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On the perturbations in a time-optimal closed-loop system

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Abstract. From the solution of the time-optimal problem of an object described by: $\dot{x} = y$, $\dot{y} \in [f^-(y), f^+(y)]$, there results the following closed-loop system

$$\dot{x} = y, \quad \dot{y} = \begin{cases} f^+(y), & (x, y) \in T_+^* \cup R_+^*, \\ f^-(y), & (x, y) \in T_-^* \cup R_-^*, \end{cases}$$

where T_+^*, T_-^*, R_+^*, R_-^* have the standard meaning (see Fig. 1).

The paper deals with the influence of both f^+, f^- perturbations and the time-optimal switching curves T_+^*, T_-^* deformations upon the solutions of the above system. In particular, the assumptions concerning the perturbations and deformations are formulated under which for any initial state the closed-loop system has Carathéodory solution [or Filippov solution, respectively] reaching a target state in finite time.

Some suggestions as to practical applications have been made.

1. Introduction

Dynamics of a certain class of industrial devices can be described as:

$$(1) \qquad \dot{x} = y(t), \quad x(0) = x_0,$$

$$(2) \qquad \dot{y} \in F(y), \quad y(0) = y_0,$$

where $F(y) = [f^-(y), f^+(y)]$, $f^-: J \to R^1$, $f^+: J \to R^1$, $J = [a, b]$, $-\infty < a < 0 < b < \infty$.

We assume that the functions f^- and f^+ are piecewise continuous on J, i.e., they have at most the finite number of the first kind of discontinuities and that there exist $0 < m < M < \infty$ such that the following restrictions are fulfilled:

$$(3) \qquad \begin{aligned} &f^-: (a, b] \to (-M, -m); \quad && f^-(a) = 0, \\ &f^+: [a, b) \to (m, M); && f^+(b) = 0. \end{aligned}$$

Paper [6] shows that the model (1), (2), (3) represents some mechanisms at which the discontinuity of motion resistances and boundary of speed y and acceleration $\dot{y}$ appear.

As the functions f^- and f^+ can be discontinuous we will look for the solution of the inclusion (2) either in a Carathéodory class ($\mathscr{C}$-solution) or in Filippov one ($\mathscr{F}$-solution), defined in Section 2.

The problem of time-optimal control of the object whose motion is described by (1), (2) can be formulated as follows [3]: For any initial point $z_0 = (x_0, y_0)$, $y_0 \in J$ and any final point $z_1 = (x_1, 0)$ we seek a $\mathscr{C}$-solution $q^*(t; z_0) = (x^*(t, z_0), y^*(t, z_0))$, $t \in [0, t^*]$ of the system (1), (2) displacing this system from z_0 to z_1 in minimal time $t^* < \infty$.

Paper [6] reveals that if only conditions (3) are fulfilled then there exists a time-optimal switching curve $T^* = T_+^* \cup T_-^* \cup z_1$ and region R_+^* and R_-^* (see Fig. 1) which role is the same as in the case of an object described by the equation $\ddot{x} = f(\dot{x}) + ku$, $f \in C^1(R^1)$ ([1], §7.11).

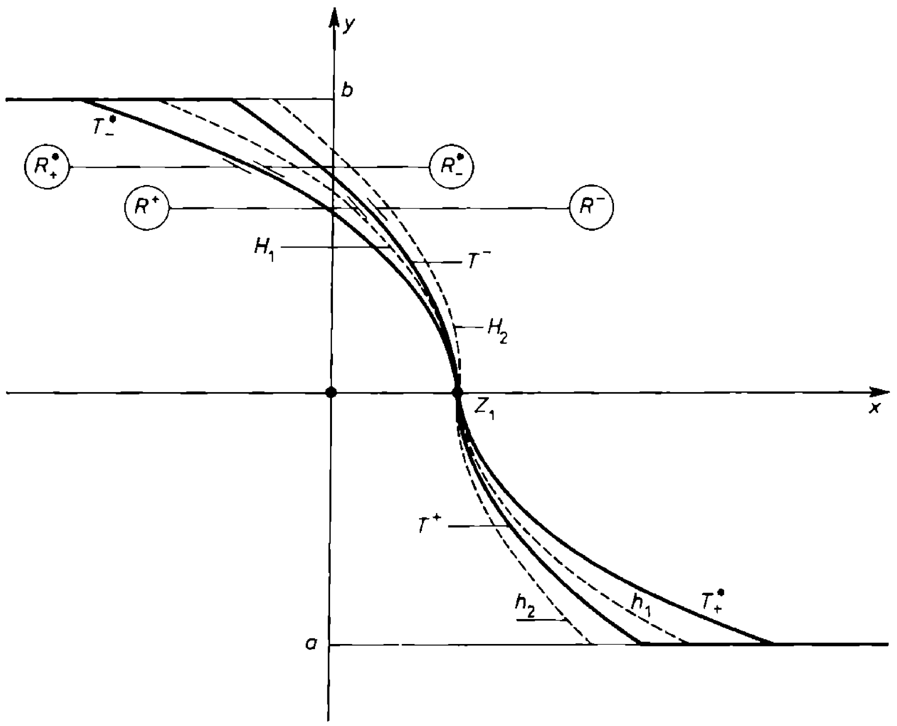

Fig. 1. Switching-curves

In the same paper [6], it has been also shown that for each z_0 there exists a time-optimal $\mathscr{C}$-solution $q^*(t; z_0)$, $t \in [0, t^*]$ of the system (1), (2), (3) which is the unique $\mathscr{C}$-solution of the closed system defined as:

$$(4) \qquad \dot{x} = y, \quad x(0) = x_0,$$

$$(5) \qquad \dot{y} = F(x, y) = \begin{cases} f^+(y), & (x, y) \in T_+^* \cup R_+^*, \quad y(0) = y_0, \\ f^-(y), & (x, y) \in T_-^* \cup R_-^*, \quad y(0) = y_0. \end{cases}$$

The practical identification of real motion resistances, i.e., determination of the functions f^+ and f^-, runs into difficulties and in some cases the values of those functions can be roughly estimated. Besides, a switching-curve created by a real time-optimal regulator may differ from the T^* considerably because of technical reasons.

