COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 1

THE EXISTENCE OF GLOBAL SOLUTIONS OF A FUNCTIONAL-DIFFERENTIAL EQUATION

BY

S. CZERWIK (KATOWICE)

1. Introduction. In the present paper we are concerned with the functional-differential equation of the form

$$\varphi'(x) = h\Big(x, \varphi(x), \varphi\big[f_1\big(x, \varphi(x)\big)\big], \ldots, \varphi\big[f_n\big(x, \varphi(x)\big)\big]\Big), \tag{1}$$

with the initial condition

$$\varphi(0) = y_0, \tag{2}$$

where φ is an unknown function, and h and f_i $(i = 1, \ldots, n)$ are known functions.

We consider the problem of the global existence of solutions of equation (1) in the interval $\langle 0, \infty)$. We shall prove (under suitable assumptions) that the problem (1)-(2) has at least one solution defined in the interval $\langle 0, \infty)$ which belongs to a certain function class G, defined in the sequel.

For the equation

$$\varphi'(x) = h\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big)$$

the corresponding problem (as well as the problem of uniqueness) has been investigated by the author in [2].

The problem of the local existence of solutions of equation (1) for $n = 1$ has been investigated by Oberg [4].

2. Existence theorem. In this section we establish a theorem asserting the existence of global solutions of the initial-value problem (1)-(2) in special class functions.

We assume that

(i) The functions $h\colon I \times R^{n+1} \to R$, $f_i\colon I \times R \to I$ $(i = 1, \ldots, n)$, where $I = \langle 0, \infty)$, $R = (-\infty, +\infty)$, are continuous in $I \times R^{n+1}$ and $I \times R$, respectively.

(ii) There exist continuous functions $L_i\colon I \to I$ $(i = 0, 1, \ldots, n+1)$ and constants $a \in \langle 0, 1\rangle$, $\beta_i \in \langle 0, 1\rangle$, $i = 1, \ldots, n$, such that for every

$x \in I$ and $z_i \in R$ $(i = 1, \ldots, n+1)$ we have

$$|h(x, z_1, \ldots, z_{n+1})| \leqslant eL_0(x) + eL_1(x)|z_1|^{\alpha} + L_2(x)|z_2|^{\beta_1} + \ldots + L_{n+1}(x)|z_{n+1}|^{\beta_n},$$

where e is the Euler number.

(iii) There exist continuous functions $g_i\colon I \to I$ $(i = 1, \ldots, n)$ such that

$$f_i(x, t) \leqslant g_i(x), \quad x \in I, \ t \in R \ (i = 1, \ldots, n),$$

$$\beta_i L[g_i(x)] \leqslant L(x) + e^{-1}, \quad x \in I \ (i = 1, \ldots, n),$$

where

$$L(x) = \int_0^x [L_0(s) + \ldots + L_{n+1}(s)]\,ds. \tag{3}$$

Now we shall prove the main result.

THEOREM. *If* (i)-(iii) *are satisfied, then equation* (1) *with initial condition* (2) $(y_0 \in R)$ *has at least one solution* $\varphi\colon I \to R$ *satisfying the condition*

$$|\varphi(x)| \leqslant a \exp\big(eL(x)\big), \quad x \in I, \tag{4}$$

where a *is a real constant and*

$$a \geqslant \max(1, |y_0|). \tag{5}$$

Proof. Let X be the space of all functions $\varphi\colon I \to R$ which are continuous in I and

$$\sup_{x \in I}\Big(|\varphi(x)| \exp\big(-eL(x) - x\big)\Big) < \infty.$$

One can verify that X with the norm

$$\|\varphi\| = \sup_{x \in I}\Big(|\varphi(x)| \exp\big(-eL(x) - x\big)\Big) \tag{6}$$

is a Banach space (cf. [1]). We omit simple calculations.

Now define G as the space of these functions $\varphi \in X$ which satisfy the condition

$$|\varphi(x)| \leqslant a \exp\big(eL(x)\big), \quad x \in I, \tag{7}$$

where a satisfies (5).

One can easily verify that the set G is a nonempty, bounded, convex and closed subset of the space X (cf. also [3]).

