CONTRACTORS AND FIXED POINTS

BY

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1. Introduction. With a view to providing a unified approach to solving general equations in abstract spaces by iterative methods, Altman [1] introduced the theory of contractors (cf. also [2] and its references). For example, in this manner the Banach contraction principle (BCP) and Krasnosel'skiĭ's fixed point theorem have been unified. Further recent results of Reddy and Subrahmaniam [14]–[16] unify Altman's contractor theorem (Theorem 2.3 below) and fixed point theorems of Matkowski [10], Krasnosel'skiĭ [8] and Czerwik [5]. However, some nice generalizations of the BCP and Nadler's multivalued contraction principle [11] (e.g., Theorems 2.1 and 2.2 below) cannot be obtained from [1], [2] or [15]–[17].

Herein, we prove, in Section 3, a general contractor theorem which includes Altman's Theorem 2.3 and several fixed point theorems and other results for contractive type single- and multivalued operators.

The Mann iteration scheme to approximate solutions of operator equations and fixed points of contractive operators has been widely studied (see, e.g., [3], [6], [9], [13], [19]). In Section 4 an iteration scheme is introduced which generalizes the Mann iteration and we show that if it converges, then it converges to a solution (Theorem 4.1).

2. Contractors. Consistent with [12], p. 620, we will use the following notation where \( Y \) is a Banach space:

\[
\text{CL}(Y) = \{ A \subseteq Y : A \neq \emptyset \text{ and is closed} \}.
\]

For \( A, B \in \text{CL}(Y) \) and \( \varepsilon > 0 \),

\[
N(\varepsilon, A) = \{ y \in Y : \| y - a \| < \varepsilon \text{ for some } a \in A \},
\]

\[
E_{A,B} = \{ \varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A) \},
\]

\[
H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset, \\ +\infty & \text{if } E_{A,B} = \emptyset, \end{cases}
\]

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and for $y \in Y$

$$D(y, A) = \inf \{ ||y - a||: a \in A \}. $$

$H$ is called the generalized Hausdorff metric for $\text{CL}(Y)$.

The following is a Banach space version of Pal and Maiti’s result [14] (see also [19], p. 42):

**Theorem 2.1.** Let $T$ be an operator on a Banach space $Y$ such that, for any two elements $x, y \in Y$, at least one of the following is true:

\begin{align*}
(2.1) \quad ||x - Tx|| + ||y - Ty|| & \leq a ||x - y||, \quad 1 < a < 2; \\
(2.2) \quad ||x - Tx|| + ||y - Ty|| & \leq b (||x - Ty|| + ||y - Tx|| + ||x - y||), \quad 1/2 < b < 2/3; \\
(2.3) \quad ||Tx - Ty|| & \leq k \max \left\{ ||x - y||, ||x - Tx||, ||y - Ty||, \frac{||x - Ty|| + ||y - Tx||}{2} \right\}, \quad 0 < k < 1; \\
(2.4) \quad ||x - Tx|| + ||y - Ty|| + ||Tx - Ty|| & \leq c (||x - Tx|| + ||y - Ty||), \quad 1 < c < 3/2.
\end{align*}

Then $T$ has a fixed point.

**Theorem 2.2** (Čirić [4]). Let $Y$ be a Banach space and $F: Y \to \text{CL}(Y)$ satisfy

$$H(Fx, Fy) \leq k \max \left\{ ||x - y||, D(x, Fx), D(y, Fy), \frac{D(x, Fy) + D(y, Fx)}{2} \right\}$$

for all $x, y \in Y$ and some $k \in (0, 1)$. Then $F$ has a fixed point.

These theorems are proved for orbitally complete metric spaces (see [4] and [14]).

