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Szegö-type properties in a non-commutative case

by WACŁAW SZYMAŃSKI (Kraków)

Abstract. In the present paper we introduce two kinds of Szegö-type properties (S_1, S_2) for a von Neumann algebra and for a positive functional on a C^*-algebra. These properties generalize the notion of the Szegö measure introduced in [3] for function algebras to a non-abelian case. First we prove that a continuous functional on a C^*-algebra A which vanishes on its, not necessarily symmetric, subalgebra B is $B-(S_2)$. Next we construct a canonical decomposition of a von Neumann algebra on $K-(S_1)$ and Szegö-singular parts. Finally we obtain a theorem which describes a similar decomposition for a positive functional on a C^*-algebra.

I. Introduction. Let X be a compact Hausdorff space, and let $C(X)$ denote the C^*-algebra of all continuous, complex functions on X. If μ is a positive (regular, Borel) measure on X and $B \subset C(X)$ is a function algebra, then $L^2(\mu)$ stands for the Hilbert space of all square μ-integrable complex functions on X and $H^2(\mu, B)$ denotes the $L^2(\mu)$ closure of B. In [3] Foiaş and Suciu introduced the following definition: A positive measure μ on X is called a *Szegö measure with respect to B* if for every measurable set $E \subset X$

$$\chi_E L^2(\mu) \subset H^2(\mu, B) \quad \text{implies} \quad \mu(E) = 0.$$

Here χ_E denotes the characteristic function of E.

In what follows we will denote by $L^\infty(\mu)$ the algebra of all μ-essentially bounded complex functions on X.

The present paper deals with a generalization of that Szegö-type property to a non-commutative case. Looking for this generalization, we find the following theorem due to Wiener [4]. Every (linear, bounded) operator T in the Hilbert space $L^2(\mu)$, which commutes with all operators of the form $T_f u = fu$ for some $f \in L^\infty(\mu)$ and for all $u \in L^2(\mu)$ is itself of the form T_{f_0} with some $f_0 \in L^\infty(\mu)$. If T is an (orthogonal) projection, then f_0 is a characteristic function of a measurable set.

Let P be the projection from $L^2(\mu)$ onto $H^2(\mu, B)$. Now we can read the above Szegö property as follows:

(A) *μ is a Szegö measure with respect to B if and only if every projection Q which commutes with all operators T_f, $f \in L^\infty(\mu)$, and such that $Q \leqslant P$ is the zero operator.*

This formulation will be a starting point for more general situations.

II. Szegö-type properties. Let H be a complex Hilbert space. By a subspace of H we always mean a closed subspace, all projections are orthogonal and von Neumann algebras have the identity. If $\mathscr{S}$ is a subset of $L(H)$, then $\mathscr{S}'$ denotes its commutant. If Z is a subset of H, we write $[\mathscr{S}Z]$ for the subspace of H spanned by vectors Sx, where $S \in \mathscr{S}$, $x \in Z$. I_H stands for the identity operator in $L(H)$.

Let K be an arbitrary subspace of H and let P be the projection H on K.

DEFINITION 1. A von Neumann algebra $\mathfrak{A}$ in $L(H)$ *has the property* $K-(S_1)$ if for every projection $Q \in \mathfrak{A}'$ such that $Q \leqslant P$ we have $Q = 0$.

Equivalently: $\mathfrak{A}$ has the property $K-(S_1)$ if no non-zero subspace of K reduces $\mathfrak{A}$.

DEFINITION 2. A von Neumann algebra $\mathfrak{A}$ in $L(H)$ *has the proprety* $K-(S_2)$ if for every projection $Q \in \mathfrak{A} \cap \mathfrak{A}'$ such that $Q \leqslant P$ we have $Q' = 0$. For brevity, we will write $\mathfrak{A}$ is $K-(S_1)$ or (S_2) instead of the formulations: $\mathfrak{A}$ has the property $K-(S_1)$ or (S_2), respectively. The abbreviation $\mathfrak{A}$ is $K-(S_i)$ ($i = 1$ or 2) derives from the following phrase: $\mathfrak{A}$ has the property Szegö 1 or 2 with respect to K.