Our main task is to analyse the influence of the perturbations of the functions f^+ and f^- and the deformation of the curve T^* upon the solutions of the system (4), (5). The problem has been solved for the final state z_1 which lies on the x-axis only. If z_1 lies out of the x-axis the solutions of the system (4), (5) can exhibit phenomena which are a subject of the next publication.

By the *perturbation of the functions* f^+ and f^- we will understand any pair of functions which are piecewise continuous on J and fulfil conditions (3). By the term of the *deformation of the switching curve* T^* we will understand any switching curve $T = T^+ \cup T^- \cup \{z_1\}$,

(6) $$T^+ = \{(x, y)\colon\ y \in [a, 0),\ x = \int_0^y h(s)\,ds + x_1\},$$

(7) $$T^- = \{(x, y)\colon\ y \in (0, b],\ x = \int_0^y H(s)\,ds + x_1\},$$

where the functions $h\colon [a, 0] \to (-\infty, 0]$, $H\colon [0, b] \to (-\infty, 0]$ are piecewise continuous and bounded.

The switching curve T, similarly to T^*, divides the state plane into two regions R^+ and R^- (Fig. 1):

$$R^+ = \{(x, y)\colon\ x < x',\ y \in [a, b],\ (x', y) \in T\},$$
$$R^- = \{(x, y)\colon\ x' < x,\ y \in [a, b],\ (x', y) \in T\}.$$

2. Solution of discontinuous differential equations

Assume given a differential equation

(8) $$\dot{x}(t) = g(x(t)), \qquad g\colon R^n \to R^n.$$

We define the operator

(9) $$Fg(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(Z) = 0} \overline{\mathrm{cvx}}\, g((x + \varepsilon B) \setminus Z),$$

where B is the open unit ball, $\mu(Z)$ is a Lebesgue measure of the set Z, $\overline{\mathrm{cvx}}\, M$ denotes closure of the convex hull of $M \subset R^n$.

DEFINITION 1. A function $x\colon I \to R^n$ (I is an interval in R^1 containing more than one point) is called a *Carathéodory solution* (or $\mathscr{C}$*-solution*) of (8) iff it is absolutely continuous (AC) on each compact subinterval of I and (8) holds almost everywhere (a.e.) in I. ∎

Definition 2. Let $x: J \to R^n$ (J an interval in R^1) be AC on each compact subinterval of J. Then x is called a *Filippov solution* (or *$\mathcal{F}$-solution*) of (8) iff

$$\dot{x}(t) \in Fg(x(t)) \quad \text{a.e. in } J. \ ∎$$

Theorem 1. *Let equation* (8) *satisfy the following assumption*: *there exists a disjoint decomposition*

$$R^n = \bigcup M_i, \quad \textit{with } M_i \subset \overline{\operatorname{Int} M_i},\ i = 1, 2, \ldots$$

and continuous g_i: $R^n \to R^n$ such that $g = g_i$ on M_i. Then each $\mathcal{C}$-solution is an $\mathcal{F}$-solution ($\mathcal{C} \subset \mathcal{F}$) ([5], Lemma 2.8). ∎

Theorem 2. *Let g: $G \to R^n$, G being an open region in R^n, be a measurable function such that $|g(x)| \leqslant A$ a.e. on G. Then for any initial state $x(0) = x_0 \in G$ there is an $\mathcal{F}$-solution of* (8) *defined at least on an interval $t \in [-d, d]$, $d > 0$, such that the n-dimentional ball $B(x_0, Ad)$ is contained in G* ([4], Theorem 4). ∎

Theorem 3. *Let S be a sector of a surface in R^n defined by an equation $\Phi(x) = 0$. Let Φ be C^1 in a neighbourhood of S and change its sign when passing through S. Denote by U^+ $[U^-]$ this part of the neighbourhood of S where $\Phi > 0$ $[\Phi < 0]$. If g: $G \to R^n$, $G \subset R^n$ is a measurable function and satisfies inequalities*

$$|g(x)| \leqslant A, \quad \textit{a.e. on } G,$$

$$(\operatorname{grad} \Phi(x) \cdot g(x)) \leqslant 0, \quad \textit{a.e. on } U^+ \ [U^-],$$

then none of the $\mathcal{F}$-solutions of (8), *for increasing t, can pass through S from $U^- \cup S$ to U^+ [or from U^- to $U^+ \cup S$, respectively]* ([4], Theorem 12). ∎

Definition 3. The system (8) has the *property of $\mathcal{C}$-permeability* [*$\mathcal{F}$-permeability*] on $G \subset R^n$ if:

(i) for each $x_0 \in G$ there exists a unique $\mathcal{C}$ [$\mathcal{F}$] solution of (8) determined on a certain interval $t \in [0, \varepsilon)$, $\varepsilon > 0$,

(ii) each $\mathcal{C}$ [$\mathcal{F}$] solution of (8) can be extended to the boundary of G, that is, for any $x_0 \in G$ there exists $0 < \bar{t} < \infty$ such that $\mathcal{C}$ [$\mathcal{F}$] solution $x(t; x_0) \in G$, $t \in [0, \bar{t})$ and $x(\bar{t}; x_0) \notin G$. ∎

When substituting T for T^* in the system (4), (5) we get:

$$(10) \qquad \begin{aligned} &\dot{x} = y, \quad x(0) = x_0, \\ &\dot{y} = F(x, y) = \begin{cases} f^+(y), & (x, y) \in T^+ \cup R^+, \quad y(0) = y_0, \\ f^-(y), & (x, y) \in T^- \cup R^-, \quad y(0) = y_0. \end{cases} \end{aligned}$$

From here on when talking about the solution of (10) we shall mean the solution for increasing t only. For the sake of convenience we write

$$B_\alpha^\beta = \{(x, y) \in R^1 \times [\alpha, \beta],\ a \leqslant \alpha \leqslant \beta \leqslant b\}.$$

LEMMA 1. *If the functions f^+ and f^- are piecewise continuous on $[a, b]$ and satisfy* (3), *then for any $z_0 \in B_a^b$ each of the following systems*

$$\text{(11)} \qquad \begin{aligned} \dot{x} &= y, \quad x(0) = x_0; \quad \dot{y} = f^+(y), \quad y(0) = y_0, \\ \dot{x} &= y, \quad x(0) = x_0; \quad \dot{y} = f^-(y), \quad y(0) = y_0 \end{aligned}$$

has the unique $\mathscr{C}$-solution defined on $[0, \infty)$.