**Definition 2.1** (Altman [1]). Let $X$ and $Y$ be Banach spaces, $P: D(P) \subset X \to Y$ be a nonlinear map with domain $D(P)$ and $\Gamma(x): Y \to X$ be a bounded linear operator associated with $x \in X$. The map $P$ is said to have a contractor $\Gamma(x)$ if there is $k \in (0 < k < 1)$ such that

\begin{align*}
(2.5) \quad x + \Gamma(x) y & \in D(P) \quad \text{for } x \in D(P), \ y \in Y; \\
(2.6) \quad ||P(x + \Gamma(x) y) - Px - y|| & \leq k ||y|| \quad \text{for } x \in D(P), \ y \in Y.
\end{align*}

A contractor $\Gamma(x)$ is said to be regular if (2.6) is satisfied for all $y \in Y$ and $D(P) = \Gamma(x) Y$. The operator $P$ is said to be closed on $D(P)$ if the graph of $P$ is closed, i.e., if $x_n \in D(P)$, $x_n \to x$ and $Px_n \to y$, then $x \in D(P)$ and $y = Px$. In the case of a nonlinear multivalued operator $P: D(P) \to \text{CL}(Y)$, $P$ is called closed on $D(P)$ if $x_n \to x$, $y_n \in Px_n$ and $y_n \to y$ imply $x \in D(P)$ and $y \in Px$ (see [17]).
The following existence theorem, due to Altman, is fundamental to the theory of contractors.

**Theorem 2.3** ([1], p. 13). Suppose that the closed nonlinear operator \( P: D(P) \subseteq X \to Y \) has a bounded contractor \( \Gamma \) satisfying (2.5), (2.6) and

\[
\| \Gamma(x) \| \leq B \quad \text{for all } x \in D(P).
\]

Then the equation \( Px = y \) has a solution for each \( y \in Y \). When \( \Gamma \) is regular, (2.5) always holds and the solution is unique.

3. Multivalued mappings and general contractors. The following lemma (cf. [20]) will be used:

**Lemma 3.1.** Let \( A, B \in \text{CL}(Y) \) and \( a \in A \). Then for \( k \in (0, 1) \) and \( \lambda \in [0, 1) \) there exists \( b \in B \) such that

\[
\|a - b\| \leq k^{-\lambda} H(A, B).
\]

(Note that for \( k \in (0, 1) \) it is always possible to choose \( \lambda \in [0, 1) \) such that \( 1 \leq k^{-\lambda} \leq 2 \).

Let \( X \) and \( Y \) be Banach spaces, \( P: D(P) \subseteq X \to \text{CL}(Y) \) and \( \Gamma(x): Y \to X \) be a bounded linear operator. For convenience, define \( t_i = t_i(x, y) \), \( i = 1, \ldots, 5 \), for \( x \in D(P), y \in Y \) as follows:

\[
\begin{align*}
t_1 &= (P(x + \Gamma(x))y, y + Px), \\
t_2 &= (y, y - Px), \\
t_3 &= (x, x - \Gamma(x)(P(x + \Gamma(x))y)), \\
t_4 &= (y, -Px), \\
t_5 &= (x, x - \Gamma(x)(-y + P(x + \Gamma(x))y)).
\end{align*}
\]

**Theorem 3.1.** Suppose \( P: D(P) \subseteq X \to \text{CL}(Y) \) satisfies the following: there exists a bounded linear operator \( \Gamma(x): Y \to X \) associated with \( x \in X \) such that

\[
\begin{align*}
\| \Gamma(x) \| &\leq B, \quad B > 0, \ x \in D(P); \\
x + \Gamma(x)y &\in D(P) \quad \text{whenever } x \in D(P), \ y \in Y.
\end{align*}
\]

