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A WEIGHTED MULTIPLIER THEOREM FOR ROCKLAND OPERATORS

BY

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1. Introduction. A simply connected nilpotent Lie group G is called *graded* if its Lie algebra $\mathfrak{G}$ is endowed with a vector space decomposition $\mathfrak{G} = \bigoplus_{1}^{n} V_j$ such that $[V_i, V_j] \subseteq V_{i+j}$. Then $\mathfrak{G}$ and correspondingly G admits a one-parameter group of automorphisms δ_r, $r > 0$, called *dilations*. Let Q be the homogeneous dimension of G that is the positive number defined by

$$r^Q \int_G f(\delta_r x)\,dx = \int_G f(x)\,dx, \quad f \in L^1(G),\ r > 0.$$

A left-invariant differential operator L on G is called a *Rockland operator* if it is homogeneous with respect to the dilations and for every irreducible unitary representation π of G the operator $\pi(L)$ is injective on the space of C^∞-vectors. It is known, by Helfer and Nourrigat, that L is hypoelliptic and as such it is essentially self-adjoint on $C_c^\infty(G)$ in $L^2(G)$.

Let L be a positive Rockland operator on G and let $E(\lambda)$ be the spectral resolution of the identity for a positive self-adjoint extension of L,

$$Lf = \int_0^\infty \lambda\, dE(\lambda) f, \quad f \in \mathrm{Dom}(L).$$

If m is a bounded measurable function on $\boldsymbol{R}^+$ we write $m(L)$ for the multiplier operator

$$m(L) f = \int_0^\infty m(\lambda)\, dE(\lambda) f, \quad f \in L^2(G)$$

which is bounded on $L^2(G)$.

Recently a Hörmander–Myhlin-type multiplier theorem has been proved by A. Hulanicki and E. M. Stein for the case when L is a sublaplacian on a stratified group, cf. [3], and as remarked by A. Hulanicki in [4] it can be generalized to Rockland operators on graded groups. It says the following:

THEOREM A. *Let L be a positive Rockland operator on G. There exists an integer N such that if $m \in C^N(\boldsymbol{R}^+)$ satisfies*

$$\|m\|_N^* = \max_{0 \leqslant j \leqslant N} \sup_{\lambda > 0} |\lambda^j m^{(j)}(\lambda)| < \infty,$$

then for every p, $1 < p < \infty$, the operator $m(L)$ extends to the bounded operator on $L^p(G)$; moreover,

$$\|m(L)f\|_p \leqslant C\|m\|_N^* \|f\|_p, \quad f \in L^p \cap L^2(G)$$

with a constant $C = C(p)$ independent of m.

On the other hand, Kurtz and Wheeden [7], [8] have obtained weighted multiplier theorems for classical multipliers on $\boldsymbol{R}^n$.

The aim of this note is to prove a similar weighted multiplier theorem for a positive Rockland operator. To state our main result let us remind the following notions.

A topological space X equipped with a continuous pseudometric ϱ and a measure μ which, for a constant C, satisfies

$$\mu(B_{2r}(x)) \leqslant C\mu(B_r(x)), \quad x \in X, \ r > 0,$$

is called a *space of homogeneous type*. Here $B_r(x)$ denotes the ball, $B_r(x) = \{y: \varrho(x, y) < r\}$.

For a locally integrable function f on X we define the maximal function Mf of f by

$$Mf(x) = \sup_{r>0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y)|\, d\mu(y). \tag{1.1}$$

We say that a weight function $w(x)$, $w(x) > 0$ on X, belongs to the *Muckenhoupt class* $A_p(X)$ if for a constant C

$$\Big[\int_B w(x)\,d\mu\Big] \cdot \Big[\int_B w(x)^{-1/(p-1)}\,d\mu\Big]^{p-1} \leqslant C\mu(B)^p, \quad 1 < p < \infty,$$

$$\int_B w(x)\,d\mu \leqslant C\mu(B) \cdot \operatorname*{ess\,inf}_{x \in B} w(x), \quad p = 1,$$

for all balls B.

A. P. Calderón [1] has extended the Muckenhoupt theory, cf. [2], from the Euclidean to the general spaces of homogeneous type. In particular, the following weighted L^p-norm inequality for the maximal function (1.1) holds: for $1 < p < \infty$

$$\int_X Mf(x)^p w(x)\,d\mu \leqslant C \int_X |f(x)|^p w(x)\,d\mu \tag{1.2}$$

with a constant C independent of f if and only if $w \in A_p$.