Proof. Is given in [2], Theorems 4.1 and 5.3. ∎

LEMMA 2. *Assume the functions f^+ and f^- are piecewise continuous on $[a, b]$ and satisfy* (3), *the curves T^+, T^- are defined by* (6), (7), *respectively, and the functions h: $[a, 0] \to (-\infty, 0]$, H: $[0, b] \to (-\infty, 0]$ are piecewise continuous and bounded.*

THESIS 1. *If the following inequality is fulfilled*

$$\text{(12)} \qquad \frac{y}{f^+(y)} \leqslant h(y) \quad \text{on } [a, 0],$$

then the system (10) *has the property of $\mathscr{C}$-permeability on $R^+ \cup T^+$.*

THESIS 2. *If the following inequality is fulfilled*

$$\text{(13)} \qquad \frac{y}{f^-(y)} \leqslant H(y) \quad \text{on } [0, b],$$

then the system (10) *has the property of $\mathscr{C}$-permeability on $R^- \cup T^-$.*

Proof of Thesis 1. It results from Lemma 1 that for each $z_0 \in R^+ \cup T^+$ the system (11) has the unique $\mathscr{C}$-solution $q(t; z_0) = (x(t, z_0), y(t, z_0))$ on $[0, \infty)$.

We note that if the $\mathscr{C}$-solution of (11), $q(t; z_0) \in R^+ \cup T^+$ on interval $[0, \bar{t})$, $\bar{t} > 0$ then $q(t; z_0)$ is also the $\mathscr{C}$-solution of (10) on the same interval. Thus, to complete the proof it is sufficient to show that for each $z_0 \in R^+ \cup T^+$ there exists the finite time $\bar{t} > 0$ such that the $\mathscr{C}$-solution of (11), $q(t; z_0) \in R^+ \cup T^+$, $t \in [0, \bar{t})$ and $q(\bar{t}; z_0) \in T^- \cup \{z_1\}$.

First we consider the case $z_0 \in R^+ \cap B_0^b$. By (3) and (7) there exists the finite time $\bar{t} > 0$ such that for the $\mathscr{C}$-solution of (11) we have $q(t; z_0) \in R^+$, $t \in [0, \bar{t})$, $q(\bar{t}; z_0) \in T^-$.

Now, we are going to consider the case $z_0 \in (R^+ \cup T^+) \setminus B_0^b$. It follows from (3), (6), (12) that there exists the finite time $t' > 0$ such that $\mathscr{C}$-solution of (11), $q(t; z_0) \in (R^+ \cup T^+) \setminus B_0^b$ on $[0, t')$, $q(t'; z_0) = (x(t', z_0), 0)$ and $x(t', z_0) \leqslant x_1$.

If $x(t', z_0) = x_1$ then $\bar{t} = t'$, whereas if $x(t', z_0) < x_1$, then $q(t'; z_0) \in R^+ \cap B_0^b$. Thus, for the latter state we can repeat the analysis of the case $z_0 \in R^+ \cap B_0^b$.

Proof of Thesis 2. Similar to the proof of Thesis 1. ■

LEMMA 3. *Let the assumptions of Lemma* 2 *be fulfilled.*

THESIS 1. *If there exists* $\alpha_1 > 0$, $\beta_1 > 0$ *such that*

$$-\alpha_1 |y|^{\beta_1} \leqslant h(y) < y/f^+(y) \quad \text{on } [a, 0), \tag{14}$$

then the system (10) *has the property of* $\mathcal{F}$*-permeability on* T^+. *Moreover, for any* $z_0 \in T^+$ *the system* (10) *has no* $\mathcal{C}$*-solution.*

THESIS 2. *If there exists* $\alpha_2 > 0$, $\beta_2 > 0$ *such that*

$$-\alpha_2 |y|^{\beta_2} \leqslant H(y) < y/f^-(y) \quad \text{on } (0, b], \tag{15}$$

then the system (10) *has the property of* $\mathcal{F}$*-permeability on* T^-. *Moreover, for any* $z_0 \in T^-$ *the system* (10) *has no* $\mathcal{C}$*-solution.*

Proof of Thesis 1. Denote by $a < y_1 < y_2 < \ldots < y_l < 0$ the points at which the function h is discontinuous, $y_0 = a$, $y_{l+1} = 0$ and $T_j^+ = T^+ \cap \cap (R^1 \times [y_j, y_{j+1}))$, $j = 0, 1, \ldots, l$, a segment of the curve T^+. As $T^+ = \bigcup_{j=0} T_j^+$ and $z_1 \notin T_j^+$ for each $j = 0, 1, \ldots, l$ it is sufficient to prove Thesis 1 for each T_j^+, $j = 0, 1, \ldots, l$. In order to prove the above we define

$$h_j(y) = \begin{cases} \lim\limits_{s \to y_{j+1}-0} h(s), & y = y_{j+1}, \\ h(y), & y \in (y_j, y_{j+1}), \\ \lim\limits_{s \to y_j+0} h(s), & y = y_j, \end{cases}$$

$$\Phi_j(x, y) = \begin{cases} x + \int\limits_{y_{j+1}}^{0} h(s)\,ds - x_1 + (y_{j+1} - y)\, h_j(y_{j+1}), & (x, y) \in R^1 \times (y_{j+1}, \infty), \\ x + \int\limits_{y}^{0} h(s)\,ds - x_1, & (x, y) \in B_{y_j}^{y_{j+1}}, \\ x + \int\limits_{y_j}^{0} h(s)\,ds - x_1 + (y_j - y)\, h_j(y_j), & (x, y) \in R^1 \times (-\infty, y_j). \end{cases}$$

Note that the function $\Phi_j(x, y)$ defined in such a way is of $C^1(R^2)$ class, satisfies the inequalities

$$\Phi_j(x, y) < 0, \quad (x, y) \in R^+ \cap B_{y_j}^{y_{j+1}},$$

$$\Phi_j(x, y) > 0, \quad (x, y) \in R^- \cap B_{y_j}^{y_{j+1}},$$

and

$$T_j^+ \subset \{(x, y)\colon \Phi_j(x, y) = 0\}.$$

Denoting the right-hand side of (10) by $V(x, y) = (y, F(x, y))$, we have

(16) $$(\operatorname{grad} \Phi_j(x, y) \cdot V(x, y)) = \begin{cases} y - h_j(y) f^+(y), & (x, y) \in (T^+ \cup R^+) \cap B_{y_j}^{y_{j+1}}, \\ y - h_j(y) f^-(y), & (x, y) \in R^- \cap B_{y_j}^{y_{j+1}}. \end{cases}$$