Now, for $\varphi \in G$, we define the transformation $\Phi = T\varphi$ by the formula

$$\Phi(x) = y_0 + \int_0^x h\Big(s, \varphi(s), \varphi\big[f\big(s, \varphi(s)\big)\big]\Big)\,ds, \quad x \in I, \tag{8}$$

where

$$\varphi[f(s,\varphi(s))] \overset{\text{df}}{=} \big(\varphi[f_1(s,\varphi(s))],\ldots,\varphi[f_n(s,\varphi(s))]\big).$$

We prove that (8) transforms G into itself. Indeed, let $\varphi\in G$. In view of (i), Φ is continuous in I. From (8), (ii), (iii), (3) and (5) we obtain

$$\begin{aligned}|\Phi(x)| &\leqslant |y_0|+\int_0^x \Big|h\big(s,\varphi(s),\varphi[f(s,\varphi(s))]\big)\Big|ds\\ &\leqslant |y_0|+\int_0^x \big\{eL_0(s)+ea^{\alpha}L_1(s)\exp\big(eaL(s)\big)+\\ &\quad +a^{\beta_1}L_2(s)\exp\big(e\beta_1 L[g_1(s)]\big)+\ldots+a^{\beta_n}L_{n+1}(s)\exp\big(e\beta_n L[g_n(s)]\big)\big\}ds\\ &\leqslant |y_0|+a\int_0^x e[L_0(s)+\ldots+L_{n+1}(s)]\exp\big(eL(s)\big)ds\\ &\leqslant |y_0|+a\big[\exp\big(eL(x)\big)-1\big]\leqslant a\exp\big(eL(x)\big).\end{aligned}$$

Thus T maps G into itself. Next we prove that T is a continuous transformation, i.e. $\|\varphi_m-\varphi\|\to 0$ implies $\|T\varphi_m-T\varphi\|\to 0$.

Let $\varphi_m,\varphi\in G$ $(m=1,2,\ldots)$ and let $\|\varphi_m-\varphi\|\to 0$ as $m\to\infty$. Hence $|\varphi_m(x)-\varphi(x)|\to 0$ as $m\to\infty$ uniformly in $\langle 0,b\rangle$ for every $b>0$. Now,

$$\begin{aligned}&|T\varphi_m(x)-T\varphi(x)|\\ &\qquad\leqslant\int_0^x \Big|h\big(s,\varphi_m(s),\varphi_m[f(s,\varphi_m(s))]\big)-h\big(s,\varphi(s),\varphi[f(s,\varphi(s))]\big)\Big|ds.\end{aligned}$$

Since

$$|\varphi_m(x)-\varphi(x)|\to 0 \quad\text{ and }\quad \big|\varphi_m[f(x,\varphi_m(x))]-\varphi[f(x,\varphi(x))]\big|\to 0$$

as $m\to\infty$ uniformly in $\langle 0,b\rangle$, it follows from the continuity of the function h that for every $\varepsilon>0$ there exists an m_0 such that for $m\geqslant m_0$ and $x\in\langle 0,b\rangle$ we have

$$\Big|h\big(x,\varphi_m(x),\varphi_m[f(x,\varphi_m(x))]\big)-h\big(x,\varphi(x),\varphi[f(x,\varphi(x))]\big)\Big|<\varepsilon.$$

Hence

$$|T\varphi_m(x)-T\varphi(x)|<\varepsilon b \quad\text{ for } x\in\langle 0,b\rangle \text{ and } m\geqslant m_0.$$

Moreover,

$$|T\varphi_m(x)-T\varphi(x)|\exp\big(-eL(x)-x\big)\leqslant 2a\exp(-x)<\varepsilon$$

for all sufficiently large values of x. Arbitrariness of ε shows that $\|T\varphi_m-T\varphi\|\to 0$ as $m\to\infty$ and so T is a continuous transformation.

Now we prove that the set $G_1 = T(G)$ is compact. Let $\Phi_m \epsilon G_1$ for each $m = 1, 2, \ldots$ and let $\{b_i\}_{i=1}^{\infty}$ be an arbitrary increasing sequence such that $b_i > 0$ $(i = 1, 2, \ldots)$ and $\lim_{i\to\infty} b_i = \infty$.

