \in [-1, 1]$, close to the unstable manifold of Feigenbaum's fixed point.

Suppose $\mu = 0$ is the limit point of doubling bifurcation parameter values and for $\mu < 0$ f_μ has zero topological entropy. Assuming that for $\mu > 0$ a certain transversality condition is satisfied we prove (Theorem 3) that in any interval $[0, \varepsilon]$ the relative measure of μ corresponding to f_μ with an absolutely continuous invariant measure is not less then $c \cdot \varepsilon$ where c depends neither on ε nor on the family under consideration.

§ 1. Universal behavior for one parameter families

1.1. We begin by formulating the main results of the Feigenbaum's universality theory (see for example [1]). For a given domain $\mathcal{D} \subset \boldsymbol{C}$ let $\mathcal{H}_{\mathcal{D}}$ be the Banach space of bounded analytic maps from $\mathcal{D}$ to $\boldsymbol{C}$ equipped with the norm

$$\|f\|_{\mathcal{D}} = \sup\{|f(z)|\colon z \in \mathcal{D}\}.$$

We suppose that $I = [-1, 1] \subset \mathcal{D}$. Let $\mathfrak{M}_{\mathcal{D}}$ be the subset of $\mathcal{H}_{\mathcal{D}}$ consisted of f satisfying the following conditions:

$$f(I) \subset I, \tag{1}$$

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) \neq 0, \quad 0 > f(1) > -1. \tag{2}$$

Let us write $\alpha(f) = f(1)$ and define the doubling transformation by

$$\mathcal{T} f(x) = \alpha^{-1}(f) \cdot f \circ f(\alpha(f) \cdot x)$$

Then:

I. There exists a domain $\mathscr{D}_1 \supset I$ such that the equation

$$\mathscr{T}f = f$$

has a solution φ_0 in $\mathfrak{M}_{\mathscr{D}_1}$. There exists a neighbourhood $\mathscr{U} \subset \mathscr{K}_{\mathscr{D}_1}$ of φ_0 such that $\mathscr{T}|_{\mathscr{U}}$ is a C^2 transformation and $\mathscr{T}\mathscr{U} \subset \mathscr{K}_{\mathscr{D}_1}$.

II. $D\mathscr{T}_{\varphi_0}$ is a compact operator from $\mathscr{K}_{\mathscr{D}_1}$ into itself. $D\mathscr{T}_{\varphi_0}$ restricted to $\{\psi(0) = 0, \psi'(0) = 0\}$ has one eigenvalue $\delta > 1$ and the remainder part of the spectrum is inside the unit circle.

III. $S\varphi_0(x) < 0$ where

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is Schwarzian derivative.

From now on we consider the restriction of $\mathscr{T}$ to $\mathfrak{M}_{\mathscr{D}_1}$. Using the stable manifold theorem for hyperbolic maps in Banach spaces (see for example [2]) we obtain

IV. In a neighborhood $\mathscr{U}_0$ of φ_0 a local unstable one dimensional manifold $W^u(\varphi_0)$ and a local stable manifold of codimension 1 $W^s(\varphi_0)$ are defined. Besides the one dimensional map $\mathscr{T}|_{W^u(\varphi_0)}$ may be linearized and in the appropriate coordinate system $W^u(\varphi_0)$ may be identified with the one parameter family of maps Φ: $\mu \to \varphi_\mu$, $\mu \in [-1, 1]$, so that

$$\mathscr{T}\varphi_\mu = \varphi_{\delta \cdot \mu}.$$

Notice that $S\varphi_0(x) < 0$ implies that for any map sufficiently close to $\varphi_0(x)$ in C^3 topology the Schwarzian derivative is also negative. We chose the neighborhood $\mathscr{U}_0$ so small that $Sf(x) < 0$ for any $f \in \mathscr{U}_0$.

1.2. Define an operator similar to the doubling transformation but acting not on maps but on one parameter families of maps (see [3], [4]).

