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On generalized ordered spaces

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1. Introduction

A linearly ordered topological space (abbreviated LOTS) is a triple $(X, \lambda, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and where λ is the usual open-interval topology of the order $\leqslant$. While LOTS seem to be well behaved, their subspaces have provided many pathological examples in general topology: the Sorgenfrey line and the spaces considered by E. Michael in [19] and [20] are well-known spaces of this type. In this paper, we investigate certain aspects of the topology of subspaces of LOTS, namely paracompactness, metrizability, local compactness and Nagami's Σ-space property. Our approach is based on Čech's observation that a space X can be embedded in a LOTS iff X is a generalized ordered space (1).

We are interested in three general problems. First, which theorems for LOTS are true for subspaces of LOTS, i.e., for generalized ordered spaces? A basic tool in the study of the first problem is the construction, for a generalized ordered space X, of a LOTS X^* which contains X as a closed subspace. This leads us to our second problem: which topological properties does X^* inherit from X? Our third problem is suggested by the fact that the topology of any generalized ordered space can be obtained by strengthening, in a prescribed manner, the open-interval topology of the given linear ordering. It seems natural, therefore, to ask whether a generalized ordered space X can be studied in terms of the weaker open-interval topology on X.

The paper is organized as follows. In Section 2 we reproduce the basic definitions and summarize the results of Frolík and Katětov. Section 3 contains technical lemmas which are used throughout the paper. Section 4 studies paracompactness in generalized ordered spaces, and in Section 5 we consider metrization and related problems. In Section 6 we present some results dealing with local compactness and provide a partial answer to a question of E. Michael concerning Nagami's Σ-spaces. Section 7 contains a sequence of examples to which preceding sections refer.

Before beginning our study, let us agree on some terminology. Let $(X, \leqslant)$ be a linearly ordered set. A subset C of X is said to be *convex in X*

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(1) This characterization appears in the recent version of Čech's *Topological Spaces* edited by Z. Frolík and M. Katětov [9]. Professor Frolík ascribes the original idea to E. Čech.

if, whenever $a, b \in C$ and $a \leqslant b$, then $\{x \in X\colon a \leqslant x \leqslant b\}$ is a subset of C. If there is no danger of confusion, we shall simply say that C is convex. An *interval of* X is a convex subset of X having two endpoints in X. Intervals will be denoted by $]a, b[$, $]a, b]$, etc. If $a \in X$, the set $\{x \in X\colon x < a\}$ will be called an *open half-line* and will be denoted by $]\leftarrow, a[$. The sets $]\leftarrow, a]$, $[a, \rightarrow[$ and $]a, \rightarrow[$ are defined analogously. The usual open-interval topology of the linear ordering $\leqslant$ will be denoted by $\lambda(\leqslant)$. Finally, the sets of real numbers, of integers, and of positive integers (not necessarily with their usual topologies) will be denoted, respectively, by $\boldsymbol{R}$, $\boldsymbol{Z}$, and $\boldsymbol{N}$.

2. Characterization of generalized ordered spaces

The material found in (2.1) to (2.8) essentially appears in [9]. Because of its fundamental nature, we repeat it here.

(2.1) DEFINITION. A *generalized ordered space* (abbreviated GO space) is a triple $(X, \tau, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and where τ is a topology on X such that:

(a) $\lambda(\leqslant) \subseteq \tau$, where $\lambda(\leqslant)$ is the open-interval topology of $\leqslant$;

(b) every point of X has a local τ-base consisting of (possibly degenerate) intervals of X.

If $(X, \leqslant)$ is a linearly ordered set and if τ is a topology on X such that $(X, \tau, \leqslant)$ is a GO space, then τ is called a *GO topology on* X.

Where it will cause no confusion, we shall omit mention of τ and $\leqslant$ and write simply "Let X be a GO space."

(2.2) EXAMPLES. It is easily seen that the following general construction always produces a GO space and that, in fact, any GO space can be obtained from such a construction. Let $(X, \leqslant)$ be a linearly ordered set and let L, R, and I be disjoint subsets of X. Let τ be the topology on X having the following collection as a subbase:

$$\big\{\{x\}\colon x \in I\big\} \cup \{]\leftarrow, x]\colon x \in L\} \cup \{[x, \rightarrow[\colon x \in R\} \cup \lambda(\leqslant).$$

This simple construction makes it clear that each of the following is a GO space:

(a) any LOTS;

(b) the Sorgenfrey line (cf. 7.2) – let $X = R = \boldsymbol{R}$ and $L = I = \emptyset$;

(c) the space $(\boldsymbol{R}, \mu, \leqslant)$ obtained from the real numbers with their usual topology and order by making the irrationals discrete (cf. 7.3) – let $X = \boldsymbol{R}$, $I = \{\text{irrationals}\}$ and $L = R = \emptyset$;

(d) the space $(X, \delta, \leqslant)$, where $(X, \leqslant)$ is any linearly ordered set and where δ is the discrete topology on X.

It is well known that a subspace of a LOTS — even an open or a closed subspace —need not be linearly orderable in its relative topology. However, we have:

(2.3) PROPOSITION (Čech). *Any subspace of a LOTS is a GO space.*

The converse of (2.3) is also true, but requires a little more work. We begin by observing that a point of a GO space has a local base consisting of intervals all having the same shape, in the sense of the next lemma.

(2.4) LEMMA (Čech). *Let X be a GO space and let $p \in X$. Suppose that p is not an isolated point of X and that $[p, \to[$ is open in X. Then p has no immediate successor in X, p is not the right endpoint of X and the collection $\{[p, b[: b \in X$ and $b > p\}$ is a local base at p.*

(2.5) DEFINITION ([2]). Let $X = (X, \tau, \leqslant)$ be a GO space. Let $\lambda = \lambda(\leqslant)$ be the usual order topology on X. Define a subset $X^* = (X, \tau, \leqslant)^*$ of $X \times \mathbf{Z}$ by

$$X^* = (X \times \{0\}) \cup \{(x, n): [x, \to[\in \tau \setminus \lambda \text{ and } n \leqslant 0\} \cup \cup \{(x, m):]\leftarrow, x] \in \tau \setminus \lambda \text{ and } m \geqslant 0\}.$$

(2.6) Convention. Throughout this paper, $(X, \tau, \leqslant)^*$ will be ordered lexicographically and will carry the usual open-interval topology of this lexicographic order.

(2.7) PROPOSITION (Čech). *Let X be a GO space. Then the function e: $X \to X^*$ defined by $e(x) = (x, 0)$ is an order-preserving homeomorphism from X onto the subspace $X \times \{0\}$ of X^*.*

(2.8) Convention. Except in situations where clarity requires that we distinguish between X and $X \times \{0\}$, we shall use the map e of (2.7) to identify X with a subspace of X^*.

The equivalence of (a) and (c) in our next result was established in [9].

(2.9) THEOREM. *The following properties of a topological space (X, τ) are equivalent:*

(a) *there is a linear ordering $\leqslant$ of X such that $(X, \tau, \leqslant)$ is a GO space;*

(b) *(X, τ) is a closed subspace of a LOTS;*

(c) *(X, τ) is a subspace of a LOTS;*

(d) *(X, τ) is a dense subspace of a compact LOTS.*

([2]) It should be pointed out that our set X^* is a (possibly proper) subset of the linearly ordered set $(T, <_T)$ constructed in [9]. However, the proof that $x \to (x, 0)$ is a homeomorphism is the same and the use of X^* instead of T seems to be justified by (2.11) and by the fact that if X is actually a LOTS in the given order, then $X^* = X$ while T may be a much larger set. For example, let X be the set $[0, 1] \times \{0, 1\}$ and let X have the lexicographic order and the usual open-interval topology.

Proof. (a) $\to$ (b). Observe that every point of $X^*\setminus X$ has both an immediate predecessor and an immediate successor in X^*. Hence every point of $X^*\setminus X$ is an isolated point of X^*, so X is closed in X^*.

(b) $\to$ (c) is trivial.

(c) $\to$ (d). Suppose (X, τ) is a subspace of a LOTS $(Y, \lambda, \leqslant)$. Let $(Y^+, \tau^+, \leqslant^+)$ be the order-compactification of Y and let Z be the closure of X in Y^+. Topologizing Z as a subspace of Y^+ we obtain a compact GO space which contains X as a dense subspace. Since Z is compact, Z is actually a LOTS (cf. (6.1)).

(d) $\to$ (a). This follows from (2.3).

In this paper we shall be primarily concerned with X as a subspace of X^*. However, the equivalence of (a) and (d) in (2.9) can also be used to study GO spaces. For example:

(2.10) Proposition. *Let X be a GO space. Then:*

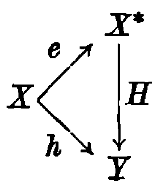
(a) *X is separable iff X is hereditarily separable;*

(b) *X satisfies the countable chain condition* (i.e., *every disjoint collection of open sets is countable*) *iff X is hereditarily Lindelöf.*

Proof. Statements (a) and (b) were proved for LOTS in [17]. Observe that if the GO space X has a countable dense subset (respectively, satisfies the countable chain condition), then so does any compactification of X. In particular this is true of the compact LOTS found in (2.9).

Let us conclude this section with two remarks concerning our definition of X^* (which differs slightly from the definition of a similar space which is given in [9]). First, observe that if the topology of X coincides with the usual open interval topology of the given order on X, then $X^* = X$. This is clear from the definition of X^* in (2.5). Second, X^* is, in some sense, the smallest LOTS which contains X as a closed subspace. This statement is made precise in Proposition (2.11), but since the proposition is not needed in later sections of the paper, we omit its proof (which is straightforward but tedious).

(2.11) Proposition. *Let X be a GO space and suppose that h is an order-preserving homeomorphism from X onto a closed subspace of a LOTS Y. Let e: $X \to X^*$ be the embedding defined in* (2.7). *Then there is an order-preserving homeomerphism H from X^* into Y such that the following diagram commutes:*



(2.12) Remark. a) The reader might ask whether (2.11) could be proved without the assumption that h is order-preserving. Example 7.1 provides a negative answer to this question.

b) It is natural to ask whether the correspondence $X \to X^*$ could be made functional by properly defining a map $f^*\colon X^* \to Y^*$ corresponding to an order-preserving continuous map $f\colon X \to Y$. If we make the additional assumption (and it seems reasonable to do so) that f^* must extend f, then the answer is negative as Example 7.5 shows.

3. Technical lemmas

In this section we develop certain technical results which will be used repeatedly in subsequent sections.