Further, for \( x \in D(P), y \in Y \), at least one of the following holds:

\[
\begin{align*}
Ht_2 + Ht_3 &\leq a\|y\|, \quad 1 < a < 1 + B; \\
Ht_2 + Ht_3 &\leq b [Dt_4 + Dt_5 + \|y\|], \quad (1 + B)^{-1} < b < 1 - B(1 + 2B)^{-1}; \\
Ht_1 &\leq k \max \left\{\|y\|, Dt_2, Dt_3, \frac{Dt_4 + Dt_5}{2}\right\}, \quad 0 < k < 1, \ Bk^{-\lambda} < 1, \text{ where } \lambda \in [0, 1) \text{ is such that } k^{-\lambda} \leq 2; \\
Ht_1 + Ht_2 + Ht_3 &\leq c [Dt_4 + Dt_5], \quad 1/B < c < 1 + B/2.
\end{align*}
\]

Then there exists \( z \in D(P) \) such that \( 0 \in Pz \).
Proof. Construct two sequences \( \{x_n\} \subseteq D(P) \) and \( \{y_n\} \subseteq Y \) as follows: Choose \( x_0 \in D(P) \) and \( y_0 \in Px_0 \). Set
\[
x_1 = x_0 - \Gamma(x_0)y_0 \in D(P)
\]
and note that \( 0 \in Px_0 - y_0 \). By Lemma 3.1, choose \( y_1 \in Px_1 \) such that
\[
\|y_1\| \leq k^{-1} H(Px_1, Px_0 - y_0).
\]
Set \( x_2 = x_1 - \Gamma(x_1)y_1 \) and choose \( y_2 \in Px_2 \) such that
\[
\|y_2\| \leq k^{-1} H(Px_2, Px_1 - y_1).
\]
Having chosen \( x_n \) and \( y_n \in Px_n \), let \( x_{n+1} = x_n - \Gamma(x_n)y_n \) and choose \( y_{n+1} \in Px_{n+1} \) such that
\[
(3.7) \quad \|y_{n+1}\| \leq k^{-1} H(Px_{n+1}, Px_n - y_n).
\]

Now for \( x = x_n \) and \( y = -y_n \) we will consider each of the cases (3.3)–(3.6). Frequent use will be made of the fact that, for any \( u \in D(P) \) and any \( v \in A \in CL(Y) \),
\[
\|u - v\| \leq H(u, A).
\]

In case (3.3) we obtain
\[
H(-y_n, -y_n - Px_n) + H(x_n, x_n - \Gamma(x_n)Px_{n+1}) \leq a\|y_n\|,
\]
that is, \( \|y_n\| + B\|y_{n+1}\| \leq a\|y_n\| \). Therefore
\[
(3.3a) \quad \|y_{n+1}\| \leq q_1 \|y_n\|, \quad \text{where} \quad q_1 = \frac{a - 1}{B}.
\]

In case (3.4) we have
\[
\|y_n\| + B\|y_{n+1}\| \leq b [\|y_n + y_{n+1}\| + \|y_n\|].
\]
Thus
\[
(3.4a) \quad \|y_{n+1}\| \leq q_2 \|y_n\|, \quad \text{where} \quad q_2 = \frac{b(1 + B) - 1}{B(1 - b)}.
\]

In case (3.5) we obtain
\[
H(Px_{n+1}, -y_n + Px_n) \leq k \max \left\{ \|y_n\|, D(-y_n, y_n - Px_n), D(x_n, x_n - \Gamma(x_n)Px_{n+1}), \frac{D(-y_n, -Px_n) + D(x_n, x_n - \Gamma(x_n)(y + Px_{n+1}))}{2} \right\}
\]
\[
\leq k \max \left\{ \|y_n\|, B\|y_{n+1}\|, \frac{B}{2}\|y_n + y_{n+1}\| \right\}.
\]
Using (3.7), this yields

\[(3.5a) \quad \|y_{n+1}\| \leq q_3 \|y_n\|, \quad \text{where} \quad q_3 = \max \left\{ k^{1-\lambda}, \frac{Bk^{1-\lambda}}{2-Bk^{1-\lambda}} \right\}. \]

