ANNALES
POLONICI MATHEMATICI
XVII (1965)

On invariant points of monotone transformations in partially ordered spaces

by A. PELCZAR (Kraków)

The author's previous paper [2] discussed the problem of the existence of the extremal invariant points of the transformation

$$y = V(x)$$

of a partially ordered set P into P, which is equivalent to the problem of the existence of the extremal solutions of the equation

$$z = V(z) .$$

In the present paper we make some remarks concerning this problem and the theorems proved in [2] and (in § 2) the connections between them and the theorems of A. Tarski and L. E. Ward (cf. [4] and [5]).

§ 1. We shall consider a partially ordered set P (cf. [1]) making use of the notation and definitions introduced in [2]. We shall formulate the following theorems, which are a more general form of the results of [2]:

THEOREM 1. *Let P be a non-empty partially ordered set, let V be an increasing map of P into P; further let the subset Q of the set P defined by the formula*

$$Q = \{z \in P:\ z \leqslant V(z)\} \tag{1.1}$$

be non-empty and let there exist in P $\sup Q$ *(where by* $\sup Q$ *we shortly denote the least upper bound of Q in P, cf.* [2]).

Then $\hat{z} = \sup Q$ is the maximal solution of the equation

$$z = V(z) \tag{1.2}$$

in P, i.e. such a solution that for each solution $z \in P$ of (1.2) *we have $z \leqslant \hat{z}$.*

THEOREM 2. *Let P be a non-empty partially ordered set, let V be an increasing map of P into P, let the set Q' defined by the formula*

$$Q' = \{z \in P:\ V(z) \leqslant z\} \tag{1.3}$$

be non-empty and let there exist in P $\inf Q'$ *(where by* $\inf Q'$ *we shortly denote the greatest lower bound of Q' in P, cf.* [2]).

Then $\check{z} = \inf Q'$ *is the minimal solution of the equation* (1.2) *in P, i.e. such a solution that for each solution $z \in P$ of* (1.2) *we have* $\check{z} \leqslant z$.

Theorem 1 (Theorem 2) is a more general version of Theorem A (Theorem A′) in [2], but the proof is based on the same idea as the proof of Theorem A (resp. Theorem A′). It is easy to see that in the proof of Theorem A, we made use of the following condition II: each non-empty subset Q of P has $\sup Q$ in P only in the case of the subset Q defined by (1.1); the other assumptions of Theorem A were also made there.

Remark 1. Of course if we suppose the conditions of Theorem 1 and, moreover, assume that for some $y \in P$ is $y \leqslant V(y)$, then $y \leqslant \hat{z}$, where $\hat{z}$ is the maximal solution of (1.2) in P (cf. Remark 4 in [2]).

§ 2. A. Tarski proved in paper [4], which was unfortunately not known to the author during the preparation of paper [2], a theorem (Theorem 1 in [4]) equivalent to the theorem which has as its assumptions all the assumptions of Theorems A and A′ (of [2]) simultaneously, and draws both conclusions of these theorems simultaneously. The idea of the proofs of Theorems A and A′ is the same as the idea of the proof of A. Tarski's theorem. It is easy to see that a theorem which has as its assumptions all assumptions of Theorems 1 and 2, and draws both conclusions of Theorems 1 and 2 is a more general version of the theorem of A. Tarski.

L. E. Ward Jr. in his paper [5] (which was not known to the author during the preparation of paper [2]), considered semi-lattices. A partially ordered space X is said to be a *semi-lattice* if $\sup L(x) \cap L(y)$ exists for each x and y in X, where $L(x) = \{a\colon a \leqslant x\}$. A semi-lattice is *complete* (cf. [5]) if for each non-empty subset $A \subset X$ $\sup \bigcap \{L(a)\colon a \in A\}$ exists in X. It is easy to see that a set P fulfils condition II of the assumptions of Theorem A (cf. [2]) or condition II′ of Theorem A′ iff it is a complete semi-lattice; if P fulfils the assumptions of Theorem 1 (Theorem 2), then it has not need to be a semi-lattice.

The interval topology (cf. [1], p. 60) is a topology generated by taking all of the sets $L(x)$ and $M(x)$, where $M(x) = \{a\colon x \leqslant a\}$, $x \in X$, as a sub-basis for the closed sets. L. E. Ward Jr. proved in [5] the following

THEOREM W (Theorem 2 in [5]). *If X is a semi-lattice, f is an increasing map of X into X, and X is compact in the interval topology, then the set F of fixed points of f is non-empty. If X is a complete semi-lattice and F is non-empty, then F is a complete semi-lattice.*

Neither Theorem A of [2] nor Theorem 1 (resp. Theorem A′ and Theorem 2) are special cases of Theorem W or any generalizations of its.

§ 3. In this paragraph we shall consider real functions of one real variable x and real functions of two real variables x, y. We write

$$(3.1)\qquad S = \{x, y\colon\ 0 < x \leqslant a,\ 0 < \varphi(x) < y \leqslant \psi(x)\},$$

where $\varphi(x)$ and $\psi(x)$ are known functions continuous in $(0, a\rangle$, such that $0 < \varphi(x) < \psi(x)$.