It follows from (3) and Theorem 2 that for each $z_0 \in T_j^+$ the system (10) has $\mathscr{F}$-solution $q(t; z_0)$ on $[0, \infty)$. Combining (14), (16) and the assumption that $f^-(y) < 0$, we then have immediately that $(\operatorname{grad} \Phi_j(x, y) \cdot V(x, y)) < 0$, $(x, y) \in R^- \cap (R^1 \times [y_j, y_{j+1}))$. Thus, from Theorem 3, with $S = T_j^+$ and $\Phi = \Phi_j$, we receive that the $\mathscr{F}$-solution $q(t; z_0)$ cannot penetrate through T_j^+ into R^-.

Assuming $\Phi_j'(x, y) = -\Phi_j(x, y)$, we get from (14) and (16) the following

$$(\operatorname{grad} \Phi_j'(x, y) \cdot V(x, y)) = -f^+(y)\left(\frac{y}{f^+(y)} - h_j(y)\right) < 0,$$
$$(x, y) \in R^+ \cap (R^1 \times [y_j, y_{j+1})).$$

Again, from the above and Theorem 3, with $S = T_j^+$ and $\Phi = \Phi_j'$, it follows that the $\mathscr{F}$-solution cannot penetrate through T_j^+ into R^+. A consequence of the above statement is that for each $z_0 \in T_j^+$ the $\mathscr{F}$-solution $q(t; z_0)$ on a certain interval $[0, t')$ lies on the curve T_j^+.

We show that there exists $\bar{t} < \infty$ such that $q(t; z_0) \in T_j^+$, $t \in [0, \bar{t})$ and $q(\bar{t}; z_0) = z_{j+1} = (x_{j+1}, y_{j+1}) \notin T_j^+$. Write

(17) $$\bar{t} = \sup \{t'\colon q(t; z_0) \in T_j^+,\ t \in [0, t')\},$$

From the fact that the $\mathscr{F}$-solution $q(t; z_0)$, $t \in [0, \bar{t})$, lies on the curve T_j^+ it follows that on interval $[0, \bar{t})$, $q(t; z_0)$ is also the $\mathscr{C}$-solution of the following equation

$$\frac{dx}{dt} = y, \quad x(0) = x_0; \quad \frac{dy}{dt} = \frac{y}{h(y)}, \quad y(0) = y_0, \quad (x_0, y_0) \in T_j^+.$$

Owing to (14),

$$\frac{dy}{dt} \geqslant \frac{1}{\alpha_1} |y|^{1-\beta_1}, \quad y(0) = y_0, \quad t \in [0, \bar{t});$$

hence

$$\bar{t} \leqslant \frac{\alpha_1}{\beta_1} (|y_0|^{\beta_1} - |y_{j+1}|^{\beta_1}).$$

The above, together with (17), implies that $\bar{t} < \infty$ and $q(\bar{t}; z_0) \in T_j^+ \cup z_{j+1}$.

Suppose that $q(\bar{t}; z_0) \neq z_{j+1}$. Following the same argument for the new initial state $z_0' = q(\bar{t}; z_0)$ we come up with a contradiction with (17). Therefore, $q(\bar{t}; z_0) = z_{j+1}$.

We now prove that for any $z_0 \in T_j^+$ the system (10) has no $\mathscr{C}$-solution. Assume that there exists $\bar{z}_0 = (\bar{x}_0, \bar{y}_0) \in T_j^+$ such that the system (10) has the $\mathscr{C}$-solution $q(t; \bar{z}_0)$, $t \in [0, t')$, $t' > 0$. Theorem 1 reveals that $q(t; \bar{z}_0)$ is also the $\mathscr{F}$-solution on this interval.

Since the system (10) has the property of $\mathscr{F}$-permeability on T_j^+, we get that $q(t; \bar{z}) \in T_j^+$, $t \in [0, t')$. This would imply that there exists $\bar{y} \in (\bar{y}_0, y_{j+1})$ such that

$$y/f^+(y) = h(y), \qquad y \in [\bar{y}_0, \bar{y}],$$

contradicting (14). This complets the proof for T_j^+ and consequently complets the proof of Thesis 1.

Proof of Thesis 2. Similar to the proof of Thesis 1. ■

3. Problem statement. We assume the following estimations of the functions f^+ and f^- (see Fig. 2).

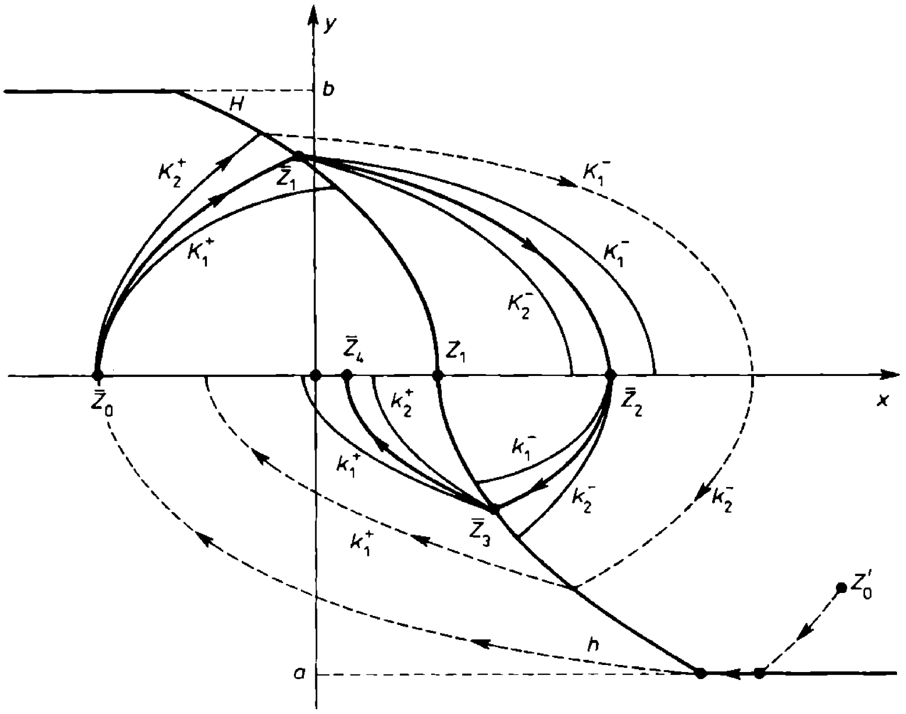

Fig. 2. Estimating functions

$$(18)\qquad m \leqslant K_1^+(y) = f^+(y) \leqslant K_2^+(y) \leqslant M \quad \text{on } [0, b),\ K_1^+(b) = f^+(b) = K_2^+(b) = 0,$$