The above two definitions immediately imply what follows:

(1) *If $\mathfrak{A}$ is $K-(S_1)$, then $\mathfrak{A}$ is $K-(S_2)$.*

(2) *$\mathfrak{A}$ is $K-(S_2)$ if and only if $\mathfrak{A}'$ is $K-(S_2)$.*

The converse of (1) is not true. Consider the following example:

EXAMPLE. Let $\mathfrak{A} \subsetneqq L(H)$ be a factor. Then $\mathfrak{A}$ is $K-(S_2)$ with an arbitrary subspace $K \neq H$. But $\mathfrak{A}$ and $\mathfrak{A}'$ generate $L(H)$ as a von Neumann algebra ([1], p. 3). Hence, there is a projection P which belongs to $\mathfrak{A}'$ and does not belong to $\mathfrak{A}$. Then $\mathfrak{A}$ is $PH-(S_2)$, but $\mathfrak{A}$ is not $PH-(S_1)$.

Let $\mathscr{S} \subset L(H)$ be an arbitrary subset and let $E \in L(H)$ be a projection. We will denote by $\mathscr{S}_E$ the *compression* of $\mathscr{S}$ to E given by the formula

$$\mathscr{S}_E = \{ES|_{EH},\ S \in \mathscr{S}\} \subset L(EH) \quad \text{(see [6])}.$$

Here we have a list of more or less simple facts concerning two kinds of Szegö properties:

(3) *Let $K \subset K_1 \subset H$ be two subspaces. If $\mathfrak{A}$ is $K_1-(S_1)$ (resp. $K_1-(S_2)$), then $\mathfrak{A}$ is $K-(S_1)$ (resp. $K-(S_2)$).*

(4) *For a projection $P \in \mathfrak{A}'$ we have: $\mathfrak{A}$ is $PH-(S_1) \Leftrightarrow P = 0$.*

(5) *Let E be a projection, which belongs to $\mathfrak{A}'$ and let K be a subspace of EH. Then $\mathfrak{A}_E$ is $K-(S_1)$ if and only if $\mathfrak{A}$ is $K-(S_1)$.*

(6) *Under the assumptions of* (5): *if* $\mathfrak{A}_E$ *is* $K-(S_2)$, *then* $\mathfrak{A}$ *is* $K-(S_2)$. *The converse is false.*

A few words are needed about (5) and (6). Let E and K be as in (5). If $Q \in \mathfrak{A}'$ ($Q \in \mathfrak{A} \cap \mathfrak{A}'$ resp.) and $Q \leqslant E$, then the projection $\tilde{Q} = Q_{EH} | \in L(EH)$ belongs to $(\mathfrak{A}_E)'$ ($\mathfrak{A}_E \cap (\mathfrak{A}_E)'$ resp.) (see [6]), which proves (6) and half of (5). Now, if $\tilde{Q} \in (\mathfrak{A}_E)'$ is a projection, then

$$Q = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}$$

is also a projection and $Q \in \mathfrak{A}'$, which proves (5).

It can happen that under the assumptions of (5) $\mathfrak{A}$ is $K-(S_2)$ but $\mathfrak{A}_E$ is not. To see that consider a factor $\mathfrak{A} \subset L(H)$ and a projection $E \in \mathfrak{A}'$, $E \neq 0$, $E \neq I_H$. As we saw above, $\mathfrak{A}$ is $EH-(S_2)$, but E belongs to the center of $\mathfrak{A}_E$ and $E \neq 0$; hence $\mathfrak{A}_E$ is not $EH-(S_2)$.