Let $\mathfrak{N}$ be the set of one parameter families:

$$F: \mu \to f_\mu \in \mathscr{U}_0, \quad \mu \in [-1, 1]$$

normalized by the condition $f_0 \in W^s(\varphi_0)$. We assume that $f_\mu(x) = f(\mu, x)$ coincides with the restriction to $Q = [-1, 1] \times [-1, 1]$ of an analytic bounded map

$$f: \mathscr{D}_0 \times \mathscr{D}_1 \to C$$

where $\mathscr{D}_0 \supset [-1, 1]$, $\mathscr{D}_1 \supset [-1, 1]$ are some domains in C.

We consider $\mathfrak{N}$ as a subset of the Banach space of analytic maps equipped with the norm

$$\|f\| = \sup_{(\mu x)} |f(\mu, x)|.$$

Let $\Sigma_1 \subset \mathscr{U}_0$ be some surface of codimension 1 which transversally intersects $W^u(\varphi_0)$ at the point φ_1 and let $\Sigma_1^{-1} = \mathscr{T}^{-1}\Sigma_1 \cap \mathscr{U}_0$. For any family F sufficiently close to Φ a parameter value $\mu_1(F)$ is uniquely defined by the condition $f_{\mu_1(F)} \in \Sigma_1^{-1}$.

We define the *transformation* $\mathscr{T}_1$ acting on families by the formula

$$\mathscr{T}_1(F)\colon\ \mu \to \mathscr{T} f_{\mu\cdot\mu_1(F)}.$$

It follows from the definition of $\mathscr{T}_1$ and from $\mathscr{T}\varphi_\mu = \varphi_{\delta\mu}$ that Φ is a fixed point of $\mathscr{T}_1$.

PROPOSITION 1. *There exists a neighborhood $\mathscr{V} \subset \mathfrak{R}$ of the family Φ such that $\mathscr{T}_1 \mathscr{V} \subset \mathscr{V}$ and $\mathscr{T}_1$ is contracting on $\mathscr{V}$ thus Φ is the unique stable fixed point of $\mathscr{T}_1|_{\mathscr{V}}$.*

The proof of Proposition 1 is similar to the proof of the stable manifold theorem (see [2]).

V. Let $\tilde{\Phi}(\mu, x)$ be the polynomial approximation of $\Phi(\mu, x) = W^u(\varphi_0)$ constructed in [3]. When μ varies $\tilde{\Phi}(\mu, x) = \tilde{\varphi}_\mu(x)$ behave like a family of polynomial-like mappings of degree two.

For a certain parameter value, which after rescaling may be chosen equal to 1, the straightforward calculation shows that $\tilde{\varphi}_1(x)$ maps some interval $I_0 = [-x_0, x_0]$ twice onto itself and satisfies the following conditions:

(V^a) $$|\Phi'_x(1, x_0)| = A_0 > 1,$$

(V^b) $$\Phi'_\mu(1, 0) = \frac{\Phi'_\mu(1, x_0)}{\Phi'_x(1, x_0) - 1}.$$

where the unstable fixed point $-x_0(\mu)$ and its preimage $x_0(\mu)$ are the ends of $I_0(\mu) \supset \tilde{\varphi}_\mu I_0(\mu)$.

It is natural to assume that $\Phi(\mu, x)$ satisfies (V^{a,b}) as well as its approximation $\tilde{\Phi}(\mu, x)$. Then (V^b) would mean that $W^u(\varphi_0)$ has a point of transversal intersection with the surface of codimension 1 consisting of transformations which map some interval twice onto itself. A similar transversality condition concerning the intersection of $W^u(\varphi_0)$ with the surface, consisting of maps which have a neutral fixed point, is proved in [4]. Here we formulate an assumption concerning only a local part of $W^u(\varphi_0)$ (if the local unstable manifold may be extended up to the global one, then both assumptions are equivalent).

(V*). $\mathscr{U}_0$ contains a point Φ_{μ_0} of transversal intersection of $W^u(\Phi_0)$ with some surface $\Sigma_{n_0} = \{f\colon$ there exists an interval $I_0 \subset I$ such that $f^i I_0$, $i \in [0, 2^{n_0} - 1]$, are disjoint, $f^{2^{n_0}}$ maps I_0 twice onto itself$\}$ and for $\Phi^{2^{n_0}}|_{I_0}$ (V^a), (V^b) hold.