THEOREM 3. *Let $F(x, y)$ be a continuous function defined in S, increasing with respect to y and such that $\varphi(x) < F(x, y) \leqslant \psi(x)$, let the set*

$$Q^* = \{g\colon g(x) \text{ is a continuous function such that } \varphi(x) < g(x) \leqslant \psi(x) \text{ and } g(x) \leqslant F(x, g(x))\}$$

be non-empty and let there exist $\sup Q^*$, *which is a continuous function $f(x)$. Then in the set P^* of all continuous functions $y(x)$ defined in the interval $(0, a\rangle$ and satisfying the condition $\varphi(x) < y(x) \leqslant \psi(x)$ there exists a maximal solution $\hat{y}(x)$ of the equation*

$$(3.2)\qquad y(x) = F(x, y(x)).$$

In order to prove Theorem 3 we can apply Theorem 1, which finishes the proof. In this case, however, we could not apply Theorem A because condition II of the assumptions of Theorem A (cf. [2]) does not hold. Similarly we could not apply here the theorem of A. Tarski or Theorem W, which was cited in § 2. An equation similar to (3.2) was considered in [3].

Remark 2. Of course if a function $u(x)$ fulfils the inequality $u(x) \leqslant F(x, u(x))$ and is continuous, then $u(x) \leqslant \hat{y}(x)$. As a simple example we can consider the following functions:

$$\varphi(x) = \sqrt{2}x, \qquad \psi(x) = 2x, \qquad F(x, y) = \frac{2x}{\ln 2} \ln \frac{y}{x}.$$

Here $y = 2x$ is the maximal solution of the equation

$$y(x) = \frac{2x}{\ln 2} \ln \frac{y(x)}{x}.$$

§ 4. Now we shall give some remarks concerning the equation (1.2), where $V(z)$ is decreasing. At first we must say that Remark 6 in paper [2] is not correctly formulated. Theorems concerning (1.2) in the case of decreasing map V are not quite analogous to Theorem A (Theorem A′). Indeed, let $P = \{z_1, z_2\}$, where $z_1 \leqslant z_2$ and let $V(z_1) = z_2$, $V(z_2) = z_1$. In this case V is decreasing and the assumptions I-III, V, VI, II′, VI′ of Theorems A and A′ are evidently satisfied, but there are no solutions of the equation (1.2) in P.

Remark 3. It is easy to see that if P is a non-empty, partially ordered set, V is a decreasing map of P into P, and two elements $z_1, z_2 \in P$ are such that $z_1 \leqslant z_2$ and $V(z_i) = z_i$ $(i = 1, 2)$, then $z_1 = z_2$.

COROLLARY 1. *If there exists in P a maximal (minimal) solution of* (1.2), *where V is a decreasing map of P into P, then there exists exactly one solution of* (1.2) *in P.*

THEOREM 4. *Let us assume that the set P is a non-empty, partially ordered set, V is a decreasing map of P into P, the set Q defined by* (1.1) *is non-empty and there exist in P: $\sup Q$, $\inf V(Q)$ and, moreover, $\sup Q \in Q$, $\inf V(Q) \leqslant \sup Q$, then there exists in P exactly one solution of* (1.2).

Proof. Let $z^* = \sup Q$. From the assumptions it follows that

$$z^* \leqslant V(z^*)\,. \tag{4.1}$$

Let $\bar{z} = \inf V(Q)$. We are going to prove that

$$V(z^*) \leqslant \bar{z}\,. \tag{4.2}$$

For each $x \in Q$ is $x \leqslant z^*$, $V(z)$ is decreasing; then we have

$$V(z^*) \leqslant V(x)$$

for each $x \in Q$, which means that

$$V(z^*) \leqslant y \tag{4.3}$$

for each $y \in V(Q)$. From (4.3) and from the definition of the infimum it follows that (4.2) holds. In consequence we have

$$z^* \leqslant V(z^*) \leqslant \bar{z}\,. \tag{4.4}$$

On the other hand, from the assumptions we have

$$\bar{z} \leqslant z^*\,. \tag{4.5}$$

From (4.4) and (4.5) it follows that $\bar{z} = z^*$ is the solution of (1.2). From the definition z^* it follows that it is the maximal solution of (1.2) in P. From Corollary 1 it follows that it is the unique solution of (1.2) in P.

It is possible to prove in an analogous way the following

THEOREM 5. *Let us assume that P is a non-empty partially ordered set, V is a decreasing map of P into P, the set Q' defined by* (1.3) *is non-empty and there exist in P: $\inf Q'$, $\sup V(Q')$ and, moreover, $\inf Q' \in Q'$, $\sup V(Q') \leqslant \inf Q'$, then there exists in P exactly one solution of* (1.2).

Remark 4. If all assumptions of Theorem 4 are satisfied and for some $y \in P$ we have $y \leqslant V(y)$, then $y \leqslant z$, where z is the unique solution of (1.2).

This proposition is formally analogical to Remark 4 in [2].

If all assumptions of Theorem 5 are satisfied and for some $y \in P$ we have $V(y) \leqslant y$, then $z \leqslant y$, where z is the unique solution of (1.2) in P.

Remark 5. If $\sup Q$ exists in P and $\sup Q \in Q$ (or $\inf Q'$ exists in P and $\inf Q' \in Q'$) and if there exists in P a solution z of (1.2), where V is decreasing, then z is the unique solution in P.

References

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

[2] A. Pelczar, *On the invariant points of a transformation*, Ann. Polon. Math. 11 (1961), pp. 199-202.

[3] — *On the extremal solutions of a functional equation*, Zeszyty Naukowe U. J., Prace Mat., 55 (1962), pp. 8-11.

[4] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific Journ. Math. 5 (1955), pp. 285-309.

[5] L. E. Ward, Jr., *Completeness in semi-lattices*, Canad. Journ. Math. 9 (1957), pp. 578-592.

Reçu par la Rédaction le 20. 5. 1963