Finally, in case (3.6) we have

\[
k^{1-\lambda} (\|y_n\| + B \|y_{n+1}\|) + \|y_{n+1}\|
\leq k^{1-\lambda} [Ht_1(x_n, -y_n) + Ht_2(x_n, -y_n) + Ht_3(x_n, -y_n)]
\leq k^{1-\lambda} c [0 + B \|y_n + y_{n+1}\|],
\]

whence

\[(3.6a) \quad \|y_{n+1}\| \leq q_4 \|y_n\|, \quad \text{where} \quad q_4 = \frac{Bc - 1}{k^{1-\lambda} - Bc + B}. \]

By (3.3a)-(3.6a), \(\|y_{n+1}\| \leq q \|y_n\|\) for all \(n\), where \(q = \max \{q_1, q_2, q_3, q_4\}\) < 1. Hence, since \(x_{n+1} = x_n - \Gamma (x_n) y_n\), \(\{x_n\}\) is a Cauchy sequence and \(x_n \to z\) and \(y_n \to 0\). Consequently, \(0 \in P_z\), since \(P\) is closed.

Several contractor theorems follow as corollaries to Theorem 3.1.

**Corollary 3.1.** If \(P\) and \(\Gamma(x)\) are as above and satisfy (3.1), (3.2) and

\[(3.8) \quad Ht_1 \leq k \max \left\{ \|y\|, D_{t_2}, D_{t_3}, \frac{D_{t_4} + D_{t_5}}{2} \right\} \]

for \(x \in D(P), y \in Y\) and some \(k \in (0, 1)\) such that \(Bk^{1-\lambda} < 1\) for \(\lambda \in [0, 1)\), then there exists \(z \in D(P)\) such that \(0 \in P_z\).

Note that in Theorem 3.1 and Corollary 3.1, if all members of \(C L(Y)\) are compact, then one may choose \(\lambda = 0\). Moreover, the conclusion of Corollary 3.1 remains true if (3.8) is replaced by

\[(3.9) \quad Ht_1 \leq \alpha \|y\| + \beta D_{t_2} + \gamma D_{t_3} + \delta (D_{t_4} + D_{t_5}) \]

for \(x \in D(P), y \in Y\) and nonnegative numbers \(\alpha, \beta, \gamma, \delta\) with \(0 < k = \alpha + \beta + \gamma + 2\delta < 1\). In fact, we obtain the following

**Corollary 3.2.** If \(P\) and \(\Gamma(x)\) are as above and satisfy (3.1), (3.2) and (3.9) for \(x \in D(P), y \in Y\) and \(\alpha, \beta, \gamma, \delta \geq 0\) such that

\(k = \alpha + \beta + \gamma + 2\delta < 1\) \quad and \quad \((\alpha + \beta + B(\gamma + 2\delta))k^{-\lambda} < 1\)

for some \(\lambda \in [0, 1)\), then there exists \(z \in D(P)\) such that \(0 \in P_z\).

Corollary 3.2 with \(\beta = \gamma = \delta = 0\) yields a multivalued version of Altman's Theorem 2.3. Also, the main result of Reddy and Subrahmanyan ([17], Theorem 3.1) with \(n = 1\) is obtained by setting \(\delta = 0\).

If the hypotheses (3.1)-(3.6) of Theorem 3.1 are satisfied, we may say that \(\Gamma(x)\) is a general contractor for \(P\) and that Theorem 3.1 is a multivalued
contractor theorem. Similarly, for $P$ and $\Gamma(x)$ satisfying the hypotheses of the next corollary, $\Gamma(x)$ may be called a general contractor of the single-valued mapping $P$ and the result a general contractor analog of Theorem 3.1.