Below we shall use Σ_1 to denote $\Sigma_{n_0} \cap \mathscr{U}_0$.

1.3. According to Proposition 1 for any family $F \in \mathscr{V}$ the sequence $\mathscr{T}_1^k F = F_k$ converges to Φ in $\mathfrak{N}$. Because of the analytic dependence on μ and x this implies that F_k converge to Φ with all partial derivatives. We shall use the convergence of derivatives up to the third order.

It follows from III and V that for the family Φ the measure of the set of parameter values corresponding to the stochastic behavior is positive. Namely, the following is true (see [5]):

THEOREM 1. *There exists a positive measure set* $\mathscr{M}$ *of parameter values such that for* $\mu \in \mathscr{M} | \Phi_\mu$ *has absolutely continuous invariant measure with positive entropy, with the support inside* 2^{n_0} *cyclically permuted intervals.*

According to Ledrappier's theorem ([6]) the natural extension of $\Phi_\mu^{2^{n_0}}|_{I_0}$ is Bernoulli. If a family F is sufficiently close to Φ in C^3 topology then f_μ also have negative Schwarzian derivative and Theorem 1 is true for F. But we shall prove a stronger result.

THEOREM 2. *For any* $\varepsilon > 0$ *there exist* $\delta_0 > 0$ *and a* C^2 *neighborhood* $\mathscr{V}_0$ *of* Φ *such that if* F: $\mu \to f_\mu$ *is in* $\mathscr{V}_0$, f_1 *cyclically permutes* 2^{n_0} *intervals* $J_0, \ldots, J_{2^{n_0}-1}$ *and* $f_1^{2^{n_0}}|_{J_0}$ *doubly covers* J_0, *then for any positive* $\delta < \delta_0$

$$\delta^{-1} \cdot \operatorname{mes}\{\mathfrak{M}_F \cap [1-\delta, 1]\} > 1-\varepsilon$$

where $\mathscr{M}_F$ *is a set of parameter values* μ *such that* f_μ *has an absolutely continuous invariant measure.*

Theorem 2 which essentially relies on the inductive construction of $\mathscr{M}_F$ from [5] will be proved below. Now we shall use it to make some conclusions about the relative measure of stochastic behavior near and above Feigenbaum's parameter value.

1.4. Suppose F: $\mu \to f_\mu$ belongs to a domain $\mathscr{W} \subset \mathfrak{N}$ which is uniformly contracting under $\mathscr{T}_1$ to the universal family $\Phi = \mathscr{T}_1 \Phi$. Then $\mathscr{T}_1^k F = F_k$: $\mu \to f_\mu^{(k)}$ is exponentially converging to Φ in $\mathfrak{N}$, and F_k satisfy the conditions of Theorem 2 for sufficiently large k.

Let us denote by $\mathscr{M}(F_k)$ the set of parameter values μ such that $f_\mu^{(k)}$ has an absolutely continuous invariant measure supported by cyclically permuted 2^{n_0} intervals. It follows from Theorem 2 that there exist c_0 and k_0 depending only on the domain $\mathscr{W}$ such that for $k \geqslant k_0$

$$\operatorname{mes} \mathscr{M}(F_k) > c_0.$$

Let $\mu_1, \mu_2, \ldots, \mu_k$ be a sequence of renormalization parameter values arising from the definition of $\mathscr{T}_1$. If $f_\mu^{(k)}$ has an absolutely continuous invariant measure supported by 2^{n_0} intervals, then $f_{\mu\mu_1\ldots\mu_k}$ has the corresponding *a.c.i.m.* supported by 2^{n_0+k} intervals. Thus we obtain:

THEOREM 3. *For any neighborhood* $\mathscr{W}$ *of the universal family* Φ *where* $\mathscr{T}_1$ *acts as a contraction there exist constants* c_0, $\varepsilon_0 > 0$ *such that if* $F\colon \mu \to f_\mu$ *is in* $\mathscr{W}$, *then in any interval* $[0, \varepsilon] \ni \mu$, $\varepsilon < \varepsilon_0$, *the set of parameter values* μ *such that* f_μ *has an absolutely continuous invariant measures is not less than* $c_0 \cdot \varepsilon$.