(3.1) Definition. Let X be a GO space and let $S \subseteq X$ be convex in X. Define

$$I(S) = \{x \in S\colon \exists a, b \in S \text{ with } a < x < b\}.$$

Define a subset $S^{\sim}$ of X^* by

$$S^{\sim} = \{(x, k) \in X^*\colon x \in I(S)\} \cup \{(x, 0)\colon x \in S \setminus I(S)\}.$$

(3.2) Lemma. *Let X be a GO space.*

(a) *If $S \subseteq T$ are convex in X, then $S^{\sim} \subseteq T^{\sim}$.*

(b) *If S is convex in X, then $S^{\sim}$ is open in X^* iff S is open in X.*

(c) *If J is convex in X^* and if $S \subseteq J$, where S is convex in X, then $S^{\sim} \subseteq J$.*

Proof. (a) Let $(x, k) \in S^{\sim}$. If $x \in I(S)$, then $x \in I(T)$ so that $(x, k) \in T^{\sim}$. If $x \notin I(S)$, then $k = 0$ so that $(x, k) \in T^{\sim}$ since $x \in T$.

(b) Suppose S is open in X. Let $(x, k) \in S^{\sim}$. If x is an isolated point of X or if $k \neq 0$, then $\{(x, k)\}$ is an open subset of X^* which is contained in $S^{\sim}$. Consider the case where x is not an isolated point of X and $k = 0$. If $x \in I(S)$, choose $a, b \in S$ such that $a < x < b$. If $]a, x[= \emptyset$, let $p = (a, 0)$; otherwise choose $a' \in]a, x[$ and let $p = (a', 0)$. In either case, $p \in X^*$, $p < (x, 0)$ and $]p, (x, 0)] \subseteq S^{\sim}$. Similarly, choose $q \in X^*$ such that $q > (x, 0)$ and $[(x, 0), q[\subseteq S^{\sim}$. Then $(x, 0) \in]p, q[\subseteq S^{\sim}$. If $x \notin I(S)$, then either $x \in S \subseteq [x, \to[$ or $x \in S \subseteq]\leftarrow, x]$. We consider only the first possibility. Since S is open in X, either x has an immediate predecessor x' in X (in which case we let $p = (x', 0)$) or else the point $p = (x, -1)$ is a point of X^*. Since, by assumption, x is not an isolated point of X, we may choose points x_1 and $x_2 \in S$ with $x < x_1 < x_2$. Let $q = (x_1, 0)$. Then $(x, 0) \in]p, q[\subseteq S^{\sim}$. Therefore, $S^{\sim}$ contains an X^*-neighborhood of each of its points, so $S^{\sim}$ is open in X^*. The converse is clear since $S^{\sim} \cap X = S$.

(c) Suppose $S \subseteq J$. Let $(x, k) \in S^{\sim}$. If $x \in I(S)$, choose points $a, b \in S$ with $a < x < b$. Then $(a, 0)$, $(b, 0) \in J$ and $(a, 0) < (x, k) < (b, 0)$. Since J is a convex subset of X^*, $(x, k) \in J$. If $x \notin I(S)$, then $k = 0$ so that $S \subseteq J$ implies $(x, k) \in J$.

We now extend definition (3.1) to arbitrary subsets of X. It is well known that any non-void subset G of X can be uniquely represented as a union of its maximal convex subsets, which are called *convex components of* G. This fact justifies the following definition.

(3.3) DEFINITION. Let X be a GO space. Let $G \subseteq X$. If $G = \emptyset$, let $G^{\sim} = \emptyset$. If $G \neq \emptyset$, let $G = \bigcup \{S_i : i \in I\}$ be the (unique) representation of G as a union of its convex components. Define $G^{\sim} = \bigcup \{S_i^{\sim} : i \in I\}$.

(3.4) Remark. Suppose G is a non-empty subset of a GO space $(X, \tau, \leqslant)$. Then $(G, \tau_G, \leqslant)$ is also a GO space (where τ_G denotes the relativized topology on G), so we can construct $G^* = (G, \tau_G, \leqslant)^*$. If $G^{\sim}$ is defined with respect to X^*, the following relationships are evident:

(a) $G^{\sim} \subseteq G^*$ and $G^{\sim}$ need not coincide with G^* even if G is convex in X.

(b) If G is convex in X, then $G^* \subseteq X^*$; in general, G^* may contain points which do not belong to X^*. (If X is a LOTS and if $G \subseteq X$ is not a LOTS under the given order, then $X^* = X \times \{0\}$ while G^* must contain points (x, k) with $k \neq 0$.)

The following properties of the correspondence $G \to G^{\sim}$ will be frequently used.

(3.5) PROPOSITION. *Let X be a GO space.*

(a) *If $G \subseteq H \subseteq X$, then $G^{\sim} \subseteq H^{\sim}$.*

(b) *If G is open in X, then $G^{\sim}$ is open in X^*.*

(c) *If $\mathscr{J}$ is a collection of convex subsets of X^* and if $\mathscr{G}$ is a collection of subsets of X which refines $\mathscr{J}$, then so does the collection $\mathscr{G}^{\sim} = \{G^{\sim} : G \in \mathscr{G}\}$.*

(d) *If $\mathscr{G}$ is a point-countable (respectively, point-finite) collection of subsets of X, then $\mathscr{G}^{\sim} = \{G^{\sim} : G \in \mathscr{G}\}$ is a point-countable (respectively, point-finite) collection of subsets of X^*.*

Proof. Statement (a) follows directly from (3.2) (a), and statement (b) follows from the fact that if G is open in X, then so is every convex component of G. Statement (c) is verified by observing that if G is contained in a convex subset J of X^*, then $S \subseteq J$ for each convex component S of G whence $S^{\sim} \subseteq J$ by (3.2) (c). Therefore $G^{\sim} \subseteq J$. Statement (d) follows from the fact that if S_1 and S_2 are convex subsets of X and if $(x, k) \in S_1^{\sim} \cap S_2^{\sim}$, then $x \in S_1 \cap S_2$.

(3.6) Remark. Observe that if S is convex in X and if $(x, k) \in X^* \setminus S^{\sim}$, then S cannot contain points on both sides of x, even though x itself may be a point of S (in case $k \neq 0$). This fact will be used in the proof of (4.4), below.

In Section 6 we shall use the notion of a coherent collection of sets.

(3.7) DEFINITION. A collection $\mathscr{C}$ of subsets of a set X is *coherent* if whenever $\mathscr{D}$ is a proper, non-empty subcollection of $\mathscr{C}$, the sets $\bigcup\mathscr{D}$ and $\bigcup(\mathscr{C}\setminus\mathscr{D})$ are *not* disjoint.

(3.8) DEFINITION. Let $\mathscr{G}$ be a collection of subsets of a set X and let $p\in X$. Then $\mathscr{S}(p,\mathscr{G})$ denotes $\{G\in\mathscr{G}: p\in G\}$ and $\mathrm{St}(p,\mathscr{G})$ denotes $\bigcup\mathscr{S}(p,\mathscr{G})$.

The following two lemmas are known [6] and can be proved by repeated application of Zorn's lemma and (finite) induction, respectively.

(3.9) LEMMA. *Let $\mathscr{G}$ be a non-empty collection of non-empty subsets of a set X. Then:*

(a) *the family Ψ of maximal coherent subcollections of $\mathscr{G}$ is non-void, and $\mathscr{G}=\bigcup\Psi$;*

(b) *if $\mathscr{C}_1$ and $\mathscr{C}_2$ are distinct elements of Ψ, then $\mathscr{C}_1\cap\mathscr{C}_2=\emptyset$ and $(\bigcup\mathscr{C}_1)\cap(\bigcup\mathscr{C}_2)=\emptyset$.*

(3.10) LEMMA. *Let $\mathscr{C}$ be a coherent collection of convex subsets of a linearly ordered set X and let $p\in X_0=\bigcup\mathscr{C}$. Then it is possible to define subsets $\{x(n)\}$ and $\{y(m)\}$, possibly finite, of X_0 such that:*

(a) $x(1)=p=y(1)$;

(b) *if $x(n+1)$ is defined, then $x(n+1)<x(n)$, $x(n+1)\notin\mathrm{St}(x(n),\mathscr{C})$ and $\mathrm{St}(x(n+1),\mathscr{C})\cap\mathrm{St}(x(n),\mathscr{C})\neq\emptyset$;*

(c) *if $y(m+1)$ is defined, then $y(m+1)>y(m)$, $y(m+1)\notin\mathrm{St}(y(m),\mathscr{C})$ and $\mathrm{St}(y(m+1),\mathscr{C})\cap\mathrm{St}(y(m),\mathscr{C})\neq\emptyset$;*

(d) $X_0=\bigcup\{\mathrm{St}(x(n),\mathscr{C}):\ x(n)$ *is defined*$\}\cup\bigcup\{\mathrm{St}(y(m),\mathscr{C}):\ y(m)$ *is defined*$\}$.

4. Paracompactness in GO spaces

We begin this section by summarizing some known results concerning paracompactness in LOTS. First, any LOTS is hereditarily collectionwise normal [26] and (hereditarily) (3) countably paracompact ([3] and [13]). Therefore a subparacompact (4) LOTS is paracompact [8], as is a metacompact (5) LOTS ([18] and [21]). Bennett [6] and Fedorčuk [11] improved

(3) The papers [3] and [13] actually prove that any LOTS is countably paracompact. *Hereditary* countable paracompactness follows from the fact that any open subspace of a LOTS is homeomorphic to a disjoint union (or topological sum) of LOTS, namely the convex components of the subspace. This argument is used in [26].

(4) A space is *subparacompact* if every open cover has a σ-locally finite closed refinement. D. K. Burke [8] proved that subparacompactness is equivalent to the property "Every open cover has a σ-discrete closed refinement" and it is clear that a collectionwise normal space with this latter property is paracompact.

(5) A space is *metacompact* if every open cover has an open, point-finite refinement. This property is also called *pointwise paracompactness* or *weak paracompactness*.

the latter result, proving that if a LOTS is *metalindelöf* (i.e., every open cover has an open point-countable refinement), then it is paracompact. In a slightly different direction, one can prove that a perfectly normal LOTS is paracompact; this result can be established using the fundamental characterization of paracompactness in LOTS due to Gillman and Henriksen [13]. We will give a different proof below.

Let us begin our study of paracompactness in GO spaces with the following observation.

(4.1) PROPOSITION. *Any GO space is collectionwise normal and countably paracompact.*

Proof. If X is a GO space, then X is a subspace of the LOTS X^* which is hereditarily collectionwise normal and hereditarily countably paracompact.

(4.2) THEOREM. *Let X be a GO space. Then the following are equivalent:*

(a) *X is metalindelöf;*

(b) *X^* is paracompact;*

(c) *X is paracompact.*

Proof. Trivially, (c) implies (a), and (b) implies (c) since X is a closed subspace of X^*. We show that (a) implies (b). Suppose that X is a metalindelöf GO space. In order to show that the LOTS X^* is paracompact, it suffices (by the Bennett-Fedorčuk result mentioned above) to show that X^* is metalindelöf.