III. Szegö functionals. Suppose we are given a C^*-algebra A with unit and a positive (linear) functional f on A. It is well known (see [1], p. 32) that there is a unique (up to a unitary isomorphism) triple (H_f, π_f, ξ_f) with the following properties:

a. H_f is a Hilbert space,

b. π_f: $A \to L(H_f)$ is a cyclic *-representation of A with the cyclic vector ξ_f, i.e., $H_f = [\pi_f(A)\xi_f]$,

c. for every $a \in A$, $f(a) = \big(\pi_f(a)\xi_f, \xi_f\big)$.

Let B be a closed (not necessarily symmetric) subalgebra of A. We define the subspace $K_f(B) = [\pi_f(B)\xi_f]$.

DEFINITION 3. A positive functional f on a C^*-algebra A with unit is called $B-(S_1)$ (resp. $B-(S_2)$) if the weak operator closure of $\pi_f(A)$ in $L(H_f)$ is $K_f(B)-(S_1)$ (resp. $K_f(B)-(S_2)$).

A simple example is the following one: Consider an arbitrary pure state f on a C^*-algebra A. The *-representation π_f corresponding to it is irreducible. Hence, either $K_f(B) = H$ or f is $B-(S_1)$, with an arbitrary closed subalgebra B of A.

If A is a commutative C^*-algebra with unit, then we can identify it (by the Gelfand–Naimark theorem) with $C(X)$, where X is a suitable compact Hausdorff space. By the Riesz–Kakutani theorem for a positive functional f on A there is a unique positive measure μ on X such that for each $u \in C(X)$ $f(u) = \int u d\mu$. In this case we have the following triple (H_f, π_f, ξ_f) for f: $H_f = L^2(\mu)$, $\pi_f(u)v = uv$ for $u \in A$, $v \in L^2(\mu)$ and $\xi_f = 1-$ the constant function. Now we see that there is no difference between the above two kinds of Szegö properties in the commutative case. The Wiener theorem quoted in the introduction says namely that if $\mathfrak{A}$ is the algebra of all operators T_u with $u \in L^\infty(\mu)$, then $\mathfrak{A} = \mathfrak{A}'$.

It has been proved by Foiaş and Suciu in [3] that if B is a function algebra in $C(X)$ and μ is a complex regular, Borel measure on X which is orthogonal to B, then μ is a Szegö measure with respect to B. Our next purpose is to prove a similar theorem in a non-commutative situation. First we recall some theorems concerning the von Neumann envelope of a C^*-algebra. In what follows we refer to [1] and [2] for details and basic definitions.

If A is a C^*-algebra (with unit), we denote by Q the set of all positive functionals on A. To every element f of Q there corresponds a triple (H_f, π_f, ξ_f). The *-representation $\pi = \bigoplus_{f\in Q} \pi_f$ of A in $L(H_\pi)$, where $H_\pi = \bigoplus_{f\in Q} H_f$, is called the *universal *-representation of A*. This representation is isometric, i.e., $\|\pi(a)\| = \|a\|$ for $a\in A$. The weak operator closure $\tilde{A}$ of $\pi(A)$ in $L(H_\pi)$ is called the *von Neumann envelope of A*. It is interesting for us to see how functionals on $\tilde{A}$ depend on functionals on A. This dependence is the following one: For every continuous functional f on A there is a unique ultraweakly continuous functional $\tilde{f}$ on $\tilde{A}$ such that $\tilde{f}(\pi(a)) = f(a)$ for all $a\in A$. If f is positive, then $\tilde{f}$ is positive and normal. Moreover, every positive normal functional p on $\tilde{A}$ is a vector state, i.e., there is a $\xi\in H_\pi$ such that for every $T\in\tilde{A}$: $p(T) = (T\xi, \xi)$. It follows also that every ultraweakly continuous functional on A is weakly continuous.