$$(19)\qquad -M \leqslant -K_2^-(y) \leqslant f^-(y) \leqslant -K_1^-(y) \leqslant -m \quad \text{on } (0, b],$$

$$(20)\qquad -M \leqslant -k_2^-(y) \leqslant f^-(y) \leqslant -k_1^-(y) \leqslant -m \quad \text{on } (a, 0],\ k_1^-(a) = f^-(a) = k_2^-(a) = 0,$$

$$(21)\qquad m \leqslant k_1^+(y) \leqslant f^+(y) \leqslant k_2^+(y) \leqslant M \quad \text{on } [a, 0).$$

For the curve T executed by a real controller we assume the estimations

$$(22)\qquad -\alpha_1 |y|^{\beta_1} \leqslant \frac{y}{h_1(y)} \leqslant h(y) \leqslant \frac{y}{h_2(y)}, \quad \alpha_1 > 0,\ \beta_1 > 0, \text{ on } [a, 0].$$

$$(23)\qquad -\alpha_2 |y|^{\beta_2} \leqslant -\frac{y}{H_1(y)} \leqslant H(y) - \frac{y}{H_2(y)}, \quad \alpha_2 > 0,\ \beta_2 > 0, \text{ on } [0, b].$$

The functions k_i^+, k_i^-, K_i^+, K_i^-, h_i, H_i, $i = 1, 2$, appeared in (18)–(23) will be called *estimating functions*.

Notice that for $y \geqslant 0$ they are denoted in capital letters and those for $y \leqslant 0$ in small letters.

An object of this paper is to formulate assumptions about estimating functions for each of the following cases:

(i) the system (10) has the property of $\mathcal{C}$-permeability on $B_a^b \backslash z_1$, that is, for any z_0 the $\mathcal{C}$-solution of (10) reaches z_1 in finite time;

(ii) for any z_0 there exists a unique $\mathcal{C}$-solution $q(t; z_0)$, $t \in [0, \infty)$; moreover, there exists a ball $B(z_1, \delta)$ and a time $t = t_\delta < \infty$ such that:

$$q(t; z_0) \neq z_1, \quad t \in [0, \infty) \quad \text{and} \quad q(t; z_0) \notin B(z_1, \delta), \quad t \geqslant t_\delta;$$

(iii) the system (10) has the property of $\mathcal{F}$-permeability on $B_a^b \backslash \{z_1\}$, that is, for any z_0 the $\mathcal{F}$-solution of (10) reaches z_1 in finite time.

In what follows the following notations will be applied:

$$(24)\qquad p = \sup_{\substack{y \in (0,b] \\ z \in [a,0)}} \frac{\int_0^y s\big(1/K_1^-(s) - 1/H_2(s)\big)ds}{\int_0^y s\big(1/K_2^+(s) + 1/H_2(s)\big)ds} \cdot \frac{\int_0^z s\big(1/k_1^+(s) - 1/h_2(s)\big)ds}{\int_0^z s\big(1/k_2^-(s) + 1/h_2(s)\big)ds},$$

$$(25)\qquad r = \inf_{\substack{y \in (0,b] \\ z \in [a,0)}} \frac{\int_0^y s\big(1/K_2^-(s) - 1/H_1(s)\big)ds}{\int_0^y s\big(1/K_1^+(s) + 1/H_1(s)\big)ds} \cdot \frac{\int_0^z s\big(1/k_2^+(s) - 1/h_1(s)\big)ds}{\int_0^z s\big(1/k_1^-(s) + 1/h_1(s)\big)ds}.$$

4. Problem solution

THEOREM 4. *Let the following assumptions be fulfilled*:

(a) *Right-hand side of* (10) *satisfies inequalities* (18)–(21).

(b) *A switching-curve* T *is defined by* (6), (7) *and satisfies inequalities* (22), (23).

(c) *Functions appearing in* (18)–(23) *are piecewise continuous on their domains.*

(d)

$$h_1(y) \geqslant k_2^+(y) \quad on\ [a, 0], \tag{26}$$

$$H_1(y) \geqslant K_2^-(y) \quad on\ [0, b]. \tag{27}$$

THESIS 1. *If* $0 \leqslant p < 1$, *then the system* (10) *has got the property of* $\mathcal{C}$-*permeability on* $B_a^b \backslash z_1$ *and therefore, for each* $z_0 \in B_a^b \backslash z_1$, *there exists the time* $\bar{t} < \infty$ *such that the* $\mathcal{C}$-*solution* $q(t; z_0)$ *of the system* (10) *satisfies relation*:

$$q(t; z_0) \neq z_1, \quad t \in [0, \bar{t}), \quad q(\bar{t}; z_0) = z_1.$$

THESIS 2. *If* $r > 1$, *then for each* $z_0 \in B_a^b \backslash \{z_1\}$, *the system* (10) *has the unique* $\mathcal{C}$-*solution* $q(t; z_0)$, $z \in [0, \infty)$ *and there exists a ball* $B(z_1, \delta)$, $\delta > 0$ *and a time* $t_\delta < \infty$ *such that*

$$q(t; z_0) \neq z_1, \quad t \in [0, \infty) \quad and \quad \|q(t; z_0) - z_1\| \geqslant \delta, \quad t \geqslant t_\delta.$$

Proof. From (26), (22) together with (21), and from (27), (23) together with (19), it arises that

$$\frac{y}{f^+(y)} \leqslant h(y) \quad \text{on } [a, 0], \quad \frac{y}{f^-(y)} \leqslant H(y) \quad \text{on } [0, b]. \tag{28}$$

From these inequalities and Lemma 2 it follows that for each $z_0 \in B_a^b \backslash z_1$ there exists a unique $\mathcal{C}$-solution of (10), $q(t; z_0) \in B_a^b \backslash z_1$, $t \in [0, t')$, $0 < t' < \infty$.