Let $\mathscr{U}$ be an open cover of X^* by convex sets. Then $\mathscr{G} = \{U \cap X: U \in \mathscr{U}\}$ is a relatively open cover of X. Let $\mathscr{H}$ be a relatively open, point-countable cover of X which refines $\mathscr{G}$. Let $\mathscr{V} = \{H^{\sim} : H \in \mathscr{H}\} \cup \big\{\{(x, n)\}: (x, n) \in X^* \setminus X\big\}$. Then $\mathscr{V}$ is a collection of open subsets of X^* which refines $\mathscr{U}$, is point-countable, and which covers X^* (cf. (3.5)).

In [16] we showed that a LOTS with a G_δ-diagonal is metrizable, and in Section 5 we will generalize this result to a certain class of GO spaces. In arbitrary GO spaces, however, the existence of a G_δ-diagonal insures little more than hereditary paracompactness, as Examples (7.2) and (7.3) and our next theorem show. The theorem requires a lemma, a corollary of which will be needed in Section 5.

(4.3) LEMMA. *Suppose $\mathscr{U}$ is an open cover of a GO space X by convex sets. Let $E = \{x \in X$: no element of $\mathscr{U}$ contains points on both sides of $x\}$. Then for each $y \in X$ there is an open neighborhood $G(y)$ of y such that $G(y) \cap E \subseteq \{y\}$.*

Proof. For each $y \in X$, let $U(y)$ be an element of $\mathscr{U}$ which contains y and which has the property that if $y \notin E$, then $U(y)$ contains points on both sides of y. Consider the following statements:

(a) $U(y)$ contains a point $z > y$ (then $]y, z[\cap E = \emptyset$);

(b) $U(y)$ contains a point $x < y$ (then $]x, y[\cap E = \emptyset$).

If neither (a) nor (b) is true, then $U(y) = \{y\}$ and we may let $G(y) = \{y\}$. If both (a) and (b) are true, let $G(y) =]x, z[$. If (a), but not (b), is true, let $G(y) = [y, z[$, and if (b), but not (a), is true, let $G(y) =]x, y]$.

(4.4) COROLLARY. *Let X be a GO space. If X is Lindelöf, so is X^*.*

Proof. Let $\mathscr{G}$ be an open cover of X^* by convex sets. Then $\mathscr{U} = \{G \cap \cap X: G \in \mathscr{G}\}$ is an open cover of X, so some countable subcollection $\mathscr{V}$ of $\mathscr{U}$ covers X. Let $E = \{x \in X$: no element of $\mathscr{V}$ contains points on both sides of $x\}$. By (4.3), E is a closed discrete subspace of X. Since X is Lindelöf, E must be countable. Let $\mathscr{H} = \{V^{\sim}: V \in \mathscr{V}\}$. By (3.5), $\mathscr{H}$ is a collection of open subsets of X^* which refines $\mathscr{G}$. Furthermore, if $(x, k) \in X^* \setminus \bigcup \mathscr{H}$, then $x \in E$ (cf. (3.6)). Hence $X^* \setminus \bigcup \mathscr{H}$ is a countable set. Therefore, $\mathscr{G}$ has a countable refinement, so X^* is Lindelöf.

(4.5) THEOREM. *A GO space having a G_δ-diagonal is hereditarily paracompact.*

Proof. Suppose X is a GO space having a G_δ-diagonal. Since any subspace of X is again a GO space with a G_δ-diagonal, it suffices to show that X is paracompact. Because a subparacompact collectionwise normal space is paracompact ([4]), it suffices to show that X is subparacompact. Burke [8] proved that the following condition is equivalent to subparacompactness in any space:

(*) If $\mathscr{U}$ is an open cover of X, then there is a sequence $\langle \mathscr{U}(n) \rangle$ of open covers of X such that for each $x \in X$ there is a set $U(x) \in \mathscr{U}$ and an integer $n = n(x)$ such that $\mathrm{St}(x, \mathscr{U}(n)) \subseteq U(x)$.

We verify that X satisfies condition (*).

Let $\mathscr{U}$ be an open cover of X by convex sets. Let $E = \{x \in X$: no element of $\mathscr{U}$ contains points on both sides of $x\}$. For each $y \in X$, choose $U(y) \in \mathscr{U}$ such that $y \in U(y)$ and such that if $y \notin E$, then $U(y)$ contains points on both sides of y. By (4.4), for each $y \in X$ there is an open set $G(y)$ such that $y \in G(y)$ and such that $G(y) \cap E \subseteq \{y\}$. We may also require that $G(y) \subseteq U(y)$.

Since X has a G_δ-diagonal, there is a sequence $\langle \mathscr{H}(n) \rangle$ of open covers of X such that for each $x \in X$, $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{H}(n)) = \{x\}$. We may assume that $\mathscr{H}(n) = \{H(n, x): x \in X\}$, where each $H(n, x)$ is a convex open set containing x and where $H(n+1, x) \subseteq H(n, x)$ for each $x \in X$ and for each $n \geq 1$.

Let $\mathscr{U}(n) = \{G(x) \cap H(n, x): x \in X\}$ for each $n \geq 1$. Clearly each $\mathscr{U}(n)$ is an open cover of X. We verify that $\langle \mathscr{U}(n) \rangle$ satisfies the condition given in (*). Let $y \in X$. If $y \notin E$, there are points $a < y < b$ such that $[a, b] \subseteq U(y)$. Choose $m \geq 1$ such that neither a nor b is a point of $\mathrm{St}(y, \mathscr{H}(m))$. Since the elements of $\mathscr{H}(m)$ are convex sets, $\mathrm{St}(y, \mathscr{H}(m))$

$\subseteq [a, b]$. Then, since $\mathscr{U}(m)$ refines $\mathscr{H}(m)$,

$$\mathrm{St}(y, \mathscr{U}(m)) \subseteq \mathrm{St}(y, \mathscr{H}(m)) \subseteq [a, b] \subseteq U(y) \in \mathscr{U}.$$

If $y \in E$, then $y \notin G(x)$ whenever $x \neq y$ because $G(x) \cap E \subseteq \{x\}$. Therefore, $\mathrm{St}(y, \mathscr{U}(1)) = G(y) \cap H(1, y) \subseteq G(y) \subseteq U(y) \in \mathscr{U}$, as required.

Another sufficient condition for paracompactness in a GO space is perfect normality. Our proof depends upon the following sequence of lemmas.

(4.6) LEMMA. *Suppose X is a perfect topological space* (6). *Suppose there is a sequence $\langle \mathscr{G}(n) \rangle$ of collections of open subsets of X with the property that, given distinct points x, y of X, there is an integer $n \geqslant 1$ such that $x \in \mathrm{St}(x, \mathscr{G}(n)) \subseteq X \setminus \{y\}$. Then X has a G_δ-diagonal.*

Proof. The proof of (4.6) parallels the proof of Theorem 1 in [5].

(4.7) LEMMA. *Let X be a perfectly normal GO space having a left endpoint p and let $\mathscr{U}$ be an open cover of X by convex sets which contain p. Then $\mathscr{U}$ has a point-countable open refinement.*

Proof. Let τ be the topology on X. If X has a right endpoint, then a single element of $\mathscr{U}$ suffices to cover X, so we assume that X has no right endpoint. Let $S = \{x_a \colon 0 \leqslant a < A\}$ be a well-ordered increasing cofinal subsets of X, where A is an initial ordinal. We may assume that $x_0 = p$ and that for each infinite limit ordinal $\mu < A$, either $\sup\{x_a \colon 0 \leqslant a < \mu\} = x_\mu$ or else the supremum of the set $\{x_a \colon 0 \leqslant a < \mu\}$ is an interior gap (7) v_μ of $(X, \leqslant)$. Then S is a closed subset of X.

Let $L = \{\mu < A \colon \mu$ is a limit ordinal and $\sup\{x_a \colon 0 \leqslant a < \mu\} = x_\mu\}$. Let $S(L) = \{x_\mu \colon \mu \in L\}$. Then $S(L)$ is closed in X, so there are open sets $V(1) \supseteq V(2) \supseteq \ldots$ in X with $S(L) = \bigcap_{n=1}^{\infty} V(n)$. Let $\mu \in L$. If $[x_\mu, \rightarrow[$ is open, define $f(n, \mu) = \mu$ for each $n \geqslant 1$. If $[x_\mu, \rightarrow[$ is not open in X, define $f(n, \mu)$ to be the first ordinal a such that $a < \mu$ and $]x_a, x_\mu] \subseteq V(n)$.

For $n \geqslant 0$, define a collection $\mathscr{W}(n)$ of relatively open subsets of S as follows:

$$\mathscr{W}(0) = \{\{x_a\} \colon a \notin L \text{ or } (a \in L \text{ and } [x_a, \rightarrow[\text{ is open in } X)\}$$

and for $n \geqslant 1$, let

$$\mathscr{W}(n) = \{]x_{f(n,\mu)}, x_{\mu+1}[\cap S \colon \mu \in L \text{ and } [x_\mu, \rightarrow[\text{ is not open}\}.$$

---

(6) A topological space X is *perfect* if every closed subset of X is a G_δ in X.

(7) An *interior gap* of a linearly ordered set $(X, <)$ is a pair (A, B) of non-void convex subsets of X such that:

(i) $A \cup B = X$;

(ii) if $a \in A$ and $b \in B$, then $a < b$;

(iii) A has no last element and B has no first element. If (A, B) is an interior gap of X and $x \in X$, we write $x < (A, B)$ (respectively $x > (A, B)$) to mean that $x \in A$ (respectively, $x \in B$).

It is easily verified that the family $\{\mathscr{W}(n): n \geqslant 0\}$ satisfies the hypotheses of Lemma (4.6). Since (S, τ_S) is perfectly normal, it follows that (S, τ_S) has a G_δ-diagonal. By (4.5), (S, τ_S) is paracompact.

Since $\mathscr{U}$ is an open cover of (S, τ_S), there is a collection $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}(n)$ of closed subsets of (S, τ_S) which refines $\mathscr{U}$ and which is σ-discrete in (S, τ_S). Since S is closed in X, $\mathscr{F}$ is also σ-discrete in X. Since X is collectionwise normal, there is a collection $\bigcup_{n=1}^{\infty} \mathscr{V}(n)$ of open subsets of X which covers S, refines $\mathscr{U}$, and is σ-discrete in X. Define a collection $\mathscr{V}(0) = \{]x_a, x_{a+1}[: 0 \leqslant a < A\} \cup \{]v_\mu, x_\mu[: \mu < A$ is a limit ordinal which is not in $L\}$. Then $\bigcup_{n=0}^{\infty} \mathscr{V}(n)$ is a point-countable open cover of X which refines $\mathscr{U}$.