Let $\mathfrak{A} \subset L(H)$ be a von Neumann algebra and let p be a positive normal functional on $\mathfrak{A}$. Then there exists a largest projection $F\in\mathfrak{A}$ such that $p(F) = 0$. The projection $E = I_H - F$ is called the *support of p*. Moreover, for every $T\in\mathfrak{A}$

$$p(ET) = p(TE) = p(ETE) = p(T).$$

If f is a functional on $\mathfrak{A}$ and $T\in\mathfrak{A}$, then we write $T.f$ for the functional on $\mathfrak{A}$ of the form: $(T.f)(S) = f(TS)$ for $S\in\mathfrak{A}$.

The following theorem is called the *polar decomposition of a functional.*

(B) *Let $\mathfrak{A} \subset L(H)$ be a von Neumann algebra and let f be an ultraweakly continuous functional on $\mathfrak{A}$. Then there is a positive, normal functional p on $\mathfrak{A}$ and a partial isometry $U\in\mathfrak{A}$ such that*

1° $f = U.p$,

2° $p = U^*.f$

3° $\|p\| = \|f\|$ *and the support of p is exactly the final projection UU^*.*

Moreover, if p' is a normal positive functional on $\mathfrak{A}$ and $U'\in\mathfrak{A}$ is a partial isometry such that $f = U'\cdot p'$, $p' = U'^*.f$ and the projection $U'U'^*$ is majorized by the support of p', then $p = p'$ and $U = U'$.

Let A be a C^*-algebra, let f be a continuous functional on A and let $\tilde{f}$ be the unique ultraweakly continuous functional on $\tilde{A}$ which corresponds to f. Let $\tilde{f} = U.p$ be the polar decomposition of $\tilde{f}$. We define $|f|(a) = p(\pi(a))$ for $a \in A$. $|f|$ is a positive functional on A. Now we are able to formulate the following theorem:

THEOREM 1. *Let A be a C^*-algebra with the unit e and B its closed subalgebra containing e. If f is a continuous functional on A which vanishes on B, then $|f|$ is $B-(S_2)$.*

Before proving this theorem we prove a simple lemma, which establishes the correspondence between functionals on A which have properties $B-(S_i)$ $(i = 1, 2)$ and suitable functionals on $\tilde{A}$.

LEMMA 1. *Let $B \subset A$ be as in Theorem 1. Let π be the universal representation of A. Then for every functional f positive on A we have: f is $B-(S_i)$ if and only if $\tilde{f}$ is $\pi(B)-(S_i)$, $i = 1, 2$.*

Proof. Let (H_f, π_f, ξ_f) be the triple corresponding to f. We can consider H_f as a subspace of H and π_f as a subrepresentation of π given by the formula $\pi_f(a) = \pi(a)|_{H_f}$ for all $a \in A$. $\tilde{f}$ is a vector state, i.e., $\tilde{f}(T) = (T\xi, \xi)$ for some $\xi \in H$ and for all $T \in \tilde{A}$. It follows from the proof of 12.1.3 in [1] that we can put $\xi = \xi_f$. Moreover, $H_f = [\pi_f(A)\xi_f] = [\pi(A)\xi_f] = [\tilde{A}\xi_f]$, because the weak and strong operator closures of a^*-operator algebra with identity are the same. If we define $\varrho(T) = T|_{H_f}$ for each $T \in \tilde{A}$, then the triple (H_f, ϱ, ξ_f) corresponds to $\tilde{f}$. It is clear that $K_f(B) = K_{\tilde{f}}(\pi(B))$. Because commutants of $\tilde{A}|_{H_f}$ and $\pi_f(A)$ in $L(H_f)$ are equal, the proof is finished.

