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BLOCK-CLOSED SUBGRAPHS AND q-PARTITIONS OF GRAPHS

Abstract. In this paper we introduce the concept of a block-closed subgraph of a graph and with help of this concept we continue the study of q-partitions of graphs which was initiated by Majcher and Płonka [2].

All graphs considered are finite undirected without loops and multiple edges.

Let G be a connected graph and let G_0 be its induced subgraph. If G_0 is connected and each block of G_0 is a block of G, then G_0 is called *block-closed*. Thus each block-closed subgraph of G is the union of some blocks of G which is connected. The empty graph K_0 (with the empty vertex set) is also considered as a block-closed subgraph of any graph.

THEOREM 1. *All block-closed subgraphs of a connected graph G form a lattice $LB(G)$ with respect to the ordering by inclusion.*

Proof. The intersection of two block-closed subgraphs G_1 and G_2 of G is evidently either empty or consists of one vertex (articulation point of G), or is a block-closed subgraph of G. If it consists of one vertex, then we define the meet $G_1 \wedge G_2$ as K_0. In other cases, the meet $G_1 \wedge G_2$ is equal to the intersection of G_1 and G_2. If the intersection of G_1 and G_2 is non-empty, then their union is connected, and thus it is a block-closed subgraph of G; in this case, we define the join $G_1 \vee G_2$ as this union. If the intersection of G_1 and G_2 is empty, then their union is disconnected. As G is connected, there exists a path connecting a vertex of G_1 with a vertex of G_2 and containing no edge of G_1 and no edge of G_2. For each of such paths, the set of blocks of G containing at least one edge of that path is the same; otherwise, there would exist a circuit containing edges of different blocks. If we add all these blocks to the union of G_1 and G_2, we obtain the least block-closed subgraph of G containing both G_1 and G_2 and this is $G_1 \vee G_2$. The greatest element of $LB(G)$ is G, the least element is K_0.

The lattice $LB(G)$ has properties analogous to those of the lattice of all subtrees of a tree [3].

THEOREM 2. *Each element of $LB(G)$ is a join of atoms.*

Proof. Atoms of $LB(G)$ are blocks of G. Each block-closed subgraph of G is the union of some blocks of G, which implies the assertion.

THEOREM 3. *If G contains at most one articulation point, then $LB(G)$ is a Boolean algebra. If G contains more than one articulation point, then $LB(G)$ is not modular.*

Proof. If G is an empty graph, then $LB(C)$ consists of one element. If G is non-empty without articulation points, then $LB(G)$ consists of two elements K_0 and G and is a Boolean algebra with one generator. If G contains exactly one articulation point, then the union of any blocks of G is a connected graph, and thus a block-closed subgraph of G. As all block-closed subgraphs of G are unions of blocks of G, there is a one-to-one correspondence between subsets of the set of blocks of G and block-closed subgraphs of G such that each block-closed subgraph of G is the union of all blocks of the corresponding set. This correspondence is evidently a lattice isomorphism of the Boolean algebra of all subsets of the set of blocks of G onto $LB(G)$, and hence $LB(G)$ is a Boolean algebra.

Now suppose that G has at least two articulation points. Then there exist blocks B_0, B_1, B_2 of G such that B_0 and B_1 have a common vertex a_1, the blocks B_0 and B_2 have a common vertex a_2, and B_1 and B_2 have no common vertex. Evidently, a_1 and a_2 are articulation points of G and $a_1 \neq a_2$. Let H_0 be the union of B_0 and B_1, and let H be the union of B_0, B_1, B_2. The graphs B_0, B_1, B_2, H_0, H belong to $LB(G)$. We have

$$B_1 \wedge B_2 = H_0 \wedge B_2 = K_0, \quad B_1 \vee B_2 = H_0 \vee B_2 = H,$$

and thus the elements K_0, B_1, B_2, H_0, H form a "forbidden pentagon" by the definition of modularity, and hence $LB(B)$ is not modular.

In the sequel we shall study trees. A graph K_1 consisting of one vertex and the empty graph K_0 are also considered as trees.

The tree of blocks [1] of a connected graph G is the tree whose vertex set is the union of the set of blocks of G and the set of articulation points of G and in which each vertex being a block of G is adjacent to all vertices being articulation points of G contained in this block.

Let $\mathscr{T}_0$ be the class consisting of all finite trees T with the property that the distance between any two terminal vertices of T is even. The graphs K_0 and K_1 also belong to $\mathscr{T}_0$.

THEOREM 4. *Let T be a finite tree. Then the following two assertions are equivalent:*

(i) *T is isomorphic to the tree of blocks of some graph G.*

(ii) $T \in \mathscr{T}_0$.