From assumption (a) it results that there exists

$$\lim_{t \to t' - 0} q(t; z_0) = z' \in B_a^b.$$

Consequently, if only $z' \neq z_1$, then the state z' can be appeared as a new initial state for a continuation of the $\mathcal{C}$-solution $q(t; z_0)$.

Let us denote by $[0, \bar{t})$, $\bar{t} \leqslant \infty$ a maximal interval on which the $\mathcal{C}$-solution $q(t; z_0)$ exists.

It follows from the above considerations that, on $[0, \bar{t})$, $q(t; z_0) \neq z_1$ and if $\bar{t} < \infty$ then

(29) $$\lim_{t \to \bar{t} - 0} q(t; z_0) = z_1.$$

To finish the proof of Thesis 1 it is therefore sufficient to show that for any z_0 the maximal interval of existence of the $\mathscr{C}$-solution $q(t; z_0)$ is finite.

Let us suppose that there exists a state z_0' such that the $\mathscr{C}$-solution $q(t; z_0')$ is defined on $[0, \infty)$. We show that there exists a state $\bar{z}_0 = (\bar{x}_0, 0)$, $\bar{x}_0 < x_1$ such that the $\mathscr{C}$-solution $q(t; \bar{z}_0)$ of (10) is defined on $[0, \infty)$ and has the following properties:

(30)
$$\begin{aligned}
&q(t; \bar{z}_0) \in R^+ \setminus B_b^b \quad \text{on } [t_{4k}, t_{4k+1}),\\
&\qquad \text{where } q(t_{4k}; \bar{z}_0) = \bar{z}_{4k} = (\bar{x}_{4k}, 0),\ \bar{x}_{4k} < x_1,\\
&\qquad\qquad q(t_{4k+1}; \bar{z}_0) = \bar{z}_{4k+1} = (\bar{x}_{4k+1}, \bar{y}_{4k+1}) \in T^-;\\
&q(t; \bar{z}_0) \in T^- \cup R^- \quad \text{on } [t_{4k+1}, t_{4k+2}),\\
&\qquad \text{where } q(t_{4k+2}; \bar{z}_0) = \bar{z}_{4k+2} = (\bar{x}_{4k+2}, 0),\ \bar{x}_{4k+2} > x_1;\\
&q(t; \bar{z}_0) \in R^- \setminus B_a^a \quad \text{on } [t_{4k+2}, t_{4k+3}),\\
&\qquad \text{where } q(t_{4k+3}; \bar{z}_0) = \bar{z}_{4k+3} = (\bar{x}_{4k+3}, \bar{y}_{4k+3}) \in T^+;\\
&q(t; \bar{z}_0) \in T^+ \cup R^+ \quad \text{on } [t_{4k+3}, t_{4k+4}),\\
&\qquad \text{where } q(t_{4k+4}; \bar{z}_0) = \bar{z}_{4k+4} = (\bar{x}_{4k+4}, 0),\ \bar{x}_{4k+4} < x_1,\\
&\qquad\qquad\qquad\qquad k = 0, 1, 2, \ldots
\end{aligned}$$

The relations above mean that the trajectory of the $\mathscr{C}$-solution $q(t; \bar{z}_0)$, $t \in [t_{4k}, t_{4k+4}]$, revolves around the target z_1, crossing: the curve T^- at the point $\bar{z}_{4k+1}$, the x-axis at the point $\bar{z}_{4k+2}$, the curve T^+ at the point $\bar{z}_{4k+3}$ and once again the x-axis at the point $\bar{z}_{4k+4}$ (see Fig. 3).

The existence of $\bar{z}_0$ such that the $\mathscr{C}$-solution $q(t; \bar{z}_0)$ is defined on $[0, \infty)$ and satisfies (30) for $k = 0$ is immediate consequence of assumptions (a), (b), (c), (d) (see Fig. 3).

Let us assume that for a fixed k the times t_{4k+i} and the states $\bar{z}_{4k}$, $\bar{z}_{4k+i}$, $i = 1, 2, 3, 4$ fulfil properties (30).

Direct calculations reveal that

(31) $$\frac{x_1 - \bar{x}_{4k+4}}{x_1 - \bar{x}_{4k}} = \frac{\int_0^{\bar{y}_{4k+3}} (s/f^+(s) - h(s))\,ds}{\int_0^{\bar{y}_{4k+3}} (h(s) - s/f^-(s))\,ds} \cdot \frac{\int_0^{\bar{y}_{4k+1}} (H(s) - s/f^-(s))\,ds}{\int_0^{\bar{y}_{4k+1}} (s/f^+(s) - H(s))\,ds}.$$

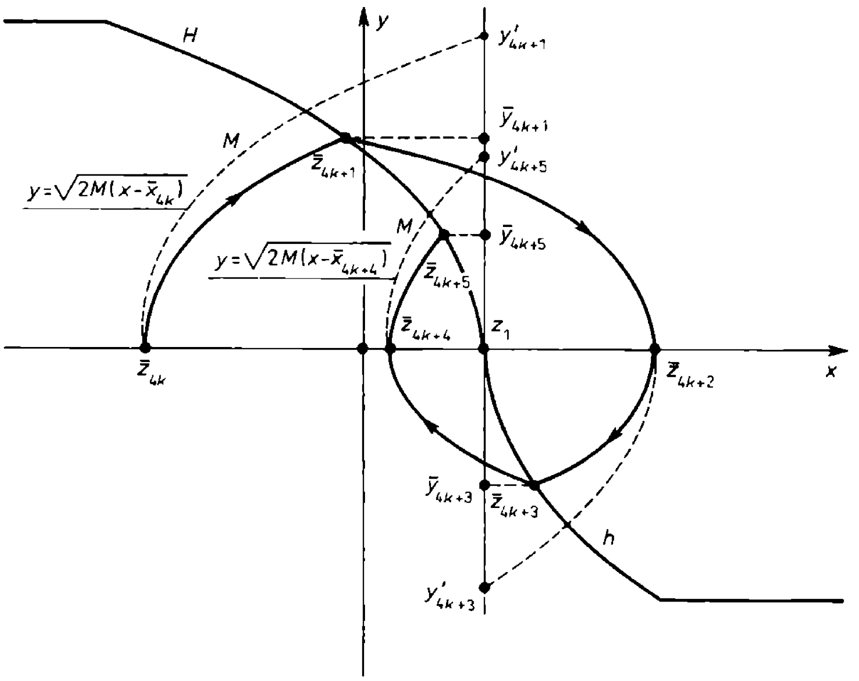

Fig. 3. Convergent trajectory

Having used assumptions (a), (b), (c), we get from (31) and (24)