On the graded group G there exists a continuous positive function $|\cdot|$ called a *homogeneous norm* such that $|x| = 0$ iff $x = e$, $|x| = |x^{-1}|$, $|xy| \leqslant$

$\gamma(|x|+|y|)$ for a $\gamma \geqslant 1$, and $|\delta_r x| = r|x|$, $r > 0$. G endowed with the Haar measure and the pseudometric generated by the homogeneous norm $d(x, y) = |y^{-1}x|$ is of homogeneous type.

For a positive weight w denote $w(E) = \int_E w(x)\,dx$, $E \subseteq G$, and

$$\|f\|_{p,w} = \Big(\int_G |f(x)|^p w(x)\,dx\Big)^{1/p}, \quad 1 < p < \infty.$$

Now we state our main result.

THEOREM 1. *Let L be a positive Rockland operator on G, $1 \leqslant p < \infty$ and $w \in A_p(G)$. There exists an integer N such that for a function $m \in C^N(\boldsymbol{R}^+)$ which satisfies*

$$\|m\|_N^* = \max_{0 \leqslant j \leqslant N} \sup_{\lambda > 0} |\lambda^j m^{(j)}(\lambda)| < \infty$$

we have: for $p > 1$

$$\|m(L)f\|_{p,w} \leqslant C\|f\|_{p,w} \tag{1.3}$$

and for $p = 1$

$$w(\{x \in G\colon |m(L)f| > \lambda\}) \leqslant \frac{C}{\lambda}\|f\|_{1,w}, \quad \lambda > 0, \tag{1.4}$$

with a constant $C = C(p)$ independent of f.

To prove Theorem 1 we apply the technique of [8] based on the sharp maximal function $f^{\#}$ of Fefferman and Stein. However, our crucial estimate contained in Theorem 2, similar to the key Lemma 1 in [8], is proved in a quite different way: in absence of Fourier transform we use a result of Hulanicki [4] (cf. also [5], [6]) which can be formulated as follows.

THEOREM B. *Let L be a positive Rockland operator on G and let $l > Q$ be given. There exists an integer N such that if $m \in C^N(\boldsymbol{R}^+)$ satisfies*

$$\|m\|_N = \max_{0 \leqslant j \leqslant N} \sup_{\lambda > 0} |(1+\lambda)^N m^{(j)}(\lambda)| < \infty, \tag{1.5}$$

then $m(L)f = f * k$ where $k \in L^1(G)$ and

$$|k(x)| \leqslant C\|m\|_N (1+|x|)^{-l} \tag{1.6}$$

with a constant C independent of m.

Note in particular that for a given $d \geqslant 0$ we can find such an N that for a function m which satisfies (1.5) we have $m(L)f = f * k$ and

$$\int_G |x|^d |k(x)|\,dx \leqslant C\|m\|_N \tag{1.7}$$

with a constant C independent of m.

Throughout this paper C will denote a positive constant not necessarily the same for each occurrence.

2. Preliminary results. Suppose L is a positive Rockland operator on G, homogeneous of degree a, $\delta_r L = r^a L$, $r > 0$. If a function m is such that $m(L)f = f * k$ and $k \in L^1(G)$ then for a function $m_r(\lambda) = m(r\lambda)$ we have $m_r(L)f = f * \alpha_r k$ where

$$\alpha_r k(x) = r^{-Q/a} k\big(\delta_{r^{-1/a}}(x)\big).$$

Till the end of this section we assume that $d \geqslant 0$ and $Q < l < Q+1$ are fixed real numbers and an integer N is such that for a function $m \in C^N(\boldsymbol{R}^+)$ which satisfies (1.5) in Theorem B, (1.6) and (1.7) are valid.

Let $m \in C^N(\boldsymbol{R}^+)$. Choose a decomposition of the identity

$$\sum_{-\infty}^{\infty} \phi_j(\lambda) = 1, \qquad \lambda > 0$$

where $\phi_j(\lambda) = \phi(2^j \lambda)$ and ϕ is infinitely differentiable, non-negative function supported in $1/2 < \lambda < 2$. Define $m_j(\lambda) = m(\lambda)\phi_j(\lambda)$. Then $m_j(L)f = f * k_j$, $k_j \in L^1(G)$. Put $K_M = \sum_{-M}^{M} k_j$, $M = 1, 2, \ldots$ As in [8], the following theorem shows how conditions on m can be interpreted as conditions on K_M.