Remark. A similar reasoning shows that a fraction of parameter values $\mu \in [0, \varepsilon]$ such that f_μ has an attractive periodic point (say a point of period 3) is also uniformly bounded away from zero. Thus in any neighborhood of Feigenbaum fixed point the maps with stable periodic behavior and those with stochastic behavior coexist both with positive probability.

§ 2. Proof of Theorem 2

2.1. The set $\mathscr{M}_F$ of μ corresponding to the stochastic behavior of f_μ is constructed in [5] as the intersection $\mathscr{M}_F = \bigcap_{n=0}^{\infty} \mathscr{M}_n$, where $\mathscr{M}_n$ is the set of parameter values allowed at the nth step of the inductive construction. The decay of mes $\mathscr{M}_n$ and mes $\mathscr{M}_0$ are defined by the preliminary (zero) step of induction, which was only sketched in [5]. Here we shall expose it with more details.

Let $J_0(\mu)$ be the interval invariant under $f_\mu^{2^n}$. After a change of variable depending on μ and a change of parameter, and using again f_μ instead of $f_\mu^{2^n}$, we reduce the problem to the investigation of a family of maps f_μ: $[-1, 1] \to [-1, 1]$, $\mu \in [-1, 1]$, satisfying the following conditions:

(i) $f_\mu(x) = f(\mu, x)$ is a C^2 map defined on $[-1, 1] \times [-1, 1]$;
(ii) $f_\mu(-1) = f_\mu(1) = -1$ for all μ;
(iii) $f_\mu(0) = \mu$;
(iv) $D_{xx} f(1, 0) = -s_0 < 0$;
(v) $D_x f(1, -1) = A_0 > 1$.

For μ sufficiently close to 1, f_μ has a fixed point $t_\mu \in (0, 1)$. Let us denote by t_μ^{-1} the preimage of t_μ and set $I_\mu = [t_\mu^{-1}, t_\mu]$. Consider the induced (the first return) map T_μ: $I_\mu \to I_\mu$. The map T_μ has an even number of monotone branches $T_{n,\mu}$, $n \in [1, N-1]$, and a central branch of parabolic type $T_{N,\mu}$, where $T_{n,\mu}$ is the composition of $n+1$ iterates of f_μ. When $\mu \to 1$, $N \to \infty$, and when $\mu = 1$ there is no parabolic branch and the number of monotone branches is infinite: The following proposition is a consequence of the implicit function theorem and straightforward calculations.

PROPOSITION 2. I. *There exists a sequence of parameter values* μ_N, $N \to \infty$, *such that for* $\mu = \mu_N$ *the central branch* $T_{N,\mu}$ *subdivides into two monotone* *branches and the new central branch* $T_{N+1,\mu}$ *is born. When* μ *varies from* μ_N *to* μ_{N+1}, $T_{N+1,\mu}(0)$ *varies from* t_μ *to* t_μ^{-1}. *There exist* $N_0 \in N$ $c_0 > 0$, $0 < \varepsilon_0 \ll 1$ *such that for* $N > N_0$

$$c_0 A_0^{-N}(1-\varepsilon_0) < \mu_N < c_0 A_0^{-N}(1+\varepsilon_0)$$

II. *There exist* $c_1, c_2 > 0$ *such that for any* n *the following estimates hold*

1. $c_1 A_0^n < |D_{xx} T_n| < c_2 A_0^n$;
2. $c_1 A_0^n x < |D_x T_x| < c_2 A_0^n x$;
3. $c_1 A_0^n < |D_\mu T_n| < c_2 A_0^n$;
4. $c_1 n A_0^n x < |D_{\mu x} T_n| < c_2 n A_0^n x$.