We have

$$|\Phi_m(x)| \leqslant a\exp\big(eL(b_1)\big), \quad x \epsilon \langle 0, b_1\rangle,\ m = 1, 2, \ldots,$$

and, for $0 \leqslant z \leqslant x \leqslant b_1$,

$$\begin{aligned}|\Phi_m(x) - \Phi_m(z)| &\leqslant \int_z^x \Big|h\Big(s, \varphi_m(s), \varphi_m\big[f\big(s, \varphi_m(s)\big)\big]\Big)\Big|\,ds \\ &\leqslant ae\int_z^x [L_0(s) + \ldots + L_{n+1}(s)]\exp\big(eL(s)\big)\,ds \\ &\leqslant ae\max_{s\epsilon\langle 0, b_1\rangle} [L_0(s) + \ldots + L_{n+1}(s)]\exp\big(eL(b_1)\big)(x - z).\end{aligned}$$

Hence the sequence $\{\Phi_m\}_{m=1}^{\infty}$ is equibounded and equicontinuous in $\langle 0, b_1\rangle$ and so there exists a subsequence $\{\Phi_m^1\}_{m=1}^{\infty}$ uniformly convergent in $\langle 0, b_1\rangle$ to some function Φ^1 such that

$$|\Phi^1(x)| \leqslant a\exp\big(eL(x)\big) \quad \text{for } x \epsilon \langle 0, b_1\rangle.$$

Let now $x \epsilon \langle 0, b_2\rangle$. Since $\{\Phi_m^1\}_{m=1}^{\infty}$ is equibounded and equicontinuous in $\langle 0, b_2\rangle$, there exists a subsequence $\{\Phi_m^2\}_{m=1}^{\infty}$ of the sequence $\{\Phi_m^1\}_{m=1}^{\infty}$ uniformly convergent in $\langle 0, b_2\rangle$ to some function Φ^2 such that

$$|\Phi^2(x)| \leqslant a\exp\big(eL(x)\big) \quad \text{for } x \epsilon \langle 0, b_2\rangle.$$

Continuing in this fashion we produce sequences $\{\Phi_m^i\}_{m=1}^{\infty}$ $(i = 1, 2, \ldots)$ uniformly convergent in $\langle 0, b_i\rangle$ to functions Φ^i such that

$$|\Phi^i(x)| \leqslant a\exp\big(eL(x)\big) \quad \text{for } x \epsilon \langle 0, b_i\rangle,\ i = 1, 2, \ldots$$

The diagonal sequence $\{\Phi_m^m\}_{m=1}^{\infty}$ is almost uniformly convergent in I to Φ and such that $\Phi(x) = \Phi^i(x)$ for $x \epsilon \langle 0, b_i\rangle$, and $\Phi \epsilon G$.

Let $\varepsilon > 0$ be arbitrary, and let $b_i \geqslant \ln(2a/\varepsilon)$. We have

$$\begin{aligned}&\sup_{x\epsilon I}\Big(|\Phi_m^m(x) - \Phi(x)|\exp\big(-eL(x) - x\big)\Big) \\ &\quad\leqslant \sup\Big\{\sup_{x\epsilon\langle 0, b_i\rangle}\Big(|\Phi_m^m(x) - \Phi(x)|\exp\big(-eL(x) - x\big)\Big), \\ &\qquad\qquad \sup_{x\geqslant b_i}\Big(|\Phi_m^m(x) - \Phi(x)|\exp\big(-eL(x) - x\big)\Big)\Big\} \\ &\quad\leqslant \sup\Big\{\sup_{x\epsilon\langle 0, b_i\rangle}\big(|\Phi_m^m(x) - \Phi(x)|\big), \sup_{x\geqslant b_i}\big(2a\exp(-x)\big)\Big\} < \varepsilon\end{aligned}$$

for m sufficiently large.

This shows that $T(G)$ is compact. On account of Schauder's fixed-point theorem there exists at least one solution $\varphi \epsilon G$ of the problem (1)-(2). This proves the theorem.

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY

Reçu par la Rédaction le 28. 5. 1975