THEOREM 2. *Let $m \in C^N(\boldsymbol{R}^+)$ satisfy*

$$\|m\|_N^* = \max_{0 \leqslant j \leqslant N} \sup_{\lambda > 0} |\lambda^j m^{(j)}(\lambda)| < \infty. \tag{2.1}$$

Then for $1 < p < \infty$ and for $|y| \leqslant R/2$

$$\Big(\int_{|x|>R} |K_M(xy^{-1}) - K_M(x)|^p\,dx\Big)^{1/p} \leqslant C \cdot R^{-(l-Q/p)} |y|^{l-Q}$$

with a constant $C = C(p, l)$ independent of M, R and y.

The proof of Theorem 2 requires several lemmas.

LEMMA 1. *Let $m \in C^N(\boldsymbol{R}^+)$ satisfy* (2.1). *Then for $1 < p < \infty$*

$$\Big(\int_{|x|>R} |k_j(x)|^p\,dx\Big)^{1/p} \leqslant C \cdot R^{-(l-Q/p)}\, 2^{ja^{-1}(l-Q)} \tag{2.2}$$

with $C = C(p, l)$ independent of j and R.

Proof. Define $\beta_j(\lambda) = m_j(2^{-j}\lambda)$. Then $\beta_j(L)f = f * B_j$, $B_j \in L^1(G)$, $B_j = \alpha_{2^{-j}} k_j$ and

$$\Big(\int_{|x|>R} |k_j(x)|^p\,dx\Big)^{1/p} = \Big(\int_{|\delta_{2^{j/a}}(x)|>R} |k_j(\delta_{2^{j/a}}(x))|^p (2^{j/a})^Q\,dx\Big)^{1/p}$$
$$= 2^{jQa^{-1}p^{-1}(1-p)} \Big(\int_{|x|>2^{-j/a}R} |B_j(x)|^p\,dx\Big)^{1/p}.$$

But β_j are supported in $1/2 < \lambda < 2$ and (2.1) implies $\|\beta_j\|_N \leqslant C$ independently of j. Hence, by (1.6), $|B_j(x)| \leqslant C|x|^{-l}$ independently of j. For the polar decomposition of x, i.e. $x = \delta_r x'$ with $r = |x|$, $|x'| = 1$ we have $dx = r^{Q-1} dr d\sigma(x')$, cf. [3]. Therefore

$$\Big(\int_{|x|>2^{-j/a}R} |x|^{-lp} dx\Big)^{1/p} = \Big(\int_{2^{-j/a}R}^{\infty} s^{Q-1-lp} ds\Big)^{1/p} = (2^{-j/a} R)^{-l+Q/p}$$

and

$$\Big(\int_{|x|>2^{-j/a}R} |B_j(x)|^p dx\Big)^{1/p} \leqslant C(2^{-j/a})^{-l+Q/p}.$$

This completes the proof of Lemma 1.

Let $T^t f = \int_0^{\infty} \exp(-t\lambda) dE(\lambda) f$ be the semi-group of operators generated by L. It is known, cf. [3], that $T^t f = f * h_t$ where $h_t \in L^1(G)$ and $\int_G |x|^\alpha |Yh_t(x)|^p dx < \infty$ for all $\alpha > 0$, $p \geqslant 1$, $t > 0$ and $Y \in \mathfrak{G}$. Denote $h = h_1$.

LEMMA 2. *Let* $1 \leqslant p < \infty$. *Then*

$$L(p, d, y) = \Big(\int_G \big||x|^d \big(h(xy^{-1}) - h(x)\big)\big|^p dx\Big)^{1/p} \leqslant C|y| \tag{2.3}$$

with a constant $C = C(p, d)$ *independent of* y.