Proof of Theorem 1. We preserve the notation of the theorem. The polar decomposition $\tilde{f} = U.p$ and the definition of $|f|$ imply that $\widetilde{|f|} = p$. As we observed in Lemma 1, it sufficies to prove that p is $\pi(B)-$ $-(S_2)$. Let p be of the form $p(T) = (T\xi, \xi)$ with some $\xi \in H$. If we denote $H_p = [\tilde{A}\xi]$, $\pi_p(T) = T|_{H_p}$, $T \in \tilde{A}$, then the triple (H_p, π_p, ξ) corresponds to p. Denote by E the projection H_π on H_p. It is clear that $E \in \tilde{A}'$. From the polar decomposition theorem and the assumption $f|_B = 0$ it follows for $b \in B$ that

$$0 = f(b) = \tilde{f}(\pi(b)) = U.p(\pi(b)) = p(U\pi(b)) = (U\pi(b)\xi, \xi).$$

By the continuity we obtain $(Ux, \xi) = 0$ for all $x \in K = [\pi(B)\xi]$. Take a projection $Q \in \tilde{A}_E \cap (\tilde{A}_E)'$ such that $QH_p \subset K$. We must show that $Q = 0$. To prove this we define a positive functional q on A_E by the formula $q(T|_{H_p}) = p(T) = (T\xi, \xi) = (T|_{H_p}\xi, \xi)$. It is well defined because $\xi \in H_p$. Moreover, the operator $U_0 = U|_{H_p}$ is a partial isometry in H_p, because $U \in \tilde{A}$ and $E \in \tilde{A}'$. From the definition of q follows

$$q(T|_{H_p}U_0U_0^*) = p(TUU^*) = p(T) = q(T|_{H_p}) \quad \text{for all } T \in \tilde{A},$$

because UU^* is the support of p. Finally we get

$$q(Q) = q(U_0U_0^*Q) = (U_0U_0^*Q\xi, \xi) = (U_0QU_0^*\xi, \xi) = (UQU_0^*\xi, \xi) = 0,$$

because $U_0 \in \tilde{A}_E$, Q is central in $\tilde{A}_E$, $QH_p \subset K$ and $(Ux, \xi) = 0$ for $x \in K$. But $q(Q) = (Q\xi, \xi) = \|Q\xi\|^2$, and hence $Q\xi = 0$.

The vector ξ, cyclic for $\tilde{A}_E$, separates $(\tilde{A}_E)'$; thus $Q = 0$ and the proof is complete.

PROBLEM. Is Theorem 1 true if we put (S_1) instead of (S_2)?

IV. Szegö singularity. Let $\mathfrak{A}$ be a von Neumann algebra in $L(H)$ and let $K \subset H$ be a subspace.

DEFINITION 4. A von Neumann algebra $\mathfrak{A} \subset L(H)$ is called *K-Szegö singular* if every projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}_E$ is $K \cap EH - (S_1)$ is the zero operator.

The following theorem gives a characterization of Szegö-singular algebras:

THEOREM 2. *The following assertions are equivalent:*

1° *$\mathfrak{A}$ is K-Szegö singular,*

2° *Every projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}_E$ is $\overline{EK} - (S_1)$ is the zero operator,*

3° *Every projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}_E$ is $\overline{EK} - (S_2)$ is the zero operator.*

4° *Every projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}_E$ is $K \cap EH - (S_2)$ is the zero operator.*

5° $K = H$.

Proof. We will prove this theorem by following the schema: $3° \Rightarrow 2° \Rightarrow 5° \Rightarrow 3°$, $4° \Rightarrow 1° \Rightarrow 5° \Rightarrow 4°$. Observe first that implications $3° \Rightarrow 2°$ and $4° \Rightarrow 1°$ are trivial consequences of remarks (1), (2). Now we prove $2° \Rightarrow 5°$. Assume $K \neq H$. Then there is a vector $x \in H \ominus K$, $x \neq 0$. The subspace $H_0 = [\mathfrak{A}x]$ is non-zero, because $I_H \in \mathfrak{A}$, and H_0 reduces $\mathfrak{A}$. Denote by E_0 the projection H on H_0. E_0 belongs to $\mathfrak{A}'$. We want to show that $\mathfrak{A}_{E_0}$ is $\overline{E_0 K} - (S_1)$, or equivalently, $\mathfrak{A}$ is $\overline{E_0 K} - (S_1)$, by (5). Take a projection $Q \in \mathfrak{A}'$ such that $QH \subset \overline{E_0 K}$. Then there is a sequence y_n of vectors in K such that $Qx = \lim_{n\to\infty} E_0 y_n$. Hence