(32)
$$\frac{|x_1-\bar{x}_{4k+4}|}{|x_1-\bar{x}_{4k}|} \leqslant \frac{\int_0^{\bar{y}_{4k+3}} \big(s/k_1^+(s)-s/h_2(s)\big)\,ds}{\int_0^{\bar{y}_{4k+3}} \big(s/k_2^-(s)+s/h_2(s)\big)\,ds}\cdot\frac{\int_0^{\bar{y}_{4k+1}} \big(s/K_1^-(s)-s/H_2(s)\big)\,ds}{\int_0^{\bar{y}_{4k+1}} \big(s/K_2^+(s)+s/H_2(s)\big)\,ds} \leqslant p.$$

From the above together with the assumption that $0 \leqslant p < 1$ we receive $\bar{x}_{4k} < \bar{x}_{4k+4}$. This inequality, together with the uniqueness of solutions, implies that the trajectory of the $\mathcal{C}$-solution $q(t;\bar{z}_0)$, $t \in [t_{4k+4}, \infty)$ lies totally in the set $R^1 \times (a, b)$. This means that we can define the times $t_{4(k+1)+i}$ and the states $\bar{z}_{4(k+1)+i}$, $i = 1, 2, 3, 4$ satisfying (30). Note that if only $\mathcal{C}$-solution of (10) is defined on $[0, \infty)$ then owing to assumptions (a), (b), (c), (d) its trajectory has to revolve around the target z_1 any number of times. Hence

(33)
$$\lim_{k\to\infty} t_{4k} \to \infty .$$

Reasoning similarly to what led us to (32), we get

(34)
$$0 \leqslant x_1-\bar{x}_{4k} \leqslant p^k(x_1-\bar{x}_0), \quad 0 \leqslant \bar{x}_{4k+2}-x_1 \leqslant p^k(\bar{x}_2-x_1),$$
$$k = 0, 1, 2, \ldots$$

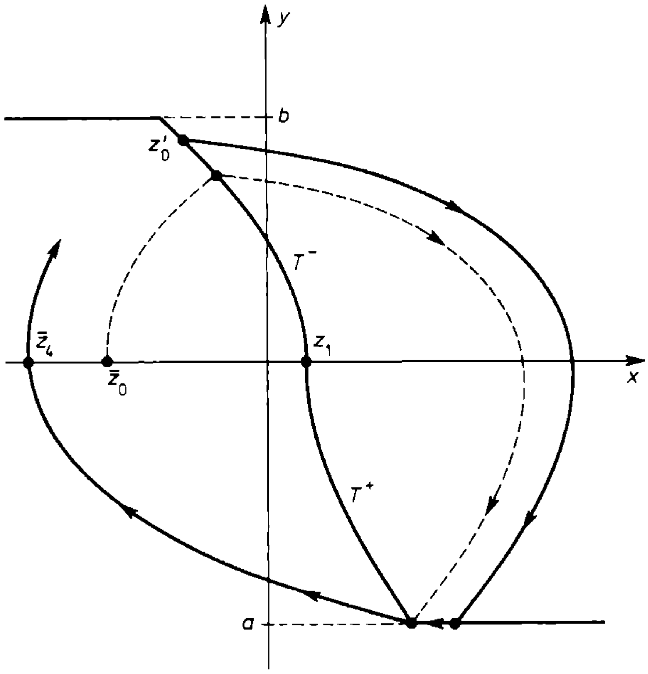

Fig. 4. Divergent trajectory

Taking the notations as those shown in Fig. 4 and using assumptions (a), (b) with inequalities (34), we have:

$$(35)\qquad \begin{aligned} \bar{y}_{4k+1} &\leqslant y'_{4k+1} = \sqrt{2M(x_1-\bar{x}_{4k})} \leqslant \sqrt{2M(x_1-\bar{x}_0)}\,(\sqrt{p})^k, \\ |\bar{y}_{4k+3}| &\leqslant |y'_{4k+3}| = \sqrt{2M(\bar{x}_{4k+2}-x_1)} \leqslant \sqrt{2M(\bar{x}_1-x_1)}\,(\sqrt{p})^k, \\ &\qquad k = 0, 1, 2, \ldots \end{aligned}$$

By assumption (a) from (10) we get the following estimation:

$$(t_{4k+1}-t_{4k})+(t_{4k+2}-t_{4k+1})+(t_{4k+3}-t_{4k+2})+(t_{4k+4}-t_{4k+3})$$

$$= (t_{4k+4}-t_{4k}) \leqslant \frac{\bar{y}_{4k+1}}{m}+\frac{\bar{y}_{4k+1}}{m}+\frac{|\bar{y}_{4k+3}|}{m}+\frac{|\bar{y}_{4k+3}|}{m}, \qquad k = 1, 2, \ldots$$

The above inequality together with (35) gives

$$\bar{t}_{4k} = \sum_{i=0}^{k-1}(t_{4i+4}-t_{4i}) \leqslant \frac{2}{m}\sum_{i=0}^{k-1}(\bar{y}_{4i+1}+|\bar{y}_{4i+3}|)$$

$$\leqslant \frac{2\sqrt{2M}}{m}\left[\sqrt{x_1-\bar{x}_0}+\sqrt{\bar{x}_2-x_1}\right]\sum_{i=0}^{k-1}(\sqrt{p})^i, \qquad k = 1, 2, \ldots$$

Since $p < 1$, therefore from the above we obtain that the sequence t_{4k}, $k = 0, 1, 2, \ldots$, is bounded, which contradicts (33).