(4.8) THEOREM. *A perfectly normal GO space is paracompact.*

Proof. Let X be a perfectly normal GO space and let $\mathscr{U}$ be an open cover of X by convex sets. By (4.2), it suffices to show that $\mathscr{U}$ has a point-countable open refinement.

Consider first the special case where $\mathscr{U}$ is a coherent collection. By (3.10) there is a countable subset C of X such that $X = \bigcup \{\mathrm{St}(x, \mathscr{U}): x \in C\}$. Therefore, it suffices to show that whenever $x \in C$, it is possible fo find a point-countable collection $\mathscr{V}(x)$ of open subsets of X which covers $\mathrm{St}(x, \mathscr{U})$ and which refines $\mathscr{U}$. But this is certainly possible, as Lemma (4.7) and its obvious left-handed analogue show when applied to the spaces $\mathrm{RSt}(x, \mathscr{U}) = \mathrm{St}(x, \mathscr{U}) \cap [x, \rightarrow[$ and $\mathrm{LSt}(x, \mathscr{U}) = \mathrm{St}(x, \mathscr{U}) \cap$ $\cap\,]\leftarrow, x]$, respectively.

Now consider the general case. Let $\{\mathscr{U}_a: a \in A\}$ be the family of maximal coherent subcollections of $\mathscr{U}$. Let $X_a = \cup \mathscr{U}_a$ and topologize X_a as a subspace of X. Each X_a is an open subset of X and distinct X_a's are disjoint. Furthermore, $\mathscr{U}_a$ is a coherent open cover of the perfectly normal GO space X_a. Using the first part of the proof, we find point-countable collections $\mathscr{V}_a$ of open subsets of X_a which cover X_a and which refine $\mathscr{U}_a$. Since each X_a is open in X, the collection $\mathscr{V} = \bigcup \{\mathscr{V}_a: a \in A\}$ is an open cover of X which refines $\mathscr{U}$ and which is point-countable because distinct X_a's are disjoint.

(4.9) Remark. One might reasonably ask whether (4.8) could be derived from the corresponding result for LOTS ([6]) using the embedding $X \to X^*$, i.e., whether X^* must be perfectly normal whenever X is. Example 7.2 provides a negative answer to this question.

([6]) That a perfectly normal LOTS is paracompact seems to be part of the folklore of LOTS; it was first pointed out to the author by Mary Ellen Rudin.

The final theorem in this section provides yet another sufficient condition for paracompactness in a GO space.

(4.10) THEOREM. *Suppose that* $(X, \leqslant)$ *is a linearly ordered set. Let* $\lambda = \lambda(\leqslant)$. *Then the following are equivalent*:

(a) (X, λ) *is hereditarily paracompact*;

(b) *whenever* τ *is a GO topology on* $(X, \leqslant)$, *the space* (X, τ) *is hereditarily paracompact*;

(c) *whenever* τ *is a GO topology on* $(X, \leqslant)$, *the space* (X, τ) *is paracompact.*

Proof. We show that (a) implies (b) and that (c) implies (a).

(a) $\to$ (b). It suffices to show that if $\mathscr{U}$ is any collection of convex τ-open sets, then there is a collection of (relatively) open subsets of $X_0 = \bigcup \mathscr{U}$ which covers X_0 and which is σ-locally finite with respect to the topology $\tau_0 = \tau_{X_0}$ on X_0.

Let $E = \{x \in X$: no element of $\mathscr{U}$ contains points on both sides of $x\}$. By (4.4) – applied to the GO space X_0 – the collection $\mathscr{E} = \{\{x\}: x \in E\}$ is discrete in (X_0, τ_0). Since X_0 is collectionwise normal, there is a collection $\mathscr{V}(0) = \{V(x): x \in E\}$ of τ_0-open sets which refines $\mathscr{U}$, is discrete in X_0 and for which $x \in V(x)$ for each $x \in E$. Let $X_1 = X_0 \setminus \bigcup \mathscr{V}(0)$. Then X_1 is τ_0-closed. If $x \in X_1$, then $x \notin E$ so there is a λ-open convex set $W(x)$ containing x and which is contained in some element of $\mathscr{U}$. Let $\mathscr{W} = \{W(x) \cap \cap X_1: x \in X_1\}$. Then $\mathscr{W}$ is an open cover of the space (X_1, λ_{X_1}) which is paracompact by assumption. Hence there is a cover $\bigcup_{n=1}^{\infty} \mathscr{F}(n)$ of X_1 which refines $\mathscr{W}$ and which is σ-discrete in the space (X_1, λ_{X_1}). Since $\lambda_{X_1} \subseteq \tau_{X_1}$, $\bigcup_{n=1}^{\infty} \mathscr{F}(n)$ in σ-discrete in (X_1, τ_{X_1}), whence also in (X_0, τ_{X_0}) since X_1 is closed in (X_0, τ_{X_0}). Since (X_0, τ_{X_0}) is collectionwise normal, we may expand each $F \in \mathscr{F}(n)$ to an open set $V(n, F)$ of (X_0, τ_0) in such a way that the collection $\mathscr{V}(n) = \{V(n, F);\ F \in \mathscr{F}(n)\}$ is still discrete in (X_0, τ_0) and refines $\mathscr{U}$. Then $\mathscr{V} = \bigcup_{n=0}^{\infty} \mathscr{V}(n)$ is a σ-discrete open cover of (X_0, τ_0) which refines $\mathscr{U}$, as required to show that (a) implies (b).

(c) $\to$ (a). To show that (X, λ) is hereditarily paracompact, it suffices to show that if S is a convex open subset of X, then (S, λ_S) is paracompact (because every open subspace of (X, λ) is homeomorphic to a disjoint union of such spaces).

Let S be a convex open subset of (X, λ). By (c), the space (X, λ) is paracompact. Therefore, if S is also λ-closed, the subspace (S, λ_S) is paracompact. If S is not λ-closed, then S has at least one endpoint in X which does not belong to S, say $p = \sup(S)$ and $p \in X \setminus S$. If S has no infimum in X, or if S contains a first element, let τ be the topology on X having the collection $\lambda \cup \{\{p\}\}$ as a base. If S has an infimum q in X

and if $q \notin S$, let τ be the topology on X generated by $\lambda \cup \{\{p\}, \{q\}\}$. In either case, S is a τ-closed subspace of the GO space $(X, \tau, \leqslant)$. By (c), (X, τ) is paracompact. Therefore, so is (S, τ_S). But $\tau_S = \lambda_S$, so (S, λ_S) is paracompact, as required to show that (c) implies (a).

We conclude this section by remarking that simple examples show that the conditions given in (4.5), (4.8) and (4.10) are not necessary conditions for paracompactness in a GO space. See Example 7.4.

5. Metrizability and related properties in GO spaces

The reader is no doubt familiar with the following classical metrization theorems:

(A) Bing's theorem [7]: A regular space is metrizable iff it is collectionwise normal and developable (9);

(B) Bing-Nagata-Smirnov theorem [24]: A regular space is metrizable iff it has a σ-locally finite base.

In the class of LOTS, both of these results can be sharpened.

(A′) G. Creede [10] proved that a semi-stratifiable (10) LOTS is metrizable.

(B′) V. V. Fedorčuk [12] proved that a LOTS having a σ-locally countable base is metrizable.

In [16], the author improved Creede's theorem by showing that

(A″) A LOTS having a G_δ-diagonal is metrizable.

In this section, we shall prove (A′) and (B′) for GO spaces and we shall investigate a class of GO spaces for which (A″) is true. (Examples 7.2 and 7.3 make it clear that (A″) is false for arbitrary GO spaces.) Our generalization of (A′) will be obtained using the notion of a quasi-developable space which was introduced by E. E. Grace and studied by H. R. Bennett [5]. The concept has a certain independent interest because of its relation to the existence of a σ-point-finite base, and we include some results on quasi-developable GO spaces at the end of this section.

Let us begin by reviewing the definition of quasi-developability and by stating a basic result on quasi-developable spaces.

(5.1) Definition. A space X is *quasi-developable* if there is a countable family Ψ of collections of open subsets of X with the property that if U

(9) Defined in (5.1).

(10) A space X is *semi-stratifiable* if each open subset U of X can be written as $U = \bigcup\{U(n) : n \geqslant 1\}$, where each $U(n)$ is closed in X, in such a way that if $U \subseteq V$ are open, then $U(n) \subseteq V(n)$ for each n. If X is metrizable, developable, semi-metrizable or stratifiable, then X is semi-stratifiable. Any semi-stratifiable space is perfect and has a G_δ-diagonal. These spaces were first defined by E. Michael and have been extensively studied by G. Creede ([10] and [11]).

is open and if $p \in U$, then $p \in \mathrm{St}(p, \mathscr{G}) \subseteq U$ for some $\mathscr{G} \in \Psi$. If, in addition, each element of Ψ covers the space X, then X is *developable.*

(5.2) THEOREM ([5]). *A regular space is developable iff it is quasi-developable and perfect* (6).

(5.3) THEOREM. *A semi-stratifiable* (10) *GO space is metrizable.*

Proof. Let X be a semi-stratifiable GO space. Then X is perfect, so by (5.2) it suffices to show that X is quasi-developable.

Being a perfect GO space, X is first countable (11) and so is semi-metrizable [11]. By a theorem of R. W. Heath [15], there is a function G: $N \times X \to \{\text{open subsets of } X\}$ such that if $p \in X$

(i) $G(1, p) \supseteq G(2, p) \supseteq \ldots$ is a local base at p;

(ii) if $\langle q(n) \rangle$ is a sequence of points of X such that $p \in G\big(n, q(n)\big)$ for each $n \geqslant 1$, then $\langle q(n) \rangle$ converges to p.

Clearly, we may also assume that:

(iii) each $G(n, p)$ is convex in X;

(iv) if $[p, \to[$ is open in X, then $G(n, p) \subseteq [p, \to[$ and if $]\leftarrow, p]$ is open in X, then $G(n, p) \subseteq\ ]\leftarrow, p]$.

Let $\mathscr{G}(n) = \{G(n, p)\colon p \in X\}$. Since every semi-stratifiable space has a G_δ-diagonal, we may assume that

(v) for each $p \in X$, $\bigcap_{n=1}^{\infty} \mathrm{St}\big(p, \mathscr{G}(n)\big) = \{p\}$.

Let τ be the topology of X and let λ be the open interval topology of the given order on X. Define four subsets of X as follows:

$A = \{x \in X\colon \{x\} \in \tau\}$;
$B = \{x \in X \setminus A\colon [x, \leftarrow [\ \in \tau \setminus \lambda\}$;
$C = \{x \in X \setminus A\colon]\leftarrow, x] \in \tau \setminus \lambda\}$;
$D = X \setminus (A \cup B \cup C)$.