**Corollary 3.3.** If $P: D(P) \subseteq X \rightarrow Y$ satisfies (3.1), (3.2) and, for $x \in D(P)$, $y \in Y$, at least one of the following

\begin{align*}
(3.10) \quad \|P x\| + \|\Gamma(x)(P(x + \Gamma(x)y))\| & \leq a \|y\|, \quad 1 < a < 1 + B; \\
(3.11) \quad \|P x\| + \|\Gamma(x)(P(x + \Gamma(x)y))\| & \leq b \left(\|y + P x\| + \|\Gamma(x)(-y + P(x + \Gamma(x)y))\| + \|y\|\right), \\
& \quad (1 + B)^{-1} < b < 1 - B (1 + 2B)^{-1}; \\
\end{align*}

\begin{align*}
(3.12) \quad \|t_1\| \leq k \max \left\{\|y\|, \|P x\|, \|\Gamma(x)P(x + \Gamma(x)y)\|, \\
\frac{\|y + P x\| + \|\Gamma(x)(-y + P(x + \Gamma(x)y))\|}{2}\right\}, & \quad 0 < k < 1, \ kb < 1,
\end{align*}

where $\|t_1\| = \|P(x + \Gamma(x)y) - y - P x\|;$

\begin{align*}
(3.13) \quad \|P x\| + \|\Gamma(x)(P(x + \Gamma(x)y))\| & + \|t_1\| \\
& \leq c \left(\|y + P x\| + \|\Gamma(x)(-y + P(x + \Gamma(x)y))\|\right), & 1/B < c < 1 + B/2.
\end{align*}

then $P x = 0$ has a solution in $D(P)$.

**Proof.** (3.3)–(3.6) reduce to (3.10)–(3.13) in the single-valued case.

**Corollary 3.4.** Suppose $P: D(P) \subseteq X \rightarrow Y$ is closed and satisfies (3.1), (3.2) and (3.12). Then $P x = 0$ has a solution in $D(P)$. Further, the solution is unique when $\Gamma$ is regular and $B \leq 1$.

**Proof.** A solution exists, so we need to show the uniqueness. Let $\alpha, \ beta$ be two solutions of $P x = 0$. Then $\ beta = \alpha + \Gamma(x)y$ for some $y \in Y$. It suffices to show that $y = 0$. If $y \neq 0$, then

$$\|\beta - \alpha\| \leq B \|y\|$$

and

$$\|y\| = \|P \beta - P \alpha - y\| = \|P(x + \Gamma(\alpha)y) - P \alpha - y\|$$

$$\leq k \max \left\{\|y\|, 0, 0, \frac{\|y\| + \|\Gamma(\alpha)(-y)\|}{2}\right\}.$$
we see by the above proof that it is not necessary that $B \leq 1$. Further, if $P$ is single-valued and $\Gamma$ is regular in Corollary 3.2, then the solution is unique provided $\delta(1 + B) < 1 - x$.

Now a fixed point theorem is derived from Theorem 3.1 which unifies several fixed point theorems including Theorems 2.1 and 2.2.

**Corollary 3.5.** Let $F: Y \to \text{CL}(Y)$ satisfy, for $x, y \in Y$, at least one of the following:

\[
H(x, Fx) + H(y, Fy) \leq a\|x - y\|, \quad 1 < a < 2;
\]

\[
H(x, Fx) + H(y, Fy) \leq b \left[ D(x, Fy) + D(y, Fx) + \|x - y\| \right],
\]

\[
1/2 < b < 2/3;
\]

\[
H(Fx, Fy) \leq k \max \left\{ \|x - y\|, D(x, Fx), D(y, Fy), \frac{D(x, Fy) + D(y, Fx)}{2} \right\}, \quad 0 < k < 1;
\]

\[
H(x, Fx) + H(y, Fy) + H(Fx, Fy) \leq c \left[ D(x, Fy) + D(y, Fx) \right],
\]

\[
1 < c < 3/2.
\]

Then $F$ has a fixed point.

**Proof.** Setting $\Gamma(x) = I$, the identity on $X = Y$, and $Px = x - Fx$ in Theorem 3.1, it is seen that $\{x_n\}$ converges to a fixed point of $F$.