Proof. (i) $\Rightarrow$ (ii). Let T be isomorphic to the tree of blocks of some graph G; we may identify it with this tree. Then the vertex set of T is the union of two disjoint sets V_0 and V_1, where V_0 is the set of blocks of G, and V_1 is the set of articulation points of G. Two vertices can be adjacent only if one of them belongs to V_0 and the other to V_1. Hence the distance between any two vertices of V_0 is even. The terminal vertices of T are always blocks of G, and therefore they belong to V_0 and the distance between any two of them is even, i.e., $T \in \mathcal{T}_0$.

(ii) $\Rightarrow$ (i). Let $T \in \mathcal{T}_0$. Choose an arbitrary terminal vertex u of T. By V_0 (or V_1) denote the set of all vertices of T whose distance from u is even (or odd, respectively). As every tree is a bipartite graph, we have $V_0 \cap V_1 = \emptyset$. As $T \in \mathcal{T}_0$, these sets are independent of the choice of u. Now we add new edges to T. Two vertices of T are joined by a new edge if and only if they both belong to V_1 and their distance in T is 2. The graph thus obtained is denoted by G. Each block of G is a clique whose vertex set consists of one vertex of V_0 and all vertices of V_1 which are adjacent to it in T. Evidently, the tree of blocks of G is isomorphic to T.

Now let $T \in \mathcal{T}_0$. We define a $\mathcal{T}_0$-subtree of T as a subtree of T, all of whose terminal vertices have even distances from terminal vertices of T. Evidently, each $\mathcal{T}_0$-subtree of T belongs to $\mathcal{T}_0$. We admit also trees having only one vertex and the empty graph K_0.

THEOREM 5. *All $\mathcal{T}_0$-subtrees of a tree $T \in \mathcal{T}_0$ form a lattice $LT_0(T)$ with respect to ordering by inclusion.*

Proof. For any two $\mathcal{T}_0$-subtrees T_1 and T_2 of T we define the meet $T_1 \wedge T_2$ and the join $T_1 \vee T_2$. The intersection of T_1 and T_2 is either empty or is a subtree of T. If it is a subtree of T not belonging to $LT_0(T)$, then $T_1 \wedge T_2$ is obtained from it by deleting all terminal vertices whose distance from terminal vertices of T is odd; such a tree is evidently the greatest $\mathcal{T}_0$-subtree of T contained in both T_1 and T_2. In other cases, $T_1 \wedge T_2$ is the intersection of T_1 and T_2. If the intersection of T_1 and T_2 is non-empty, then the union of T_1 and T_2 is a tree; each terminal vertex of this tree is a terminal vertex of T_1 or of T_2, and thus this union is in $LT_0(T)$ and $T_1 \vee T_2$ is equal to it. If the intersection of T_1 and T_2 is empty, then there exists exactly one path in T connecting a vertex of T_1 with a vertex of T_2 and containing no edge of T_1 and no edge of T_2. By adding it to the union of T_1 and T_2 we obtain a tree with the property that each terminal vertex of it is a terminal vertex of T_1 or of T_2. Therefore, this tree belongs to $LT_0(T)$ and it is the smallest tree of $LT_0(T)$ which contains both T_1 and T_2; hence it is $T_1 \vee T_2$. The greatest element of $LT_0(T)$ is T, the least is K_0.

THEOREM 6. *Let G be a graph, and T be its tree of blocks. Then*

$$LB(G) \cong LT_0(T).$$

Proof. This is an easy consequence of Theorems 4 and 5.

COROLLARY 1. *If the diameter of T is at most 2, then $LT_0(T)$ is a Boolean algebra. If it is greater than 2, then $LT_0(T)$ is not modular.*

THEOREM 7. *Let the lattice $LB(G)$ of a graph G be given as an abstract lattice. Then the tree of blocks of G can be reconstructed uniquely up to isomorphism.*

Proof. Let $LB(G)$ be given. Let $B_1, \ldots, B_n$ be its atoms; they are blocks of G. If i, j, k are pairwise different numbers from the set $\{1, \ldots, n\}$ and $(B_i \vee B_j) \wedge B_k = B_k$, then each block-closed subgraph of G containing B_i and B_j contains also B_k, and therefore B_i and B_j have no common vertex. Hence B_i and B_j have a common vertex if and only if

$$(B_i \vee B_j) \wedge B_k = \emptyset \quad \text{for all } k \in \{1, \ldots, n\} - \{i, j\}.$$

Now we can construct the family $\mathscr{S}$ of maximal sets S of blocks of G with the property that any two blocks from S have a common vertex. From the properties of blocks it follows that for each $S \in \mathscr{S}$ there exists an articulation point of G which is common to all blocks of S. Conversely, to each articulation point of G a set from $\mathscr{S}$ can be assigned which consists of blocks containing this point. This correspondence is evidently one-to-one, therefore we may reconstruct the articulation points of G and determine, for each block, which articulation points it contains; this means that we are able to reconstruct the tree of blocks of G.