$$(Qx, x) = (\lim_{n\to\infty} E_0 y_n, x) = \lim_{n\to\infty} (E_0 y_n, x) = \lim_{n\to\infty} (y_n, E_0 x) = \lim_{n\to\infty} (y_n, x) = 0,$$

because $E_0 x = x$ and $x \in H \ominus K$. It follows that $Qx = 0$ and $Q|_{H_0} = 0$, but $Q \leqslant E_0$; hence $Q = 0$. The proof of $1° \Rightarrow 5°$ is very similar to the above, so we omit it. The proofs of $5° \Rightarrow 3°$ and $5° \Rightarrow 4°$ are identical, so we carry out them together: Take a projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}_E$ is $EH - (S_2)$. This means that the center of $\mathfrak{A}_E$ contains only the zero operator. Thus $E = 0$ and the proof is finished.

Some simple observations are now at hand:

(7) *If $\mathfrak{A}$ is K-Szegö singular, then $\mathfrak{A}$ is $H \ominus K - (S_1)$.*

(8) *If $K \subset K_1 \subset H$ are two subspaces and $\mathfrak{A}$ is K-Szegö singular, then $\mathfrak{A}$ is K_1-Szegö singular.*

Let f be a positive functional on a C-*algebra A and let (H_f, π_f, ξ_f) be the triple corresponding to f. Let B be a closed subalgebra of A.

DEFINITION 5. A positive functional f on a C^*-algebra A with unit is called *B-Szegö singular* if the weak closure of $\pi_f(A)$ in $L(H_f)$ is $K_f(B)$-Szegö singular.

V. Decomposition theorems. Consider an arbitrary von Neumann algebra $\mathfrak{A} \subset L(H)$ and a subspace K of H. We will construct a decomposition of $\mathfrak{A}$ on (S_1) and Szegö-singular parts.

If $\mathscr{P}$ is a family of projections in $L(H)$, then the supremum of $\mathscr{P}$ always exists and equals the projection H on the subspace spanned by Px, where $P \in \mathscr{P}$, $x \in H$. If the family $\mathscr{P}$ is directed, i.e., for $P, Q \in \mathscr{P}$, $P \vee Q \in \mathscr{P}$ (where $P \vee Q$ stands for the projection H onto $[PH+QH]$), and strongly closed, then the supremum of $\mathscr{P}$ belongs to $\mathscr{P}$.

Our decomposition theorem is the following one:

THEOREM 3. *For every von Neumann algebra $\mathfrak{A} \subset L(H)$ and every subspace K of H there is a largest projection $E_0 \in \mathfrak{A}'$ such that $\mathfrak{A}$ is $\overline{E_0 K} - (S_1)$ and $\mathfrak{A}_{(I_H - E_0)}$ is $\overline{(I_H - E_0)K}$ Szegö singular. This decomposition is unique in the following sense: if F is a projection which belongs to $\mathfrak{A}'$ such that $\mathfrak{A}$ is $\overline{FK} - (S_1)$ and $\mathfrak{A}_{(I_H - F)}$ is $\overline{(I_H - F)K}$ Szegö singular, then $F = E_0$. Moreover, $E_1 = I_H - E_0$ is the largest projection commuting with $\mathfrak{A}$ and such that $E_1 H \subset K$.*

Proof. Let R denote the projection H on K. Define a family of projections as follows:

$$\mathscr{P} = \{P \in \mathfrak{A}',\ P \text{ is a projection},\ P \leqslant R\}.$$