III. *Let* $x_n^1(\mu)$, $x_n^2(\mu)$ *be the endpoints of the domain of definition of* $T_n(\mu, x)$. *There exist constants* $c_3, c_4 > 0$ *such that:*

1. *For the central branch* $T_{N+1}(\mu, x)$

$$c_3 \sqrt{\mu_N - \mu} < |x_n^i(\mu)| < c_4 \sqrt{\mu_N - \mu}. \tag{1}$$

2. *For* $n = N$ *let* $d(x_n^1(\mu), 0) < d(x_n^2(\mu), 0)$; *then* $x_n^1(\mu)$ *satisfies* (1) *and* $x_n^2(\mu)$ *satisfies*

$$c_3 A_0^{-n/2} < |x_n^2(\mu)| < c_4 A_0^{-n/2}. \tag{2}$$

3. *If* $n \leqslant N-1$ *then both* $x_\mu^1(\mu)$, $x_\mu^2(\mu)$ *satisfy* (2).

It follows from Proposition 2 that for $n < N$, n sufficiently large, $|D_x T_n|$ is of order $A_0^{n/2}$. A priori branches with small indices may be nonexpanding. Yet, the negative Schwarzian derivative implies (see [7])

PROPOSITION 3. *For* $\mu = 1$ *there exist* $m_0 \in N$ *and* $\varrho_0 > 1$ *such that for any* $T_{i_1}, \ldots, T_{i_{m_0}}$

$$|D_x T_{i_1} \circ \ldots \circ T_{i_{m_0}}| > \varrho_0. \tag{3}$$

We fix some $R_0 \gg 1$. Let $d_0 = [\log_{\varrho_0} R_0] + 1$, $m_1 = m_0 d_0$. For $\mu = 1$ we have for any composition of m_1 maps T_{i_s}

$$|D_x T_{i_1} \circ \ldots \circ T_{i_{m_1}}| > R_0. \tag{4}$$

Let for $\mu = 1$

$$\min_{1 \leqslant s \leqslant m_1} \inf_{i_1 \ldots i_s} |D_x T_{i_1} \circ \ldots \circ T_{i_s}| = \alpha_0. \tag{5}$$

It follows from $|D_x T_n| > c_0 A^{n/2}$ that there exists k_0 such that if one of the maps forming the composition $T_{i_1} \circ \ldots \circ T_{i_s}$ has the index $i_l \geqslant k_0$ then

$$|D_x T_{i_1} \circ \ldots \circ T_{i_s}| > R_0. \tag{6}$$

This implies

COROLLARY 1. *There exists* $\varepsilon_1 > 0$ *such that if* $\mu \in [1-\varepsilon_1, 1]$ *and* $N_0 = N_0(\mu)$ *is the index of the central branch, then* (4) *holds for any* $1 \leqslant i_1, \ldots, i_{m_1} \leqslant N_0 - 2$. *If any* T_i *with* $k_0 \leqslant i \leqslant N_0 - 2$ *enters in the composition, then* (6) *holds.*

2.2. Construction on the x-axis

We suppose that $\mu \in [1-\varepsilon, 1]$, N_0 is large, and in particular $N_0 \gg k_0$.

Let divide the maps T_i into several types. Fix some $\varrho_0 \in N$ satisfying $k_0 \ll \varrho_0 \ll N_0$, and attribute to the first type T_i with $1 \leqslant i < k_0$, to the second type T_i with $k_0 \leqslant i < \varrho_0$, to the third type T_i with $\varrho_0 \leqslant i \leqslant N_0$.

Consider T_{i_1} of the first type. If $|D_x T_{i_1}| > R_0$ for all μ under consideration, then we do not change T_{i_1}. Otherwise we construct all compositions of the form $T_{i_2} \circ T_{i_1}$, where T_{i_2} are of the first or of the second type. If $|D_x T_{i_2} \circ T_{i_1}| > R_0$, then we do not change it further. Otherwise we construct $T_{i_3} \circ T_{i_2} \circ T_{i_1}$, where $1 \leqslant i < \varrho_0$, and so on. It follows from (4) that after having repeated this procedure no more than m_1 times we obtain for any $T_{i_s} \circ T_{i_{s-1}} \circ \ldots \circ T_{i_2} \circ T_{i_1}$, $i_l \in [1, \varrho_0 - 1]$, $s \in [1, m_1]$