Observe that if $p \in B$, then no sequence of points of $]\leftarrow, p[$ can converge to p, so there is an integer $M(p)$ such that $p \notin \bigcup \{G(n, q)\colon q < p\}$ whenever $n \geqslant M(p)$. Furthermore, if $q \in B$ and $q > p$, then $p \notin\ ]p, \to[\ \supseteq G(n, q)$ for each $n \geqslant 1$ (by (iv)). Therefore, $p \notin \bigcup \{G(n, q)\colon q \in B \setminus \{p\}\}$ whenever $n \geqslant M(p)$. Similarly, if $p \in C$ there is an integer $N(p)$ such that $p \notin \bigcup \{G(n, q)\colon q \in C \setminus \{p\}\}$ whenever $n \geqslant N(p)$.

Define four subcollections of each collection $\mathscr{G}(n)$ by

$\mathscr{G}_1(n) = \{G(n, a)\colon a \in A\}$,
$\mathscr{G}_2(n) = \{G(n, b)\colon b \in B\}$,
$\mathscr{G}_3(n) = \{G(n, c)\colon c \in C\}$,
$\mathscr{G}_4(n) = \{G(n, d)\colon d \in D\}$.

(11) If a point p of a GO space X is a G_δ in X, then p has a countable local base in X. Therefore, a perfect GO space must be first countable.

We claim that the family $\Psi = \{\mathscr{G}_i(n): 1 \leqslant i \leqslant 4 \text{ and } n \geqslant 1\}$ is a quasi-development for X. For suppose U is open in X and $p \in U$. If $p \in A$, it follows from (iv) that $G(n, p) = \{p\}$ and that if $q \neq p$ is also a point of A, then $p \notin G(n, q)$ for each $n \geqslant 1$. Hence $p \in \mathrm{St}(p, \mathscr{G}_1(1)) = \{p\} \subseteq U$. If $p \in B$, then whenever $n \geqslant M(p)$ and $q \in B \setminus \{p\}$, we know that $p \notin G(n, q)$, whence $\mathrm{St}(p, \mathscr{G}_2(n)) = G(n, p)$. By (i), the sets $G(n, p)$ form a local base at p, so for n sufficiently large, $p \in \mathrm{St}(p, \mathscr{G}_2(n)) = G(n, p) \subseteq U$. The case where $p \in C$ is similar. Suppose $p \in D$. If p is not an endpoint of X, there are points $x, y \in X$ such that $p \in]x, y[\subseteq U$. By (v) and (i), there is an integer N such that neither x nor y is a point of $\mathrm{St}(p, \mathscr{G}(N)) \supseteq \mathrm{St}(p, \mathscr{G}_4(N))$. Since the latter set is a convex subset of X which contains p but does not contain either x or y, we have $p \in \mathrm{St}(p, \mathscr{G}_4(N)) \subseteq]x, y[\subseteq U$. The case where $p \in D$ is an endpoint of X is only slightly different. Therefore, Ψ is a quasi-development for the space X.

We turn now to the Fedorčuk metrization theorem.

(5.4) THEOREM. *A GO space with a σ-locally countable base is metrizable.*

Proof. Let X be a GO space having a σ-locally countable base $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}(n)$. By (B′) above, it suffices to show that the LOTS X^* also has a σ-locally countable base.

For each $n \geqslant 1$, let $A(n) = \{x \in X: \exists G \in \mathscr{B}(n) \text{ with } x \in G \subseteq [x, \rightarrow[\}$ and let $B(n) = \{x \in X: \exists G \in \mathscr{B}(n) \text{ with } x \in G \subseteq]\leftarrow, x]\}$. Let $y \in X$. Since $\mathscr{B}(n)$ is locally countable, there is an open subset $J(n, y)$ of X which meets at most countably many elements of $\mathscr{B}(n)$. We may assume that each $J(n, y)$ is an interval in X. We claim that the set $J(n, y) \cap (A(n) \cup B(n))$ is countable. For each $x \in J(n, y) \cap A(n)$, choose $G(x) \in \mathscr{B}(n)$ such that $x \in G(x) \subseteq [x, \rightarrow[$. Observe that if x and x' are distinct elements of $J(n, y) \cap \cap A(n)$, say with $x < x'$, then $x \in G(x) \setminus [x', \rightarrow[\subseteq G(x) \setminus G(x')$. Hence $G(x) \neq G(x')$, so the correspondence $X \rightarrow G(x)$ is one-to-one from the set $J(n, y) \cap A(n)$ into the set $\{G \in \mathscr{B}(n): G \cap J(n, y) \neq \emptyset\}$. Since the latter set is countable, so is $J(n, y) \cap A(n)$. Similarly, $J(n, y) \cap B(n)$ is countable.

Define collections $\mathscr{C}(n) = \{G^{\sim}: G \in \mathscr{B}(n)\} \cup \{\{(x, k)\}: (x, k) \in X^* \setminus X \text{ and } x \in A(n) \cup B(n)\}$. We claim that each $\mathscr{C}(n)$ is locally countable in X^*. To verify this, let $(y, m) \in X^*$. If $m \neq 0$, then $\{(y, m)\}$ is a neighborhood of (y, m) in X^* which meets an element $G^{\sim}$ of $\mathscr{C}(n)$ only if $y \in G \in \mathscr{B}(n)$ (cf. (3.1)). Since y is a point of at most countably many elements of $\mathscr{B}(n)$, the set $\{(y, m)\}$ meets only countably many elements of $\mathscr{C}(n)$. If $m = 0$, let $J = J(n, y)^{\sim}$. Then J is a neighborhood of $(y, 0)$ in X^*. Since $J \cap G^{\sim} \neq \emptyset$ only if $J(n, y) \cap G \neq \emptyset$, it suffices to show that J contains only countably many points $(x, k) \in X^* \setminus X$ with $x \in A(n) \cup B(n)$.

But this is certainly the case, because $\{x \in A(n) \cup B(n) \colon (x, k) \in J \setminus X\} \subseteq J(n, y) \cap (A(n) \cup B(n))$ and the latter set is countable. Therefore, $\mathscr{C}(n)$ is locally countable in X^*.

To complete the proof, let us show that $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}(n)$ is a base for the topology of X^*. Let $(x, k) \in I$, where I is a convex open subset of X^*. If $k \neq 0$, then $x \in A(n) \cup B(n)$ for some $n \geqslant 1$ whence $\{(x, k)\} \in \mathscr{C}(n)$ and $\{(x, k)\} \subseteq I$. If $k = 0$, then the set $I \cap X$ is a neighborhood of x in X. Since $\mathscr{B}$ is a base for X, there is an integer $n \geqslant 1$ and a set $G \in \mathscr{B}(n)$ such that $x \in G \subseteq I \cap X$. Since I is convex in X^*, it follows from (3.5) that $(x, 0) \in G^{\sim} \subseteq I$. Since $G^{\sim} \in \mathscr{C}(n)$, $\mathscr{C}$ is a base for X^*.

Our next proposition is an immediate consequence of the proof of (5.4). However, it can be established without reference to the Fedorčuk metrization theorem.

(5.5) Proposition. *Let X be a GO space. If X is metrizable, so is X^*.*

Proof. Let $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}(n)$ be a σ-locally *finite* base for X. Define $A(n)$ and $B(n)$ as in the proof of (5.4) and define

$$\mathscr{C}(n, m) = \{G^{\sim} \colon G \in \mathscr{B}(n)\} \cup \{\{(x, k)\} \colon (x, k) \in X^* \setminus X,\ x \in A(n) \cup B(n) \text{ and } |k| \leqslant m\}.$$

If the intervals $J(n, y)$ in the proof of (5.4) are chosen to meet only finitely many elements of $\mathscr{B}(n)$, then the rest of the proof of (5.4) shows that the collection $\mathscr{C} = \bigcup_{m,n=1}^{\infty} \mathscr{C}(n, m)$ is a σ-locally finite base for X^*.

(5.6) Corollary. *Let X be a GO space. If X is separable metrizable, then so is X^*.*

Proof. This follows immediately from (5.5) and (4.4), which asserts that if X is Lindelöf, then so is X^*.

(5.7) Remark. Example 7.2 shows that a GO space X may be either hereditarily Lindelöf or separable and yet the associated LOTS X^* may be neither. Corollary (5.6) shows, by way of contrast, that if X is second countable, then so is X^*.

Let us turn our attention to GO spaces which have a G_δ-diagonal. In [16] we proved that a LOTS with a G_δ-diagonal is metrizable. We present another proof here which is simultaneously more direct and applicable to a wider class of spaces, namely the p-embedded subspaces of LOTS.

(5.8) Definition ([1] and [11]). Suppose X is (homeomorphic to) a subspace of a topological space Y. Then X is said to be *p-embedded in Y* if there is a sequence $\langle \mathscr{U}(n) \rangle$ of covers of (the image of) X by open subsets of Y such that if $x \in X$, then $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{U}(n)) \subseteq X$. The sequence $\langle \mathscr{U}(n) \rangle$ is called a *pluming of X in Y.*

Clearly, if X is either an open subset of Y, or a G_δ-subset of Y, then X is p-embedded in Y. Spaces which are p-embedded in some compact Hausdorff space were introduced by A. V. Arhangel'skiĭ [1] under the name of *p-spaces* and have been extensively studied.

(5.9) THEOREM. *Let $(Y, \lambda, \leqslant)$ be a LOTS and let X be a p-embedded subspace of Y. Let τ be the relative topology on X. If (X, τ) has a G_δ-diagonal, then (X, τ) is metrizable.*

Proof. Let $\langle \mathscr{U}(n) \rangle$ be a pluming of X in Y. Since (X, τ) has a G_δ-diagonal, there is a sequence $\langle \mathscr{G}(n) \rangle$ of relatively open covers of X such that for each $x \in X$, $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{G}(n)) = \{x\}$. For each $x \in X$ and $n \geqslant 1$, we inductively choose convex (in Y) open subsets $V(n, x)$ of Y such that:

(i) $V(n, x)$ contains x;

(ii) $V(n, x) \cap X$ is contained in some element of $\mathscr{G}(n)$;

(iii) $V(n, x)$ is contained in some element of $\mathscr{U}(n)$;

(iv) if $n \geqslant 2$, then $V(n, x) \subseteq V(n-1, x)$.