It is clear that $\mathscr{P}$ is strongly closed. Moreover, if $P, Q \in \mathscr{P}$, then $P \vee Q \in \mathscr{P}$ (see [6]). Hence, the supremum E_1 of $\mathscr{P}$ belongs to $\mathscr{P}$. Observe first that if $P \in \mathscr{P}$, then $\mathfrak{A}_p$ is $\overline{PK}$-Szegö singular. Indeed, $\overline{PK} = PK = PRH = PH$. Put $E_0 = I_H - E_1$. We now verify that $\mathfrak{A}$ is $E_0 K = K \ominus E_1 H - (S_1)$. If Q is a projection, $Q \in \mathfrak{A}'$ and $QH \subset K \ominus E_1 H$, then, in particular, $Q \in \mathscr{P}$ and hence $Q \leqslant E_1$, because E_1 is the supremum of $\mathscr{P}$. But $Q \leqslant I_H - E_1$, which proves that $Q = 0$. Let E be a projection which belongs to $\mathfrak{A}'$ and is such that $\mathfrak{A}$ is $\overline{EK} - (S_1)$. Denote by F the projection onto $\overline{EE_1 H}$. F belongs to $\mathfrak{A}'$, because $F = E - E \wedge (I_H - E_1)$ (see [6], p. 18). Here, if P and Q are two projections, $P \wedge Q$ denotes the projection on $PH \cap QH$. Moreover, $\overline{EE_1 H} \subset \overline{EK}$, because $E_1 H \subset K$. Hence $F \in \mathfrak{A}'$ and $FH \subset \overline{EK}$, but $\mathfrak{A}$ is $\overline{EK} - (S_1)$. It follows that $F = 0$ and $EE_1 = 0$. Thus $E \leqslant E_0$. We have just proved that every projection $E \in \mathfrak{A}'$ such that $\mathfrak{A}$ is $\overline{EK} - (S_1)$ is majorized by E_0.

Finally, if $P \in \mathscr{P}$ and Q is a projection, $Q \in \mathfrak{A}'$ and $Q \leqslant P$, then $Q \in \mathscr{P}$. This observation proves the uniqueness of the decomposition (see [5], (A)), and the proof is complete.

Let us point out that Szegö-singularity characterizes in some sense all subspaces of H which reduce $\mathfrak{A}$ and which are contained in K.

Now we have a decomposition theorem for functionales on C^*-algebras:

THEOREM 4. *Let A be a C^*-algebra with the unit e, let B be its closed subalgebra containing e and let f be a positive functional on A. Then there are two positive functionals f_0, f_1 on A such that:*

1. *f_0 is $B-(S_1)$,*
2. *f_1 is B-Szegö singular,*
3. *$f = f_0 + f_1$ and the decomposition is unique relative to* 1, 2,
4. *f_0 is the supremum of all $B-(S_1)$ functionals g on A such that $g \leqslant f$.*

Proof. Let (H_f, π_f, ξ_f) be the triple corresponding to f. Let $\mathfrak{A}$ be the weak operator closure of $\pi_f(A)$ in $L(H_f)$. Define $K = [\pi_f(B)\xi_f]$. Using Theorem 3, we obtain a largest projection $E_0 \in \mathfrak{A}'$ such that $\mathfrak{A}$ is $\overline{E_0 K}-(S_1)$. Put $f_0(a) = \big(\pi_f(a)E_0\xi_f, \xi_f\big)$ for $a \in A$ and $\pi_0(a) = \pi_f(a)|_{E_0 H}$, $a \in A$. It is obvious that the triple $(E_0 H_f, \pi_0, E_0\xi_f)$ corresponds to f_0. In this case $\overline{E_0 K} = [\pi_f(B)E_0\xi_f]$; hence f_0 is $B-(S_1)$, which proves 1. Let us define $f_1(a) = \big(\pi_f(a)(I-E_0)\xi_f, \xi_f\big)$ for each $a \in A$. We write I for the identity in H_f. A short consideration similar to the above assures us that f_1 is B-Szegö singular.