$$|D_x T_{i_s} \circ \ldots \circ T_{i_1}| > R_0. \tag{7}$$

For any map f we shall use Δf to denote the domain of f. We denote by $\delta_0 = \delta_0(\mu)$ the union of ΔT_i with $\varrho_0 \leqslant i \leqslant N_0$. After the above construction we may represent I as a union

$$I = \delta_0 \cup \Big(\bigcup_{l=1}^{m_1} \delta_0^{-l} \Big) \cup \Big(\bigcup_{s=1}^{m_1} \Delta T_{i_s} \circ \ldots \circ T_{i_1} \Big) \tag{8}$$

where

$$\delta_0^{-l} = T_{i_1}^{-1} \circ \ldots \circ T_{i_l}^{-1} \delta_0 = \bigcup_{j=\varrho_0}^{N_0} \Delta T_j \circ T_{i_l} \circ \ldots \circ T_{i_1}$$

and every summand depends on μ.

It follows from the estimates of $|D_x T_i|$, $|D_{xx} T_i|$ in Proposition 2 and from the expanding property (4) that distortions of compositions $T_{i_s} \circ \ldots \circ T_{i_2} \circ T_{i_1}$, $1 \leqslant i_k < \varrho_0$ are uniformly bounded by a constant which does not depend on R_0, ϱ_0 and N_0 (see for example § 5 in [5]).

Let f_0 denote any of the maps $T_{i_s} \circ \ldots \circ T_{i_2} \circ T_{i_1}$ in (8). Then (8) becomes

$$I = \delta_0 \cup \big(\bigcup \delta_0^{-l} \big) \cup \big(\bigcup \Delta f_0 \big). \tag{9}$$

It follows from the bounded distortion that

$$\operatorname{mes} \big(\bigcup \Delta f_0 \big) > (1 - |\delta_0| \cdot c)^{m_1} \tag{10}$$

where m_1 is of the order $\log R_0$, c does not depend on R_0 or δ_0, and $\operatorname{mes} I = 1$.

2.3. On the parameter axis we deal independently on every interval $\mu \in [\mu_N, \mu_{N+1}]$, $N \geqslant N_0$. Let denote the central branch T_{N+1} by $h(\mu, x)$. When μ varies in $[\mu_N, \mu_{N+1}]$ $h(\mu, 0)$ runs along I.

Now we shall define some admissible domains $\Delta f_0(\mu)$, and we shall call μ *admissible* if $h(\mu, 0) \in \Delta f_0(\mu)$, where $\Delta f_0(\mu)$ is admissible.

According to the inductive construction of [5] $h(\mu, 0)$ is not allowed to fall into some enlarged domains $\hat{\delta}_0^{-l} \supset \delta_0^{-l}$, which we shall construct now.

Fix some α, $0 < \alpha < 1$. Without loss of generality we may consider $R_0 = A_0^z$, $z = N$, and suppose that $\alpha z \in N$. Now we assume that ϱ_0 is so large that

$$q_0 = \varrho_0 - 2\alpha z \gg k_0. \tag{11}$$

Consider the interval of the length $R_0^\alpha \cdot |\delta_0|$ homothetic to δ_0 with respect to 0. The endpoints of this interval may fall inside some domains Δf_0. In this case we enlarge the interval in such a way that it encloses such Δf_0, and we denote such an enlarged interval by $\hat{\delta}_0$. Since the construction may depend on μ, we do it for $\mu = \mu_N$ and define $\hat{\delta}_0(\mu)$ as the union of those $\Delta f_0(\mu)$ which are in $\hat{\delta}_0(\mu_N)$. Let estimate $|\hat{\delta}_0| - R_0^\alpha |\delta_0|$. According to Proposition 2 we have $|x_0^i| \sim A_0^{-\varrho_0/2}$ for the endpoints x_0^i of δ_0. Then the endpoints of R_0^α-homothetic image of δ_0 are in a distance of order $A_0^{-\varrho_0/2} \cdot R_0^\alpha = A_0^{-\varrho_0/2 + \alpha z}$ from zero. As $|\Delta T_i|$ decreases exponentially, $|\Delta T_i|$ is of the same order that the distance of ΔT_i from zero. It implies