Let $\mathscr{V}(n) = \{V(n, x)\colon x \in X\}$. For each $x \in X$, $\mathrm{St}(x, \mathscr{V}(n)) \subseteq \mathrm{St}(x, \mathscr{U}(n))$ so that $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{V}(n)) \subseteq X$. Furthermore, $\mathrm{St}(x, \mathscr{V}(n)) \cap X \subseteq \mathrm{St}(x, \mathscr{G}(n))$ so

$$x \in \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{V}(n)) = \bigcap_{n=1}^{\infty} [X \cap \mathrm{St}(x, \mathscr{V}(n))] \subseteq \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{G}(n)) = \{x\},$$

i.e.,

$$\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{V}(n)) = \{x\}.$$

Let $\mathscr{H}(n) = \{V \cap X\colon V \in \mathscr{V}(n)\}$. Because (X, τ) is collectionwise normal (4.1), the theorem will be proved if we show that $\langle \mathscr{H}(n) \rangle$ is a development for (X, τ). Let x be a point of a basic τ-open set $J \cap X$, where J is an open interval or a half line in Y. Consider the case where J is an open interval in Y, say $J =]a, b[$. (Observe that the points a and b need not be points of X.) Since $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{V}(n)) = \{x\}$, there is an integer n such that neither a nor b is a point of $\mathrm{St}(x, \mathscr{V}(n))$. Because $\mathrm{St}(x, \mathscr{V}(n))$ is a convex subset of Y containing $x \in]a, b[$ but containing neither a nor b, we conclude that $\mathrm{St}(x, \mathscr{V}(n)) \subseteq]a, b[$. Then $\mathrm{St}(x, \mathscr{H}(n)) = \mathrm{St}(x, \mathscr{V}(n)) \cap X \subseteq]a, b[\cap X$ as required.

The case where J is a half line in Y is analogous and even simpler. Hence $\langle \mathscr{H}(n) \rangle$ is a development for (X, τ).

The author does not know of any class of GO spaces which is larger that the class of p-embeddable subspaces of LOTS and for which the G_δ-diagonal metrization theorem is true.

We conclude Section 5 with a few results on quasi-developable GO spaces. The next proposition is known.

(5.10) PROPOSITION. *A LOTS is quasi-developable iff it has a σ-point finite base.*

A proof appears in [6]. We generalize (5.10) to the class of GO spaces as follows:

(5.11) PROPOSITION. *Let X be a GO space. Then the following are equivalent:*

(a) *X has a σ-point finite base;*

(b) *X^* is quasi-developable;*

(c) *X is quasi-developable.*

Proof. That (b) implies (a) follows from (5.10) and the fact that if X^* has a σ-point finite base, then so do its subspaces. A result of C. E. Aull [2], which says that any space with a σ-point finite base is quasi-developable, shows that (a) implies (c). We show that (c) implies (b). Let $\langle \mathscr{G}(n)\rangle$ be a quasi-development for X. We may assume that each $\mathscr{G}(n)$ is a collection of convex subsets of X. Define $\mathscr{H}(0) = \{\{(x, k)\}: (x, k)\in X^*$ and $k \neq 0\}$. For $n \geqslant 1$, let $\mathscr{H}(n) = \{G^{\sim} : G\in \mathscr{G}(n)\}$. Each $\mathscr{H}(n)$ is a collection of open subsets of X^*, and it follows easily from (3.5) that $\{\mathscr{H}(n): n \geqslant 0\}$ is a quasi-development for X^*.

(5.12) COROLLARY. *A quasi-developable GO space is hereditarily paracompact.*

Proof. Apply (4.2) and (5.11) (a).

Our final result in Section 5 gives a simple characterization of quasi-developable LOTS. It is reminiscent of the G_δ-diagonal metrization theorem for LOTS.

(5.13) PROPOSITION. *A LOTS Y is quasi-developable iff there is a sequence $\langle \mathscr{G}(n)\rangle$ of collections of open subsets of Y such that if $p \neq q$ are points of Y, there is an integer $n \geqslant 1$ with the property that $p\in \mathrm{St}(p, \mathscr{G}(n)) \subseteq X\setminus\{q\}$.*

Proof. To prove the non-trivial half of (5.13), suppose $\langle \mathscr{G}(n)\rangle$ exists. We may suppose that each $\mathscr{G}(n)$ is a collection of convex subsets of Y. For each pair $m, n \geqslant 1$, define $\mathscr{H}(m, n) = \{G \cap G' : G\in \mathscr{G}(m)$ and $G'\in \mathscr{G}(n)\}$. We will show that $\{\mathscr{H}(m, n): m, n \geqslant 1\}$ is a quasi-development for Y.

Suppose $p\in U$, where U is open in Y. If p is not an endpoint of Y, there are points $a, b\in Y$ with $p\in\,]a, b[\, \subseteq U$. By hypothesis, there are integers m' and n' such that $p\in \mathrm{St}(p, \mathscr{G}(m')) \subseteq X\setminus\{a\}$ and $p\in \mathrm{St}(p, \mathscr{G}(n')) \subseteq X\setminus\{b\}$. Then $\mathrm{St}(p, \mathscr{H}(m', n'))$ is a convex subset of Y containing p

but neither a nor b, so $\mathrm{St}(p, \mathscr{H}(m', n')) \subseteq\ ]a, b[\ \subseteq U$. As usual, the case where p is an endpoint of Y is even easier.

(5.14) Remark. (a) Because of the similarity of (5.13) to the G_δ-diagonal metrization theorem, the reader might conjecture that (5.13) could be proved for p-embedded subspaces of LOTS. This is, in fact, true, but the proof is more involved.

(b) Let us point out that, in spite of (5.12) and (5.13), a quasi-developable GO space need not have a G_δ-diagonal, as Example (7.3) shows.

6. Local compactness and the Σ-space property in GO spaces

The present section is divided into three parts. We begin by showing that if a GO space X is locally compact, then so is X^*. We then establish a result (6.5) which should be useful in studying normality in the product of two locally compact GO spaces. We conclude with some results concerning Nagami's Σ-spaces [22], showing that a locally compact GO space is a Σ-space and that a GO space X is a Σ-space iff X^* is a Σ-space.

Let us begin with an easy lemma.

(6.1) Lemma. *If a GO space $(X, \tau, \leqslant)$ is either compact or connected, then* $\tau = \lambda(\leqslant)$.

Proof. Suppose first that (X, τ) is compact. Then the function $1_X\colon (X, \tau) \to (X, \lambda(\leqslant))$ defined by $1_X(x) = x$ is a continuous, one-to-one mapping from the compact space (X, τ) to the Hausdorff space $(X, \lambda(\leqslant))$ and so is a homeomorphism. It follows that $\tau = \lambda(\leqslant)$.

Next suppose that (X, τ) is connected. For contradiction, suppose that $X^* \setminus X \neq \emptyset$. Let $(x, k) \in X^* \setminus X$. Then the sets $A = X \cap\,]\leftarrow, (x, k)]$ and $B = X \cap [(x, k), \rightarrow[$ are disjoint closed non-empty subsets of X which cover X; this is impossible since X is connected. Therefore, $X^* = X$, so $\tau = \lambda(\leqslant)$.

(6.2) Remark. In contrast to (6.1), a GO space may be either locally compact or locally connected without being a LOTS under any order. For example, consider $X =]0, 1[\ \cup \{2\}$ topologized as a subspace of $\boldsymbol{R}$ with the usual topology.

(6.3) Proposition. *If X is a locally compact GO space, then X is open in X^*.*

Proof. In this proof, we shall distinguish between X and $X \times \{0\}$, which is a subset of X^*. Let us begin by showing that if $x < y$ are points of X such that $[x, y]$ is compact (as a subspace of X), then, in X^*, $[(x, 0), (y, 0)] \subseteq X \times \{0\}$. For contradiction, suppose there is a point $(z, k) \in X^* \setminus X$ having $(x, 0) < (z, k) < (y, 0)$. We consider the case where $k < 0$. Then $x < z \leqslant y$ and z has no immediate predecessor in X. Furthermore, the set $[z, \rightarrow[$ is open in X so the set $J = [x, z[= [x, y] \setminus [z, \rightarrow[$ is compact. But this is impossible because, by (6.1), $\tau_J = \lambda(\leqslant_{|J})$, so $(J, \tau_J, \leqslant_J)$ is a compact LOTS and must, therefore, have a final point. The case where $k > 0$ is similar. Therefore, $[(x, 0), (y, 0)] \subseteq X \times \{0\}$.

Now suppose $(z, 0) \in X \times \{0\}$. Since (X, τ) is locally compact, there is an interval I in X which is a neighborhood of z in X and whose closure in X is compact. Letting x and y be the endpoints of the closure in X of I, we have $x \leqslant z \leqslant y$. Furthermore, even though the set $K = [x, y]$ may be strictly larger than the closure of I in X, K is still compact. From the first part of the proof, $[(x, 0), (y, 0)] \subseteq X \times \{0\}$. By (3.5), the set $I^{\sim}$ is a neighborhood of $(z, 0)$ in X^* and $I^{\sim} \subseteq [(x, 0), (y, 0)]$. Therefore $X \times \{0\}$ is open in X^*, as required.

(6.4) Corollary. *Let X be a GO space. If X is locally compact, then so is X^*, and conversely.*

Proof. Let $p \in X^*$. If $p \in X^* \setminus X$, then $\{p\}$ is a compact neighborhood of p. If $p \in X$, then there is a neighborhood U of p in X whose closure in X is compact. Since X is open in X^*, U must also be a neighborhood of p in X^* whose closure in X^* is compact. Therefore, X^* is locally compact.

The converse follows directly from (2.9), since a closed subspace of a locally compact space is locally compact.

Historically, local compactness in LOTS has been studied in connection with the theory of product spaces (e.g., [4] and [14]). Our next result should be useful in studying products of locally compact GO spaces, especially in the light of Hayashi's important paper [14].

(6.5) Proposition. *Let X and Y be locally compact GO spaces. Then $X \times Y$ is normal iff $X^* \times Y^*$ is normal.*

Proof. $\Leftarrow$: If $X^* \times Y^*$ is normal, then so is $X \times Y$ since $X \times Y$ is a closed subspace of $X^* \times Y^*$.

$\Rightarrow$: Suppose $X \times Y$ is normal. Let A and B be disjoint closed subsets of $X^* \times Y^*$. Let $Z_1 = X \times Y$, $Z_2 = (X^* \setminus X) \times Y$, $Z_3 = X \times (Y^* \setminus Y)$ and $Z_4 = (X^* \setminus X) \times (Y^* \setminus Y)$. By (6.3), X is open in X^* and Y is open in Y^*. Hence each set Z_i $(1 \leqslant i \leqslant 4)$ is open in $X^* \times Y^*$. Let $A_i = A \cap Z_i$ and $B_i = B \cap Z_i$ for $1 \leqslant i \leqslant 4$. Then A_i and B_i are relatively closed subsets of Z_i and are disjoint. Now Z_4, being the product of two discrete spaces, is certainly a normal space, and Z_1 is normal by hypothesis. The spaces Z_2 and Z_3 are each products of a discrete space and a GO space and so are normal (alternatively, apply (6.10), proved independently below). Hence there are disjoint relatively open subsets U_i and V_i of Z_i which contain A_i and B_i respectively, for $1 \leqslant i \leqslant 4$. Let $U = U_1 \cup U_2 \cup U_3 \cup U_4$ and let $V = V_1 \cup V_2 \cup V_3 \cup V_4$. Since each Z_i is open in $X^* \times Y^*$, U and V are open in $X^* \times Y^*$, and clearly $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Hence $X^* \times Y^*$ is normal.