Clearly, Theorem 2.2 is included in the above corollary, and setting $Fx = \{Tx\}$, for $T: Y \to Y$ we obtain Theorem 2.1. Further, setting $\Gamma(x) = I$, $X = Y$ and $Px = x - Fx$ in Corollary 3.1, a variant of fixed point theorems for multivalued operators due to Nadler [11] and Iseki [7] is obtained.

4. **Approximation of solutions.** In the single-valued general contractor theorem (Corollary 3.3) $\{x_n\}$, defined by $x_{n+1} = x_n - \Gamma(x_n)Px_n$, converges to a solution. This scheme was used by Altman in Theorem 2.3. We now introduce another scheme for a multivalued operator $P: D(P) \subseteq X \to \text{CL}(Y)$ (resp. single-valued operator $P: D(P) \subseteq X \to Y$), as follows:

\[
x_0 \in X;
\]

\[
x_{n+1} = x_n - \alpha_n \Gamma(x_n) y_n, \quad y_n \in P(x_n) \quad \text{(resp. } y_n = Px_n);\]

\[
0 \leq \alpha_n \leq 1, \quad \lim \alpha_n = \alpha > 0.
\]

We denote the sequence $\{x_n\}$ defined above by $M(\Gamma(x_0), \alpha_n, P)$.

For $X = Y$, $T: Y \to Y$, defining $Px = x - Tx$ and $\Gamma(x) = I$, we see that (4.2) is equivalent to

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,
\]
which is the Mann iterative process (see, e.g., [6], [18], [19]). A sequence \( \{x_n\} \) satisfying (4.1), (4.2') and (4.3) will be denoted by \( M(x_0, \alpha_n, T) \).

Recently, Kuhfittig [9] has studied the Mann iterative process for certain classes of multivalued operators; in fact, point-compact operators. In Corollary 4.1 we will consider point-closed operators satisfying a very general contractive type condition which may include operators studied in [9].

**Theorem 4.1.** Under the hypotheses of Theorem 3.1, \( M(\Gamma(x_0), \alpha_n, P) \) converges to a solution provided it converges, \( \Gamma(x) : Y \to X \) is invertible and \( \Gamma(x) \) is continuous.

**Proof.** Theorem 3.1 guarantees that \( P \) has a solution. Assume that the sequence \( M(\Gamma(x_0), \alpha_n, P) \) converges to \( z \). Then, since \( \lim \alpha_n = \alpha > 0 \) and \( \Gamma(x_n) \) has a bounded inverse, the equality

\[
\|x_{n+1} - x_n\| = \alpha_n \|\Gamma(x_n) y_n\|
\]

implies \( y_n \to 0 \). Thus \( 0 \in Pz \), since \( P \) is closed.

Evidently, not all the hypotheses are needed in the proof of Theorem 4.1. In fact, if \( P \) is any closed operator (single-valued or multivalued) on \( D(P) \) and if \( M(\Gamma(x_0), \alpha_n, P) \) converges to \( z \), then \( 0 \in Pz \) provided \( \Gamma(x) \) has a bounded inverse. This suggests that, while considering a special case of Theorem 4.1, one needs to require only those conditions which ensure that \( P \) is closed. Therefore we have the following:

Let \( X = Y \) be a normed space and \( C \subseteq X \) be closed and convex and \( F : C \to \text{CL}(C) \). Defining \( Px = x - Fx \) and \( \Gamma(x) = I \) and replacing (4.2) by

\[
(4.2'') \quad x_{n+1} = (1 - \alpha_n) x_n + \alpha_n p_n, \quad p_n \in Fx_n,
\]

a sequence defined by (4.1), (4.2'') and (4.3) will be denoted by \( M(p_0, \alpha_n, F) \).