COROLLARY 2. *There exist constants* $c_5, c_6 > 0$ *such that*

$$c_5 \cdot A_0^{-\varrho_0/2 + \alpha z} < |\hat{\delta}_0| < c_6 \cdot A_0^{-\varrho/2 + \alpha z}. \tag{12}$$

It follows from the definition of k_0 that any interval δ_0^{-l} from (9) is encircled by the domains of definition of $f_0 = T_{i_{l+1}} \circ T_{i_l} \circ \ldots \circ T_{i_1}$ with $k_0 \leqslant i_{l+1} < \varrho_0$. Because of (11) $\hat{\delta}_0^{-l} \backslash \delta_0^{-l}$ consists of such Δf_0.

2.4. Let estimate the measure of $\{\mu\colon h(\mu, 0) \in \bigcup \hat{\delta}_0^{-l}\}$. Let begin with $\hat{\delta}_0$. The endpoints $y_0^1(\mu)$, $y_0^2(\mu)$ of this interval move with the velocities

$$|v^i(\mu)| = |dy_0^i/d\mu| = \left| \left(\frac{D_\mu T_{q_0}^i}{D_x T_{q_0}^i} \right) ((\mu, y_0^i(\mu)) + \alpha^i(\mu) \right| \tag{13}$$

where y_0^i coincides with an endpoint of $\Delta T_{q_0}^i$ and $\alpha^i(\mu)$ is the velocity of that two points t_μ or t_μ^{-1} which coincides with $T_{q_0}^i(y_0^i)$, $i = 1, 2$. Using Proposition 2 we see that $|v^i(\mu)|$ is of order $A_0^{q_0/2}$ and the velocity of the top of the central branch is

$$|v(\mu)| = \left| \frac{d}{d\mu} h_\mu(0) \right| \sim A_0^{N_0} \gg A_0^{q_0/2} \sim |v^i(\mu)| \tag{14}$$

Suppose $h_\mu(0) = y_0^1(\mu)$ for $\mu = \mu_1$ and $h_\mu(0) = y_0^2(\mu)$ for $\mu = \mu_2 > \mu_1$. Set $u_1 = \max_\mu |v^2(\mu)|$, $v_1 = \max_\mu |v(\mu)|$, $v_2 = \min_\mu |v(\mu)|$, $\mu \in [\mu_N, \mu_{N+1}]$. Then for $\Delta\mu = \mu_2 - \mu_1$ we have

$$(15) \qquad \left(\frac{|\hat{\delta}_0(\mu_1)|}{v_1}\right)\left(1+\frac{u_1}{v_1}\right)^{-1} \leqslant \Delta\mu \leqslant \left(\frac{|\hat{\delta}_0(\mu_1)|}{v_2}\right)\left(1-\frac{u_1}{v_1}\right)^{-1}.$$

Using (14) we obtain

$$(16) \qquad c_7 \cdot |\hat{\delta}_0(\mu_1)| \cdot A_0^{-N_0} \cdot (1 + A_0^{q_0/2 - N_0})^{-1} \leqslant \Delta\mu \leqslant c_8 \cdot |\hat{\delta}_0(\mu_1)| \cdot A_0^{-N_0} \cdot (1 - A_0^{q_0/2 - N_0}).$$

Using the estimates of $|D_\mu F/D_x F|$ for the compositions (see § 9 of [5]), we see that the velocities of the endpoints of $\hat{\delta}_0^{-l}$, $l \in [1, m-1]$ are of the same order: $\text{const} \cdot A_0^{q_0/2}$. Thus the estimate of $\Delta\mu = \{\mu\colon h_\mu(0) \in \hat{\delta}_0^{-l}(\mu)\}$ is similar to (16) but with $|\hat{\delta}_0^{-l}(\mu_1)|$ instead of $|\hat{\delta}_0(\mu_1)|$.