Now, we prove Thesis 2. First it will be shown that for any z_0 the system (10) has a $\mathscr{C}$-solution defined on $[0, \infty)$.

Let us suppose there exists a state z_0' such that the maximal interval for which a $\mathscr{C}$-solution $q(t; z_0')$ exists, is finite. From the uniqueness of the $\mathscr{C}$-solution of (10) it arises that there exists also a state $\bar{z}_0 = (\bar{x}_0, 0)$, $\bar{x}_0 < x_1$ (see Fig. 4) such that the maximal interval $[0, \bar{t})$ of the $\mathscr{C}$-solution $q(t; \bar{z}_0)$ existence is finite and its trajectory does not reach the boundary of B_a^b. This trajectory either revolves around z_1 any number of times or it coincides with the curve T on a certain interval $[\bar{t}-\varepsilon, \bar{t})$, $\varepsilon > 0$. If the trajectory $q(t; \bar{z}_0)$, $t \in [0, \bar{t})$, revolves around z_1 any number of times then reasoning similarly to what led us to (30) we get

$$x_1 - \bar{x}_{4k} \geqslant r^k (x_1 - \bar{x}_0), \qquad k = 0, 1, 2, \ldots \tag{36}$$

From the above, together with the assumption that $r > 1$, we obtain: $\lim_{k \to \infty} |\bar{x}_{4k}| = \infty$, which contradicts (29). Therefore, the trajectory of the $\mathscr{C}$-solution $q(t; \bar{z}_0)$ cannot revolve around the state z_1 any number of times.

If the trajectory of the $\mathscr{C}$-solution $q(t; \bar{z}_0) = (x(t, \bar{z}_0), y(t, \bar{z}_0))$ of (10) coincides with T^- on a certain interval $[\bar{t}-\varepsilon, \bar{t})$, then from (10), (7), (29) we get

$$y/f^-(y) = H(y) \quad \text{ on } (0, y(\bar{t}-\varepsilon, z_0)].$$

The above and (19), (23), (27) imply that $K_2^-(s) = H_1(s)$, $s \in (0, y(\bar{t}-\varepsilon, z_0)]$. Thus, from (25) we get $r = 0$, which contradicts the assumption that $r > 1$. Therefore, the trajectory of the $\mathscr{C}$-solution $q(t; \bar{z}_0)$ cannot coincide with T^- on any interval $[\bar{t}-\varepsilon, \bar{t})$, $\varepsilon > 0$.

Similarly as above we prove that the trajectory $q(t; z_0)$ cannot coincide with T^+ on any interval $[\bar{t}-\varepsilon, \bar{t})$, $\varepsilon > 0$.

The considerations performed so far confirm that for any z_0 the maximal interval of the existence of the $\mathscr{C}$-solution $q(t; z_0)$ of (10) is $[0, \infty)$ and the trajectory of this solution revolves around z_1 any number of times.

In view of $r > 1$, for any z_0 there exists $t_0 < \infty$ such that the trajectory $q(t; z_0)$ reaches at this time the boundary of B_a^b and on $[t_0, \infty)$ forms a closed curve which is the boundary of a certain neighbourhood U of z_1. There exists therefore $\delta > 0$ such that the ball $B(z_1, \delta) \subset U$.

In order to complete the proof it is sufficient to put $t_\delta = t_0$. ■

THEOREM 5. *If assumptions* (a), (b), (c) *of Theorem* 4 *are fulfilled and also*

$$h_2(y) < k_1^+(y) \quad \text{on } [a, 0]; \qquad H_2(y) < K_1^- \quad \text{on } [0, b], \tag{37}$$

then the system (10) *has the property of the* $\mathscr{F}$*-permeability on* $B_a^b \backslash z_1$.

Proof. The property of the $\mathscr{F}$-permeability of the system (10) on $R^+ \cup R^-$ results from Theorem 1 and Lemma 2. Hence, to prove the theorem it is sufficient to show that the system (10) has the property of the $\mathscr{F}$-permeability on T^+ and on T^-.

From (21), (22), (37) we get:

$$-\alpha_1 |y|^{\beta_1} \leqslant h(y) < y/f^+(y) \quad \text{on } [a, 0).$$

This inequality, together with Thesis 1 of Lemma 3, implies that the system (10) has the property of the $\mathscr{F}$-permeability on T^+. Similarly, using (19), (23), (37) and Thesis 2 of Lemma 3 we can prove that the system (10) has the property of the $\mathscr{F}$-permeability on T^-. The proof is completed. ∎

5. Concluding remarks

1. If a real switching curve T is more steep than the time-optimal one, then for any initial state z_0 the system (10) has a $\mathscr{C}$-solution and the target state z_1 becomes a focus of its trajectory. Assumptions under which z_1 is a convergent or divergent focus were given in Theorem 4. In the case of a divergent focus, trajectories can be formed as limit-cycles.

2. If a curve T is less steep than T^*, then for any initial state z_0 not lying on T there exists a $\mathscr{C}$-solution of (10) that brings the system to T in the finite time. If an initial state z_0 lies on T, then the system (10) has no $\mathscr{C}$-solution, whereas it has the $\mathscr{F}$-solution which reaches the target z_1 in finite time. In this case the switching curve T becomes a sliding curve. The system oscillates around it with frequency and amplitude resulting from the time inherent in the switching operation.

3. In the case where the estimating functions are constant, i.e., $K_i^+(y) = K_i^+$, $K_i^-(y) = K_i^-$, $k_i^+(y) = k_i^+$, $k_i^-(y) = k_i^-$, $i = 1, 2$, and satisfy inequalities $K_2^+ < K_1^-$, $k_1^+ < k_2^-$ then from (24) we get

$$p = \frac{k_2^- K_2^+ (h_2 - k_1^+)(H_2 - K_1^-)}{k_1^+ K_1^- (h_2 + k_2^-)(H_2 + K_2^+)} < 1.$$

The above inequality and Theorem 4 imply that for each z_0 the system (10) has $\mathscr{C}$-solution which reaches the target z_1 in the finite time.

The case as shown above corresponds to the large class of industrial devices, the motion resistances of which are the passive ones (e.g. friction resistances), i.e., they cause an effect of "motion restrainting".

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