It might be interesting to know whether (6.5) can be proved without assuming local compactness.

In [22], Nagami introduced a class of spaces, called Σ-spaces, which have particular relevance to product theory. Nagami proved that if X is a countable union of closed subspaces each of which is a Σ-space, then X is also a Σ-space, and that any (countably) compact space is a Σ-space. E. Michael pointed out that a paracompact, locally compact space is a Σ-space. He then asked whether an arbitrary locally compact Hausdorff space must be a Σ-space. The author observed that an example due to D. K. Burke [8] provides a negative answer to Michael's question ([12]). However, Burke's space is not normal and the question "Must a normal locally compact Hausdorff space be a Σ-space?" is still open. Proposition (6.8) below provides an affirmative answer to this question in the special case where X is a GO space.

The following definition is equivalent to the one given by Nagami in [22].

(6.6) DEFINITION. A *Σ-network* for a space X is a σ-locally finite closed cover $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}(n)$ of X such that for each $x \in X$,

(a) the set $C(x) = \bigcap \mathscr{S}(x, \mathscr{F})$ is countably compact (cf. (3.8));

(b) $\mathscr{F}$ contains an *outer network* for $C(x)$ in X, i.e., if U is an open subset of X which contains $C(x)$, then there is a set $F \in \mathscr{F}$ such that $C(x) \subseteq F \subseteq U$.

A space which has a Σ-network is called a *Σ-space* ([13]).

Our next lemma will be used in (6.8).

(6.7) LEMMA. *Let $\mathscr{K}$ be the family of all compact convex subsets of a GO space X, and let $\mathscr{C}$ be a maximal coherent subcollection of $\mathscr{K}$. If $p \in \bigcup \mathscr{C}$, then* $\mathrm{St}(p, \mathscr{K}) = \mathrm{St}(p, \mathscr{C}) = \bigcup \mathscr{C}$.

Proof. Let $Y = \bigcup \mathscr{C}$. It is clear that $\mathscr{S}(p, \mathscr{C}) \subseteq \mathscr{S}(p, \mathscr{K})$ and that $\mathscr{S}(p, \mathscr{K})$ is a coherent subcollection of $\mathscr{K}$. Since $\mathscr{C}$ is a maximal coherent subcollection of $\mathscr{K}$ and since $\mathrm{St}(p, \mathscr{K}) \cap (\bigcup \mathscr{C}) \neq \emptyset$, it follows from (3.9) that $\mathscr{S}(p, \mathscr{K}) \subseteq \mathscr{C}$. Hence $\mathscr{S}(p, \mathscr{K}) = \mathscr{S}(p, \mathscr{C})$. Therefore, $\mathrm{St}(p, \mathscr{K}) = \mathrm{St}(p, \mathscr{C})$, so it suffices to show that $Y \subseteq \mathrm{St}(p, \mathscr{C})$. We verify that $Y \cap [p, \rightarrow[\subseteq \mathrm{St}(p, \mathscr{C})$. If $Y \cap [p, \rightarrow[\nsubseteq \mathrm{St}(p, \mathscr{C})$, choose $q \in Y$ such that

([12]) Burke shows in [8] that his example is a locally compact Hausdorff space which is metacompact but not subparacompact. If this space were a Σ-space, then, in the notation of (6.6), each set $C(x)$ would be metacompact and countably compact, whence compact. By a result of E. Michael (cf. footnote ([13])), Burke's space would then have to be subparacompact.

([13]) If each set $C(x)$ is compact, then X is called a *strong Σ-space*. E. Michael proved that a regular Σ-space is a strong Σ-space iff it is subparacompact. It follows from (4.1) that a GO space is a strong Σ-space iff it is a paracompact Σ-space.

$q > p$, $q \notin \mathrm{St}(p, \mathscr{C})$ and $\mathrm{St}(q, \mathscr{C}) \cap \mathrm{St}(p, \mathscr{C}) \neq \emptyset$. (This is possible by (3.10).) Let $x \in \mathrm{St}(q, \mathscr{C}) \cap \mathrm{St}(p, \mathscr{C})$. Choose $I, J \in \mathscr{C}$ such that $\{p, x\} \subseteq I$ and $\{q, x\} \subseteq J$. Then $K = I \cup J$ is an element of $\mathscr{K}$, so $q \in K \subseteq \mathrm{St}(p, \mathscr{K}) = \mathrm{St}(p, \mathscr{C})$, contrary to our choice of q. Therefore, $Y \cap [p, \rightarrow[\ \subseteq \mathrm{St}(p, \mathscr{C})$. The proof that $Y \cap\,]\leftarrow, p] \subseteq \mathrm{St}(p, \mathscr{C})$ is analogous.

(6.8) PROPOSITION. *Every locally compact GO space is a Σ-space.*

Proof. Let $\mathscr{K}$ be the collection of all compact convex subsets of the locally compact GO space X, and let $p \in X$. We show first that $\mathrm{St}(p, \mathscr{K})$ is open in X. Suppose $q \in \mathrm{St}(p, \mathscr{K})$, and let I be a compact convex set which is a neighborhood of q. Since $q \in \mathrm{St}(p, \mathscr{K})$, there is a $J \in \mathscr{K}$ such that $\{p, q\} \subseteq J$. Then $K = I \cup J \in \mathscr{K}$ and $q \in I \subseteq K \subseteq \mathrm{St}(p, \mathscr{K})$. Hence $\mathrm{St}(p, \mathscr{K})$ contains a neighborhood of each of its points, so $\mathrm{St}(p, \mathscr{K})$ is open in X.

Let $\Psi = \{\mathscr{C}(a) \colon a \in A\}$ be the family of all maximal coherent subcollections of $\mathscr{K}$. For each $a \in A$, choose $p(a) \in \bigcup \mathscr{C}(a)$. By (6.7), $\mathrm{St}(p(a), \mathscr{K}) = \mathrm{St}(p(a), \mathscr{C}(a)) = \bigcup \mathscr{C}(a)$. Therefore, $\{\mathrm{St}(p(a), \mathscr{C}(a)) \colon a \in A\}$ is a disjoint open cover of X, so each set $\mathrm{St}(p(a), \mathscr{C}(a))$ is both open and closed in X. Since a disjoint union (topological sum) of Σ-spaces is again a Σ-space, it suffices to show that each subspace $\mathrm{St}(p(a), \mathscr{C}(a))$ of X is a Σ-space.

Fix $a \in A$. Let $p = p(a)$ and $\mathscr{C} = \mathscr{C}(a)$. Let $S = \mathrm{St}(p, \mathscr{C}) \cap [p, \rightarrow[$ and $T = \mathrm{St}(p, \mathscr{C}) \cap\,]\leftarrow, p]$. Then S and T are closed subsets of X.

Let us show that S is a Σ-space. It will suffice to show that S is a countable union of closed subspaces, each of which is a Σ-space. There are two cases to consider. First suppose that S contains a countable cofinal subset, i.e., a countable set $D \subseteq S$ with the property that for each $x \in S$, there is an element $d \in D$ with $x \leqslant d$. Since $D \subseteq \mathrm{St}(p, \mathscr{C})$, for each $d \in D$ we may choose a compact convex set $K(d) \in \mathscr{S}(p, \mathscr{C})$ which contains d. Then $S \subseteq \bigcup \{K(d) \colon d \in D\}$, so S is σ-compact, as required. Second, consider the case where no countable subset of S is cofinal. We claim that S itself is then countably compact. Let E be a countably infinite subset of S. Since E cannot be cofinal in S, there is a point $x \in S$ such that $E \subseteq\,]\leftarrow, x]$. Since $x \in S$, there is a $K \in \mathscr{S}(p, \mathscr{C})$ which contains x. Since K is convex, $E \subseteq K$. Because K is compact, some point of $K \cap [p, \rightarrow[$ is a limit point of the set E. Therefore, any infinite subset of S has a limit point in S, so S is countably compact. Therefore, S is a Σ-space.

A similar argument shows that T is also a Σ-space. Since $\mathrm{St}(p, \mathscr{C}) = S \cup T$, it follows that $\mathrm{St}(p, \mathscr{C})$ is a Σ-space, and the proof is complete.

As one application of (6.8), we state a result which follows immediately from (6.8) and two theorems of Nagami (Theorem 5 of [23] and Theorem 2.7 of [22]) which imply that if X is a collectionwise normal

Σ-space and Y is a σ-space ([14]) such that $X \times Y$ is normal, then $X \times Y$ is collectionwise normal.

(6.9) COROLLARY. *The product of a locally compact GO space and a metrizable space is collectionwise normal.*

Next we show that if a GO space X is a Σ-space, then so is X^*, and conversely. Our proof requires two lemmas.

(6.10) LEMMA. *Let X be a GO space and let $N \geqslant 0$ be fixed. For each subset F of X, let $F^\# = \{(x, k) \in X^*: x \in F$ and $|k| \leqslant N\}$. If F is closed in X, then $F^\#$ is closed in X^*, and if $\mathscr{F}$ is a locally finite collection of subsets of X, the collection $\mathscr{F}^\# = \{F^\#: F \in \mathscr{F}\}$ is locally finite in X^*.*

Proof. Both statements follow from the fact that if $S \subseteq X$ is convex in X and if $S^{\sim} \cap F^\# \neq \emptyset$, then $S \cap F \neq \emptyset$.

(6.11) LEMMA. *Let X be a GO space and let $N \geqslant 0$. Let C be a countably compact subspace of X and define $C^\#$ as in* (6.10). *Then $C^\#$ is a countably compact subspace of X^*.*

Proof. Let $E = \{(x_j, k_j): j \geqslant 1\}$ be a countably infinite subset of $C^\#$. We show that some point of $C^\#$ is a limit point of E.

Since $|k_j| \leqslant N$ for each j, the set $C' = \{x_j: j \geqslant 1\}$ is an infinite subset of C. Therefore, some point $y \in C$ is a limit point of C'. Let us show that the point $(y, 0)$, which is certainly a point of $C^\#$, is a limit point of E. Because y is a limit point of C', y is not an isolated point of X and either (i) every X-neighborhood of y contains points to the right of y and $]y, z[\cap C' \neq \emptyset$ whenever $z > y$, or else (ii) every X-neighborhood of y contains points to the left of y and $]z, y[\cap C' \neq \emptyset$ whenever $z < y$. We assume that (i) holds. Let U be any neighborhood of $(y, 0)$ in X^*. Then there is a point $(y', m') \in X^*$ such that $(y', m') > (y, 0)$ and $[(y, 0), (y', m')[\subseteq U$. Since $]\leftarrow, y]$ is not open in X, it is not possible that $y = y'$. Hence $y < y'$. Choose $x_j \in]y, y'[\cap C'$. Then $(x_j, k_j) \in E \cap [(y, 0), (y', m')[\subseteq E \cap U$. Therefore $(y, 0)$ is a limit point of E, as required.