In order to show that $\text{mes} \bigcup \hat{\delta}_0^{-l}$ is small it suffices to notice that $\{\Delta f\}$ include any interval ΔT_i, $k_0 \leqslant i \leqslant \varrho_0$, thus $\hat{\delta}_0$ is encircled by $\tilde{\delta}_0 = \bigcup_{j=k_0}^{N_0} \Delta T_j \circ T_{i_l} \circ \ldots \circ T_{i_1}$. Setting $\Gamma_0 = \tilde{\delta}_0 \backslash \hat{\delta}_0 = \bigcup \Delta T_i$, $k_0 \leqslant i < \varrho_0 - 2\alpha z = q_0$ we have

$$(17) \qquad c_9 \cdot A_0^{(1/2)(k_0 - q_0)} < \frac{|\hat{\delta}_0^{-l}(\mu)|}{|\Gamma_0^{-l}(\mu)|} < c_{10} \cdot A_0^{(1/2)(k_0 - q_0)}$$

for any $\mu \in [\mu_N, \mu_{N+1}]$, where $q_0 - k_0$ may be chosen arbitrarily large. For an interval $\tilde{\Delta}\mu = \{\mu\colon h_\mu(0) \in \tilde{\delta}_0^{-l}(\mu)\}$ we have the same estimate as (16) but with $\tilde{\delta}_0^{-l}(\mu_1)$ instead of $\hat{\delta}_0(\mu_1)$. Finally we obtain

PROPOSITION 4. *After the zero step of the inductive construction any interval* $[\mu_N, \mu_{N+1}]$, $N \geqslant N_0$, *may be represented as a disjoint union* $\mathscr{V}_0 \cup \mathscr{M}_0$, *where*

$$\mathscr{V}_0 = \{\mu\colon h_\mu(0) \in \bigcup_{l=0}^{m_1 - 1} \hat{\delta}_0^{-l}(\mu)\} \quad \textit{is non-admissible},$$

$$\mathscr{M}_0 = \{\mu\colon h_\mu(0) \in \bigcup \Delta f_0\} \quad \textit{is admissible}$$

and

$$(18) \qquad \text{mes}\, \mathscr{V}_0 / \text{mes}\, \mathscr{M}_0 < c \cdot A_0^{(1/2)(k_0 - q_0)}$$

and c *does not depend on* k_0, q_0.

2.5. If we choose $q_0 = \varrho_0 - 2\alpha z$ sufficiently large comparatively to k_0, then we obtain that $(\text{mes}\, \mathscr{M}_0) \cdot (|\mu_N - \mu_{N+1}|)^{-1}$ is arbitrarily close to 1.

The main theorem of [5] asserts that if some μ is admissible at any step of the inductive construction, then f_μ has an absolutely continuous invariant measure. The admissible set $\mathcal{M}_n \subset \mathcal{M}_{n-1}$ constructed at the nth step satisfies

$$\text{mes}\, \mathcal{M}_n > \text{mes}\, \mathcal{M}_{n-1}(1 - R_0^{-t_0 n}) \tag{19}$$

where $t_0 > 0$.

If we choose $R_0 = A_0^z$ sufficiently large, then choose $\varrho_0 \gg z$, $|\delta_0| \sim A_0^{-\varrho_0/2}$, and finally $N_0 \gg \varrho_0$, then we obtain

PROPOSITION 4. *For any $\varepsilon > 0$ there exists $N_0 \in \mathbf{Z}_+$ such that on any interval $[\mu_N, \mu_{N+1}]$, $N \geqslant N_0$, the relative measure of parameter values corresponding to f_μ with absolutely continuous invariant measure is larger than $1 - \varepsilon$.*

Notice that if some family $\{f'_\mu\}$ is sufficiently close to $\{f_\mu\}$ then we may take the constants R'_0, ϱ'_0, N'_0 which define the relative measure of stochastic behavior for f'_μ in $[\mu'_N, \mu'_{N+1}]$ equal respectively to R_0, ϱ, N_0.

Finally, taking into account that the union of $[\mu_N, \mu_{N+1}]$, $N \geqslant N_0$, constitute a neighborhood $(1-\delta, 1)$ of 1, we finish the proof of Theorem 2.

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