**Corollary 4.1.** Suppose that \( M(p_0, \alpha_n, F) \) converges to \( z \in C \). If, for \( x, y \in M(p_0, \alpha_n, F) \cup \{z\} \), one of the following holds:

\[
(4.4) \quad H(x, Fx) + H(y, Fy) \leq a \|x - y\|, \quad 1 < a < 2;
\]

\[
(4.5) \quad H(x, Fx) + H(y, Fy) \leq b [D(x, Fx) + D(y, Fy) + \|x - y\|],
\]

\[1/2 < b < 2/3;\]

\[
(4.6) \quad H(Fx, Fy) \leq k \max \{t \|x - y\|, D(x, Fx) + D(y, Fy),
\]

\[D(x, Fx) + D(y, Fx)\}, \quad 0 < k < 1, \quad t > 0;
\]

\[
(4.7) \quad H(x, Fx) + H(y, Fy) + H(Fx, Fy)
\]

\[\leq c [D(x, Fy) + D(y, Fx)], \quad 1 < c < 3/2,
\]

then \( z \) is a fixed point of \( F \).

**Proof.** Since \( \{\alpha_n\} \) is bounded away from zero and \( x_n \to z \), it follows
from (4.2'') that
\[ \|x_n - p_n\| \to 0 \quad \text{and} \quad \|p_n - z\| \to 0. \]

Now suppose that (4.4) holds for the pair \( x = x_n, y = z \). Then
\[ (4.4a) \]
\[ \|x_n - p_n\| + D(z, Fz) \leq H(x_n, Fx_n) + H(z, Fz) \]
\[ \leq a\|x_n - z\|. \]

Similarly, if (4.5)–(4.7) are true, then correspondingly we obtain
\[ (4.5a) \]
\[ \|x_n - p_n\| + D(z, Fz) \leq b\left[\|x_n - z\| + D(z, Fz) + \|z - p_n\| + \|x_n - z\|\right], \]
\[ (4.6a) \]
\[ D(z, Fz) \leq \|z - p_n\| + H(Fx_n, Fz) \]
\[ \leq \|z - p_n\| + k \max\{t \|x_n - z\|, D(x_n, Fx_n) + D(z, Fz), D(x_n, Fz) + D(z, Fx_n)\}, \]
\[ \leq \|z - p_n\| + k \max\{t \|x_n - z\|, \|x_n - p_n\| + D(z, Fz), 2\|x_n - z\| + D(z, Fz) + \|z - p_n\|\}, \]
\[ (4.7a) \]
\[ \|x_n - p_n\| + 2D(z, Fz) \]
\[ \leq H(x_n, Fx_n) + H(z, Fz) + H(z, Fz) + H(Fx_n, Fz) + \|z - p_n\| \]
\[ \leq c\left[D(x_n, Fz) + D(z, Fx_n)\right] + \|z - p_n\| \]
\[ \leq c\left[\|x_n - z\| + D(z, Fz) + \|z - p_n\|\right] + \|z - p_n\|. \]

Letting \( n \to \infty \) in (4.4a)–(4.7a) we obtain \( D(z, fz) = 0 \), so \( z \in Fz \).

Finally, the following is a variant of a result of Rhoades ([19], Theorems 1 and 2):

**Corollary 4.2.** Let \( C \) be a closed, convex subset of a normed space, \( T: C \to C \) and \( M(x_0, \alpha_n, T) \) converge to \( z \in C \). If, for \( x, y \in M(x_0, \alpha_n, T) \cup \{z\} \), one of (2.1), (2.2), (2.4) and
\[ \|Tx - Ty\| \leq k \max\{t \|x - y\|, \|x - Tx\| + \|y - Ty\|, \|x - Ty\| + \|y - Tx\|\}, \]
\[ 0 < k < 1, \ t > 0, \]
holds, then \( z \) is a fixed point of \( T \).

**Proof.** Write \( Fx = \{Tx\}, x \in C \). Then
\[ M(x_0, \alpha_n, T) = M(p_0, \alpha_n, F) \]
and the result follows from Corollary 4.1.
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