(6.12) THEOREM. *Let X be a GO space. If X is a Σ-space, then so is X^*, and conversely.*

Proof. Suppose X is a Σ-space. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}(n)$ be a Σ-network for X. We may suppose that each $\mathscr{F}(n)$ covers X, is closed under finite intersections, and that for each $x \in X$, $C(x, n+1) = \bigcap \mathscr{S}(x, \mathscr{F}(n+1)) \subseteq C(x, n) = \bigcap \mathscr{S}(x, \mathscr{F}(n))$.

For $N \geqslant 1$, let $Y(N) = \{(x, k) \in X^*; |k| \leqslant N\}$. Then $X^* = \bigcup_{N=1}^{\infty} Y(N)$ and each $Y(N)$ is closed in X^*. Therefore it suffices to show that each $Y(N)$ is a Σ-space [22].

([14]) A space X is a *σ-space* if there is a σ-locally finite collection $\mathscr{F}$ of closed subsets of X with the property that if U is an open subset of X and if $p \in U$, then $p \in F \subseteq U$ for some $F \in \mathscr{F}$. Clearly a metrizable space is a σ-space.

Let $N \geqslant 1$ be fixed, and define collections $\mathscr{F}^{\#}(m) = \{F^{\#}: F \in \mathscr{F}(m)\}$ as in (6.10). Then each $\mathscr{F}^{\#}(m)$ is a locally finite closed cover of $Y(N)$. We will show that $\mathscr{F}^{\#} = \bigcup_{m=1}^{\infty} \mathscr{F}^{\#}(m)$ is a Σ-network for $Y(N)$. For $p \in Y(N)$ and for $m \geqslant 1$, let $K(p, m) = \bigcap \mathscr{S}(p, \mathscr{F}^{\#}(m))$ and let $K(p) = \bigcap \mathscr{S}(p, \mathscr{F}^{\#})$. If p is the point (x, j) of $Y(N)$, then $K(p, m) = C(x, m)^{\#}$ and $K(p) = C(x)^{\#}$. By (6.11) $K(p)$ is countably compact. Furthermore, $K(p, m) \in \mathscr{F}^{\#}(m)$ since $C(x, m) \in \mathscr{F}(m)$, so it suffices to show that $\{K(p, m): m \geqslant 1\}$ is an outer network for $K(p)$.

Suppose that U is an open subset of X^* which contains $K(p)$. Then $U \supseteq C(x)$ so there is an integer m' such that $C(x, m) \subseteq U$ whenever $m \geqslant m'$. For contradiction, suppose that $K(p, m) \nsubseteq U$ for each $m \geqslant 1$. Then each set $K(p, m) \setminus U$ is infinite, so we may choose a sequence of distinct points $(x_m, k_m) \in K(p, m) \setminus U$. Because $x_m \in C(x, m) \subseteq U$ whenever $m \geqslant m'$, $k_m \neq 0$ whenever $m \geqslant m'$.

Let $B = C(x) \cup \{x_m: m \geqslant m'\}$. Because $C(x)$ is countably compact and because every open subset W of X which contains $C(x)$ also contains all but finitely many points x_m, the set B is also countably compact. By (6.11), so is $B^{\#}$. Since points (y, j) of $B^{\#}$ with $j \neq 0$ are isolated in X^*, the set $E = C(x)^{\#} \cup \{(x_m, j): m \geqslant m'$ and $j = 0$ or $j = k_m\}$ is a closed subset of $B^{\#}$. Hence E is also countably compact. But then so is $E \setminus U = \{(x_m, k_m): m \geqslant m'\}$ which is impossible since $E \setminus U$ is an infinite discrete set. This contradiction establishes that for some $m \geqslant 1$. $K(p, m) \subseteq U$, as required.

The converse assertion is trivial, since any closed subspace of a Σ-space is again a Σ-space [22].

7. Examples

In this section we present several examples which have been referred to in earlier parts of the paper. The spaces of (7.2) and (7.3) are perhaps the most interesting examples of GO spaces.

(7.1) EXAMPLE. Let X be the set $[0, \Omega]$, where Ω is the first uncountable ordinal, and let $\leqslant$ be the usual ordering of X. Let τ be the topology on X obtained from the usual order topology by isolating every countable limit ordinal. Then $(X, \tau, \leqslant)$ is certainly a GO space.

Let Z be the set $([0, \Omega[\times \boldsymbol{Z}) \cup \{(\Omega, 0)\}$ and let $(\leqslant)$ be the lexicographic ordering on Z. Let λ be the open-interval topology of $(\leqslant)$. Let f be any one-to-one function from $[0, \Omega[$ onto $[0, \Omega[\times \boldsymbol{Z}$ (such functions exist since the sets have the same cardinality). Define $h: X \to Z$ by $h/[0, \Omega[= f$ and $h(\Omega) = (\Omega, 0)$. Then h is a homeomorphism from (X, τ) onto (Z, λ). However, since $X^* \setminus X \neq \emptyset$, there is no one-to-one function

$H: X^* \to Z$ which makes the mapping diagram in (2.11) commute. This example provides a negative answer to the first question in (2.12).

This space $(X, \tau, \leqslant)$ has several other interesting properties which we list. Compact subsets of X are finite and X is not a Σ-space. However, X is a P-space in the sense of Morita [24]. It is easily seen that X is Lindelöf and hereditarily paracompact, as is any finite power X^n of X. N. Noble [25] proved that $X^{\aleph_0}$, the countable product of copies of X, is Lindelöf.

(7.2) *The Sorgenfrey line.* Let $\boldsymbol{R}$ have its usual order $\leqslant$. Let σ be the topology on $\boldsymbol{R}$ having a base consisting of sets of the form $[a, b[$. The space $(\boldsymbol{R}, \sigma)$, called the Sorgenfrey line, is a separable GO space. By (2.10), the Sorgenfrey line is hereditarily separable and hereditarily Lindelöf. Furthermore, $(\boldsymbol{R}, \sigma)$ has a G_δ-diagonal and yet is not metrizable and does not have a σ-point finite base ([15]). (Cf. (5.14).) It follows from (5.9) that $(\boldsymbol{R}, \sigma)$ cannot be p-embedded in any LOTS; in particular, $\boldsymbol{R}$ is not a G_δ-subset of the LOTS $(\boldsymbol{R}, \sigma, \leqslant)^*$ which shows that the latter space is not perfect—and hence not separable — even though $(\boldsymbol{R}, \sigma)$ is both.

(7.3) Yet another line. Let $\leqslant$ be the usual ordering of $\boldsymbol{R}$ and let μ be the topology on $\boldsymbol{R}$ obtained from the usual topology by isolating each irrational. (This GO space has important uses in the study of normality in product spaces — cf. [19] and [20].) Clearly $(\boldsymbol{R}, \mu)$ has a G_δ-diagonal and is not metrizable or even perfect ([6]). Furthermore, $(\boldsymbol{R}, \mu)$ has a σ-point finite base, so $(\boldsymbol{R}, \mu)$ is quasi-developable. By (5.11), so is the LOTS $(\boldsymbol{R}, \mu, \leqslant)^*$. Since this latter space cannot be metrizable, we see that a LOTS can be quasi-developable without having a G_δ-diagonal, as claimed in (5.14). This space also shows that a space with a σ-point finite base need not have a uniform base ([16]).

(7.4) Example. **There are hereditarily paracompact GO spaces which do not satisfy the hypotheses of (4.5), (4.8) or (4.10) (which asserted, respectively, that the existence of a G_δ-diagonal, perfect normality, and hereditary paracompactness of the weaker open-interval topology are each sufficient conditions for hereditary paracompactness in a GO space).**

(a) Let X be the unit square with the lexicographic order topology. Being a paracompact first countable LOTS, X is hereditarily paracompact ([17]). Yet X, being non-metrizable, cannot have a G_δ-diagonal.

([15]) In fact, $(\boldsymbol{R}, \sigma)$ does not even have a point-countable base.

([16]) A base $\mathscr{B}$ for a space X is *uniform* if whenever $\mathscr{C}$ is an infinite subcollection of $\mathscr{B}$ each of whose elements contains a point p of X, then $\mathscr{C}$ is a local base at p. It is known that a regular space X has a uniform base iff X is developable and metacompact. Hence a regular space with a uniform base has a σ-point finite base, but the converse is true only in perfect spaces.

([17]) Sketch of proof: Using the characterization of paracompactness in LOTS given in [13], show that any convex subspace of a first countable paracompact LOTS is paracompact. Then argue as in footnote ([3]).

(b) The space of (7.3) is hereditarily paracompact by (5.12) but is not perfectly normal. The same is true for the lexicographic square in (a), above.

(c) Let Y be the set of countable ordinals with the usual order and discrete topology. Then Y is metrizable, while the countable ordinals with the usual topology are not even paracompact.

(7.5) EXAMPLE. The correspondence $X \to X^*$ is not functorial (cf. (2.12) (b)). Let X be the set $\boldsymbol{R}$ with its usual order. Let τ be the topology on X obtained from the usual topology by making the set $]\leftarrow, 0]$ open. Let $Y = \boldsymbol{R}\setminus\{1\}$ have its usual topology and order. Define $f\colon X \to Y$ and $g\colon Y \to X$ by

$$f(x) = \begin{cases} x & \text{if } x \leqslant 0, \\ x+1 & \text{if } 0 < x, \end{cases} \qquad g(y) = \begin{cases} y & \text{if } y \leqslant 0, \\ 0 & \text{if } 0 < y < 1, \\ y-1 & \text{if } 1 < y. \end{cases}$$

Then f and g are order-preserving continuous maps, and $g\circ f = 1_X$. Suppose that there were a way to define f^* and g^* so that $*$ became functorial and so that g^* extended g. Then $g^*\circ f^* = (g\circ f^*) = (1_X)^* = 1_{X^*}$. By (1.5), $Y^* = Y$ and $(0, 1)\in X^*\setminus X$. Since $f^*(0, 1)\in Y^* = Y$, we have $g^*\big(f^*(0, 1)\big)\in g^*[Y] = g[Y] \subseteq X$ because g^* is assumed to extend g. Therefore, $g^*\circ f^*(0, 1)$ cannot be $(0, 1)$, contradicting $g^*\circ f^* = 1_{X^*}$.

Observe that, for the spaces X and Y of this example, our spaces X^* and Y^* coincide with the extensions of X and Y which are defined in [9].

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