COROLLARY 2. *Let the lattice $LT_0(T)$ of a tree $T \in \mathscr{T}_0$ be given as an abstract lattice. Then T can be reconstructed uniquely up to isomorphism.*

Now we turn our attention back to q-partitions.

In [2], an induced subgraph Q of a connected graph G is called a *quasi-component* of G if the number of vertices of Q which are adjacent in G to a vertex not belonging to Q is at most one. If there is one such vertex, it is called the *control point* of Q and denoted by $c(Q)$. It is proved in [2] that every quasi-component of a graph G is either a one-vertex subgraph of G or a connected union of blocks of G. A partition P of the vertex set $V(G)$ of G is called a *q-partition* of G if each class of P induces a quasi-component of G.

It has been proved that all q-partitions of a graph G form a lattice. The ordering in this lattice is defined as follows. The symbol $[a]_P$ denotes the class of P which contains vertex a. We have $P_1 \preccurlyeq P_2$ if and only if $[a]_{P_1} \subseteq [a]_{P_2}$ for each vertex a of G.

In what follows we study the interconnections between block-closed subgraphs and q-partitions of G.

Let G be a graph and let H be its block-closed subgraph. Let $P(H)$ be the partition of $V(G)$ such that two vertices of G belong to the same class of $P(H)$ if and only if they are connected by a path in G which does not contain any edge of H.

THEOREM 8. *For each block-closed subgraph H of G the partition $P(H)$ is a q-partition of G.*

Proof. The subgraphs of G induced by the classes of $P(H)$ are connected components of the graph G' obtained from G by deleting all edges of H. Hence each of them is either a union of some blocks of G not belonging to H or a one-vertex graph whose vertex belongs to H. In the second case it is evidently a quasi-component of G. Let the first case occur and let Q be such a subgraph. Suppose that there exist two vertices a and b of Q which are incident with edges of G not belonging to Q, i.e., edges of H. Since each quasi-component of G is connected (see [2]), there exists a path P_1 in Q connecting a and b. As a and b are incident with edges of H, they belong to H. As H is connected, there exists a path P_2 in H connecting a and b. The union of P_1 and P_2 is a circuit containing edges of H and edges of Q. Thus the edges of P_1 belong to the same block of G as those of P_2. This block contains edges of H, but not all of them; hence the blocks of H containing these edges are not blocks of G and H is not block-closed, which is a contradiction. Hence Q is a quasi-component of G and $P(H)$ is a q-partition of G.

THEOREM 9. *Let P be a q-partition of G. Then there exists a block-closed subgraph H of G such that $P = P(H)$.*

Proof. Let H be the subgraph of G whose edge set is the set of all edges joining pairs of vertices from different classes of P and whose vertex set is the set of end vertices of these edges. Suppose that H is not connected. Then there exist vertices a and b in H which are connected by a path R in G, none of whose edges is in H. As each edge joining vertices of different classes of P lies in H, all edges of R lie in the subgraph of G induced by one class Q of P. But then a and b are two vertices of Q incident with edges not belonging to Q and Q is not a quasi-component of G, which is a contradiction. Hence H is connected. Suppose that there exists a block B of H which is not a block of G. Then B is a proper subgraph of a block B' of G. Let e_1 be an edge of B, and e_2 be an edge of B' not belonging to B. As both e_1 and e_2 belong to B', there exists a circuit C in B' containing both e_1 and e_2. The circuit C contains a path (containing e_2) which connects two vertices a and b of B and whose edges are not contained in H. Then we obtain the same contradiction as in the preceding case. Hence H is a block-closed subgraph of G. Evidently, $P = P(H)$.

THEOREM 10. *The lattice of q-partitions of a graph G is dually isomorphic to $LB(G)$.*

Proof. From Theorems 8 and 9 it is clear that there exists a one-to-one correspondence between the graphs H from $LB(G)$ and the q-partitions $P(H)$ of G. If H_1 and H_2 are in $LB(G)$ and H_1 is a subgraph of H_2, then each edge not belonging to H_2 does not belong to H_1 and thus $[a]_{P(H_2)} \subseteq [a]_{P(H_1)}$ for each vertex a of G. Hence $P(H_2) \preccurlyeq P(H_1)$. This implies the assertion.

COROLLARY 3. *Each element of the lattice of all q-partitions of G is a meet of dual atoms.*

COROLLARY 4. *If G contains at most one articulation point, then the lattice of all q-partitions of G is a Boolean algebra; otherwise, it is not modular.*

COROLLARY 5. *Let the lattice of all q-partitions of a graph G be given as an abstract lattice. Then the tree of blocks of G can be reconstructed uniquely up to isomorphism.*

References

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