Proof. It suffices to show (2.3) for $|y| \leqslant 1$ only. Recall, [3], that for $|y| \leqslant 1$

$$c_1 \|y\| \leqslant |y| \leqslant c_2 \|y\|^b \tag{2.4}$$

for positive constants b, c_1, c_2, where $\|\cdot\|$ means the Euclidean norm on $\mathfrak{G}$. Write $y^{-1} = \exp sY$, $|Y| = 1$. Then

$$L(p, d, y) = \Big(\int_G \Big|\int_0^s |x|^d\, Yh(x \exp tY) dt\Big|^p dx\Big)^{1/p}$$

$$\leqslant \int_0^s \Big(\int_G \big||x|^d\, Yh(x \exp tY)\big|^p dx\Big)^{1/p} dt.$$

But

$$|x|^d \leqslant C\big(|x \exp tY|^d + |\exp(-tY)|^d\big),$$

$$\sup_{|Y|=1} \||\cdot|^d\, Yh\|_p < \infty$$

and in virtue of (2.4) we get

$$L(p, d, y) \leqslant C(s \||\cdot|^d\, Yh\|_p + s |\exp sY|^d \|Yh\|_p)$$

$$\leqslant C(|y| \||\cdot|^d\, Yh\|_p + |y|^{1+d} \|Yh\|_p) \leqslant C|y|.$$

LEMMA 3. *Let* $\beta \in C^N(\boldsymbol{R}^+)$, $\operatorname{supp}\beta \subseteq (1/2, 2)$ *and* $\beta(L) f = f * B$,

$B \in L^1(G)$. *Denote* $\beta'(\lambda) = \beta(\lambda)\exp\lambda$. *Then*

$$(2.5) \qquad \Big(\int_{|x|>R} |B(xy^{-1}) - B(x)|^p dx\Big)^{1/p} \leqslant CR^{-d}|y|\,\|\beta'\|_N$$

with a constant $C = C(p, d)$ *independent of* R, y *and* β.

Proof. Let $\beta'(L)f = f * B'$, $B' \in L^1(G)$. Then $B = B' * h$ and we have

$$\Big(\int_{|x|>R} |B(xy^{-1}) - B(x)|^p dx\Big)^{1/p} \leqslant R^{-d}\Big(\int_{|x|>R} \big||x|^d\big(B(xy^{-1}) - B(x)\big)\big|^p dx\Big)^{1/p}$$
$$\leqslant R^{-d}\Big(\int_G\Big|\int_G |x|^d B'(z)\big(h(z^{-1}xy^{-1}) - h(z^{-1}x)\big)dz\Big|^p dx\Big)^{1/p}$$
$$\leqslant R^{-d}\int_G |B'(z)|\Big(\int_G \big||x|^d\big(h(z^{-1}xy^{-1}) - h(z^{-1}x)\big)\big|^p dx\Big)^{1/p} dz.$$

But $|x|^d \leqslant C(|z|^d + |z^{-1}x|^d)$ and using notations of Lemma 2 we get

$$\Big(\int_{|x|>R} |B(xy^{-1}) - B(x)|^p dx\Big)^{1/p} \leqslant CR^{-d}\int_G |B'(z)|\big(|z|^d L(p, 0, y) + L(p, d, y)\big)dz.$$

In virtue of Lemma 2 and Theorem B we obtain (2.5).

LEMMA 4. *Suppose* $m \in C^N(\boldsymbol{R}^+)$ *satisfies* (2.1). *Then for* $1 < p < \infty$

$$\Big(\int_{|x|\geqslant R} |k_j(xy^{-1}) - k_j(x)|^p dx\Big)^{1/p} \leqslant CR^{-d}(2^{-j/a})^{Q-Q/p-d+1}|y|$$

with a constant $C = C(p, d)$ *independent of* R, y *and* j.

Proof. As before define $\beta_j(\lambda) = m_j(2^{-j}\lambda)$, $\beta'_j(\lambda) = \beta_j(\lambda)\exp\lambda$. We then have $\beta_j(L)f = f * B_j$, $B_j \in L^1(G)$, $\alpha_{2^{-j}} k_j = B_j$, and (2.1) implies $\|\beta_j\|_N \leqslant C$, $\|\beta'_j\|_N \leqslant C$ independently of j. For a function k, by homogeneity we have

$$\Big(\int_{|x|\geqslant R} |k(xy^{-1}) - k(x)|^p dx\Big)^{1/p}$$
$$= s^{-(Q/p-Q)/a}\Big(\int_{|x|\geqslant Rs^{1/a}} \big|\alpha_s k\big(x\delta_{s^{1/a}}(y^{-1})\big) - \alpha_s k(x)\big|^p dx\Big)^{1/p}.$$

Putting $s = 2^{-j}$ we obtain, in virtue of Lemma 3,

$$\Big(\int_{|x|\geqslant R} |k_j(xy^{-1}) - k_j(x)|^p dx\Big)^{1/p}$$
$$= 2^{-j(Q-Q/p)/a}\Big(\int_{|x|\geqslant R2^{-j/a}} \big|B_j\big(x\delta_{2^{-j/a}}(y^{-1})\big) - B_j(x)\big|^p dx\Big)^{1/p}$$
$$\leqslant C2^{-j(Q-Q/p)/a}R^{-d}(2^{-j/a})^{-d}2^{-j/a}|y|.$$

This completes the proof of Lemma 4.