It is clear that $f = f_0 + f_1$. It remains to prove 4. Since we are in a cyclic situation, for a positive functional g on A such that $g \leqslant f$ there is a unique self-adjoint operator $T \in \mathfrak{A}'$ such that $0 \leqslant T \leqslant I$ and, for all $a \in A$, $g(a) = \big(\pi_f(a)T\xi_f, T\xi_f\big)$ (see [1], p. 35). It is not difficult to determine the triple corresponding to g. It is the following one: $H_g = [\pi_f(A)T\xi_f]$, $\pi_g(a) = \pi_f(a)|_{H_g}$ for all $a \in A$, bacause H_g reduces $\pi_f(A)$, and the cyclic vector is $T\xi_f$. Denote by E_g the projection on H_g. It is an element of $\mathfrak{A}'$ and $E_g H_f = \overline{TH_f} = H_g$; hence E_g is the projection on the closure of the range of T. Assume that g is $B-(S_1)$. We will prove that $E_1 E_g = 0$, where $E_1 = I - E_0$. From this equality it follows that $E_g \leqslant E_0$. But in order that $E_1 E_g = 0$ it is necessary and sufficient to have $E_1 T = 0$ (see [6]) and further, it suffices to show that $E_1 T\xi_f = 0$, because ξ_f separates $\mathfrak{A}'$. The assumption g is $B-(S_1)$ implies that every projection $P \in \mathfrak{A}'$ such that $PH \subset \overline{TK} = [\pi_f(B)T\xi_f]$ is a zero operator. The operator TE_1T is self-adjoint and belongs to $\mathfrak{A}'$. Hence the projection F onto the closure of its range also belongs to $\mathfrak{A}'$. (see [6]). Now we have $FH_f = \overline{TE_1TH_f} \subset \overline{TE_1H_f} \subset \overline{TK}$, and hence $F = 0$ by our assumption. It follows that $TE_1T = 0$. Finally we obtain

$$\|E_1T\xi_f\|^2 = (E_1T\xi_f, E_1T\xi_f) = (TE_1T\xi_f, \xi_f) = 0.$$

Hence $E_1 T\xi_f = 0$ and $E_g \leqslant E_0$. Let h be a positive functional on A defined by the subrepresentation π_g and vector $E_g\xi_f$. h has the form $h(a) = (\pi_g(a)E_g\xi_f, \xi_f)$. Our last task is to prove that $g \leqslant h \leqslant f_0$.

We break off our proof in order to make the following simple observation.

Let f be a positive functional on a C^*-algebra A with the triple (H_f, π_f, ξ_f). Then every $T \in \pi_f(A)'$, $T = T^*$, defines the following positive functional on A: $f_T(a) = (\pi_f(a)T\xi_f, T\xi_f)$. Let T, S be two self-adjoint positive operators in $L(H_f)$ which belong to $\pi_f(A)'$. Assume $TS = ST$. Then $0 \leqslant S \leqslant T$ implies $f_S \leqslant f_T$. Indeed, let $a \in A$:

$$(f_T - f_S)(a^*a) = (\pi_f(a^*a)\xi_f, (T^2 - S^2)\xi_f) = \|\pi_f(a)(T^2 - S^2)^{\frac{1}{2}}\xi_f\|^2 \geqslant 0$$

because under our assumption $T^2 - S^2 \geqslant 0$.

We now return to the proof of Theorem 4. The preceding part of it shows that $0 \leqslant T \leqslant E_g \leqslant E_0$. T commutes with E_g and E_g commutes with E_0. Applying the above observation, we finish the proof.

We remark that this theorem implies the Foiaş and Suciu theorem [3] concerning the decomposition of a positive measure on Szegö and Szegö-singular parts.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, CRACOW
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK, ODDZIAŁ KRAKÓW

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