Proof of Theorem 2. Denote

$$I_p(j, R, y) = \Big(\int_{|x|\geqslant R} |k_j(xy^{-1}) - k_j(x)|^p dx\Big)^{1/p}.$$

By Lemma 1, for $|y| \leqslant R/2$, we get

$$I_p(j, R, y) \leqslant CR^{-(l-Q/p)}2^{j(l-Q)/a}.$$

By Lemma 4, putting $d = l - Q/p$, we obtain

$$I_p(j, R, y) \leqslant CR^{-(l-Q/p)}(2^{-j/a})^{Q+1-l}|y|.$$

Suppose $2^{(j_0-1)/a} \leqslant |y| \leqslant 2^{j_0/a}$. Then

$$\begin{aligned}
\Big(\int_{|x| \geqslant R} |K_M(xy^{-1}) - K_M(x)|^p\,dx\Big)^{1/p}
&\leqslant \sum_j I_p(j, R, y)\\
&\leqslant CR^{-(l-Q/p)}\Big(\sum_{2^{j/a} < |y|} 2^{j(l-Q)/a} + |y| \sum_{2^{j/a} \geqslant |y|} 2^{-j(Q+1-l)/a}\Big)\\
&\leqslant CR^{-(l-Q/p)}|y|^{l-Q}.
\end{aligned}$$

This completes the proof of Theorem 2.

3. Proof of Theorem 1. For $1 < p < \infty$ define

$$M_p f(x) = \big(M(|f|^p)(x)\big)^{1/p}$$

and for a locally integrable function f on G, let f^* be defined by

$$f^*(x) = \sup_{x \in B}\Big(|B|^{-1} \int_B |f(x) - f_B|\,dx\Big),$$

where B is a ball and $f_B = |B|^{-1} \int_B f(x)\,dx$. $|B|$ means the Haar measure of B.
Denote $T_M f = f * K_M$, $M = 1, 2, \ldots$

The following lemma is obtained in the same manner as, in [8], (3.1) is derived from Lemma 1. We include its proof here for reader's convenience.

LEMMA 5. *Under the assumptions of Theorem 2, for $1 < p < \infty$, we have*

$$(T_M f)^*(x) \leqslant C M_p f(x)$$

with a constant $C = C(p)$ independent of f and M.

Proof. Fix $x \in G$ and let B be a ball centered at x and of radius δ. Write

$$f = \bar{f}_0 + \sum_{j=0}^{\infty} f_j$$

where $\bar{f}_0 = f \cdot \chi_{\bar{A}_0}$, $f_j = f \cdot \chi_{A_j}$ and

$$\bar{A}_0 = \{y \in G\colon |y^{-1}x| < \delta\}, \qquad A_j = \{y \in G\colon 2^j\delta < |y^{-1}x| < 2^{j+1}\delta\}.$$

Thus

$$f * K_M = \bar{f}_0 * K_M + \sum_{j=0}^{\infty} f_j * K_M.$$

By Theorem A we get $\|g * K_M\|_p \leqslant C\|g\|_p$ with a C independent of M, and so

$$\begin{aligned}|B|^{-1}\int_B|\bar{f}_0 * K_M|\,dy &\leqslant \Big(|B|^{-1}\int_B|f_0 * K_M|^p\,dy\Big)^{1/p}\\ &\leqslant C\Big(|B|^{-1}\int_B|f|^p\,dy\Big)^{1/p}\\ &\leqslant CM_p(f)(x)\end{aligned}$$

with a C independent of M. For any j,

$$\begin{aligned}f_j * K_M(y) &= f_j * K_M(x)+\int_G f_j(z)[K_M(z^{-1}y)-K_M(z^{-1}x)]\,dz\\ &= c_j+\varepsilon_j(y).\end{aligned}$$

Assume for a moment that

$$|\varepsilon_j(y)| \leqslant C2^{j(Q-l)}M_p(f)(x) \tag{3.1}$$

for every $y\in B$ with a C independent of j and M. Since

$$(f * K_M)_B = (f_0 * K_M)_B+\sum_j c_j+\sum_j(\varepsilon_j)_B,$$

we then have

$$\begin{aligned}|B|^{-1}&\int_B|f * K_M(y)-(f * K_M)_B|\,dy\\ &\leqslant 2(\bar{f}_0 * K_M)_B+|B|^{-1}\int_B\Big|\sum_j(f_j * K_M(y)-c_j)-\sum_j(\varepsilon_j)_B\Big|\,dy\\ &\leqslant 2CM_p(f)(x)+|B|^{-1}\int_B\Big|\sum_j\varepsilon_j(y)-\sum_j(\varepsilon_j)_B\Big|\,dy\\ &\leqslant 2CM_p(f)(x)+2\sum_j(|\varepsilon_j|)_B\\ &\leqslant 2CM_p(f)(x)+2C\Big(\sum_j(2^j)^{Q-l}\Big)M_p(f)(x)\\ &\leqslant CM_p(f)(x).\end{aligned}$$

To complete the proof we verify (3.1). By Hölder's inequality, Theorem 2 and an obvious change of variable for $y\in B$ we obtain

$$\begin{aligned}|\varepsilon_j(y)| &\leqslant \int_{2^j\delta<|z^{-1}x|<2^{j+1}\delta}|f(z)|\,|K_M(z^{-1}y)-K_M(z^{-1}x)|\,dz\\ &\leqslant \Big(\int_{|z^{-1}x|<2^{j+1}\delta}|f(z)|^p\,dz\Big)^{1/p}\Big(\int_{2^j\delta<|z^{-1}x|}|K_M(z^{-1}y)-K_M(z^{-1}x)|^{p'}\,dz\Big)^{1/p'}\\ &\leqslant C(2^{j+1}\delta)^{Q/p}M_p(f)(x)\Big(\int_{2^j\delta<|u|}\big|K_M(u(y^{-1}x)^{-1})-K_M(u)\big|^{p'}\,du\Big)^{1/p'}\end{aligned}$$

$$\leqslant C(2^{j+1}\delta)^{Q/p}(2^j\delta)^{-(l-Q/p)}\delta^{l-Q}M_p(f)(x)$$
$$\leqslant C(2^j)^{Q-l}M_p(f)(x).$$

This completes the proof of Lemma 5.

We say that a weight function w satisfies the A_∞ *condition*, $w\in A_\infty$, if there exist positive constants C and δ such that for every ball $B\subseteq G$ and a subset E in B

$$\frac{w(E)}{w(B)}\leqslant C\left(\frac{|E|}{|B|}\right)^\delta.$$

We refer to [1] and [2] for results concerning A_p and A_∞ conditions, which we use below. In particular, if $w\in A_p$ then $w\in A_\infty$ and $w\in A_{p-\varepsilon}$ for some $\varepsilon>0$. To prove Theorem 1 we need also two known results.

LEMMA 6, [1]. *If* $0<r<p$ *and* $w\in A_{p/r}(G)$ *then*

$$\|M_r f\|_{p,w}\leqslant C\|f\|_{p,w}$$

with a constant C independent of f.

LEMMA 7, [9]. *If* $1<p<\infty$ *and* $w\in A_\infty(G)$ *then*

$$\|f\|_{p,w}\leqslant C\|f^*\|_{p,w}$$

with a constant C independent of f.

Let $w\in A_p$. Choose $1<r<p$ such that $w\in A_{p/r}$. Using Lemmas 5, 6, 7 we obtain

$$\|T_M f\|_{p,w}\leqslant C\|(T_M f)^*\|_{p,w}\leqslant C\|M_r(f)\|_{p,w}\leqslant C\|f\|_{p,w}$$

with constants independent of M and f. Consequently, applying Fatou's lemma for $p>1$ we obtain

$$\|m(L)f\|_{p,w}\leqslant C\|f\|_{p,w}.$$

This completes the proof of the first part of Theorem.

The case $p=1$, i.e. the weak-type (1,1) result, can be obtained in the same way as in [8] the weak-type estimate is proved. We use an abstract version of the Calderón–Zygmund decomposition lemma for a homogeneous space and we apply our estimate contained in Theorem 2.

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