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The tangent space and the arcwise tangent space to a differential space

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Abstract. The concept of a differential space was introduced by R. Sikorski in 1967, [4]. We due to the same author the notion of the tangent space regarded as the linear space of all derivations of the differential structure at the point. This linear space is equivalent in a natural manner to the so-called Zariski's tangent space (cf. Serre [3] and also Pustelnik [2]). In the more classical differential geometry the tangent space is often meant as the linear space of all tangent vectors to smooth curves passing through the point considered. In the present paper this analogy is investigated.

1. Terminology and basic notations. Let C be a differential structure on M. Following R. Sikorski [4], its topology, i.e. the weakest topology on M for which each of functions of C is continuous, will be denoted by τ_C. The set of all real functions defined and infinitely differentiable on whole Cartesian space $\boldsymbol{R}^n$ will be denoted by E_n. This is the so-called natural differential structure of $\boldsymbol{R}^n$. In particular, E_1 (shortly E) stands for the natural differential structure of the set $\boldsymbol{R}$ of all real numbers.

For any set C of real functions on M we define $\operatorname{sc} C$ as the set of all functions of the form $\omega(\alpha_1(\cdot), \ldots, \alpha_s(\cdot))$, where $\alpha_1, \ldots, \alpha_s$ are of C, ω is of E_s and s is an arbitrary natural number. Following R. Sikorski, let us denote by C_A the set of all locally C-functions on the set A, $A \subset M$. Then $\operatorname{sc} C = C = C_M$ iff C is a differential structure on M (cf. MacLane [1]).

For a differential space (M, C) the tangent space to (M, C) at the point p will be denoted by $(M, C)_p$.

2. Non-standard differential structure on $\boldsymbol{R}$.

LEMMA 1. *Let C_0 be a set of real continuous functions α_p defined on $\boldsymbol{R}$ such that for every $p \in \boldsymbol{R}$ there exists $\delta > 0$ satisfying the conditions:*

(i) *$\alpha_p|(p-\delta;\ p+\delta)$ is one-to-one;*

(ii) *both $\alpha_p|(p-\delta;\ p+\delta)$ and $\left(\alpha_p|(p-\delta;\ p+\delta)\right)^{-1}$ have derivatives of all orders.*

Then the function $\mathrm{id}_{\boldsymbol{R}}$ belongs to the smallest differential structure C containing C_0, and τ_C coincides with the natural topology of $\boldsymbol{R}$.

Proof. Let $p \in \boldsymbol{R}$ and $\delta > 0$ satisfy (i) and (ii). Let $\alpha_p[(p-\delta;\ p+\delta)] = (a;\ b)$, $a < b$. Then $\alpha_p(p) \in (a;\ b)$.

There exists a function $\varphi \in E$ separating the point $\alpha_p(p)$ in the interval $(a; b)$, i.e. such that

$$\varphi(y) = \begin{cases} 1 & \text{for } y \in (a_0; b_0), \\ 0 & \text{for } y \in \boldsymbol{R} - \langle a_1; b_1 \rangle, \end{cases}$$

where $a < a_1 < a_0 < b_0 < b_1 < b$. Let

$$\beta_p(y) = \begin{cases} \varphi(y)\big(\alpha_p|(p-\delta;\ p+\delta)\big)^{-1}(y) & \text{for } y \in (a;b), \\ 0 & \text{for } y \in \boldsymbol{R} - (a;b). \end{cases}$$

It is easy to check that $\beta_p \in E$ and

$$\beta_p(y) = \big(\alpha_p|(p-\delta;\ p+\delta)\big)^{-1}(y) \quad \text{for } y \in (a_0; b_0).$$

This implies that $\beta_p \circ \alpha_p \in \operatorname{sc} C_0 \subset C$, and in view of the equalities

$$(\beta_p \circ \alpha_p)(x) = \big(\alpha_p|(p-\delta;\ p+\delta)\big)^{-1}\big(\alpha_p(x)\big) = x \quad \text{for } x \in \alpha_p^{-1}[(a_0; b_0)],$$

i.e.,

$$\mathrm{id}_{\boldsymbol{R}}|A = \beta_p \circ \alpha_p|A, \quad \text{where } A = \alpha_p^{-1}[(a_0; b_0)],$$

we get

$$\mathrm{id}_{\boldsymbol{R}} \in C_{\boldsymbol{R}} = C.$$

The weakest topology on $\boldsymbol{R}$ in which $\mathrm{id}_{\boldsymbol{R}}$ and all the functions which are continuous in the usual sense are continuous, coincides, of course, with the natural topology of $\boldsymbol{R}$. This completes the proof.

LEMMA 2. *Let C_0 be a family of real functions γ_p defined and continuous in the usual sense on $\boldsymbol{R}$, and suppose that for every $p \in \boldsymbol{R}$*

(iii) *the functions $\gamma_p|(-\infty;\ p)$ and $\gamma_p|(p;\ +\infty)$ are one-to-one, differentiable, and both $\big(\gamma_p|(-\infty; p)\big)^{-1}$ and $\big(\gamma_p|(p;\ +\infty)\big)^{-1}$ are infinitely differentiable.*

Then, a function α belongs to the smallest differential structure C containing C_0 if and only if for every $p \in \boldsymbol{R}$ there exist a neighbourhood $A \in \tau_C$ of p and a function $\psi \in E_2$ such that

$$\alpha(x) = \psi\big(x, \gamma_p(x)\big) \quad \textit{for } x \in A. \tag{1}$$

Proof. The functions α_p, $\alpha_p(x) = \gamma_p(x-1)$ for $x \in \boldsymbol{R}$, satisfy conditions (i) and (ii). Therefore, by Lemma 1, τ_C is the natural topology of $\boldsymbol{R}$.

Let $\alpha \in C$ and $p \in \boldsymbol{R}$. Thus there exist points $p_i \neq p$, $i = 1, \ldots, n$, a neighbourhood B of p and a function $\varphi \in E_{n+1}$ such that

$$\alpha(x) = \varphi\big(\gamma_{p_1}(x), \ldots, \gamma_{p_n}(x), \gamma(x)\big) \quad \text{for } x \in B.$$

We may assume that $B = (p-\delta; p+\delta)$, where $0 < \delta < |p-p_i|$, $i = 1, \ldots, n$. The points $p_1, \ldots, p_n$ lie outside B. Therefore the functions $\gamma_{p_i}|B$ are smooth, $i = 1, \ldots, n$. Hence it follows that we have some functions $\lambda_1, \ldots, \lambda_n$ of E_1 class such that $\lambda_i(x) = \gamma_{p_i}(x)$ for x in some neighbourhood of p, $i = 1, \ldots, n$. Setting $\psi(x, y) = \varphi(\lambda_1(x), \ldots, \lambda_n(x), y)$ for $x, y \in \boldsymbol{R}^2$ we get (1), where A is a neighbourhood of p.

THEOREM 1. *Let C_0 be a family of real functions γ_p defined on $\boldsymbol{R}$, continuous in the usual sense; assume that* (iii) *is satisfied, left-hand side as well as right-hand side derivatives $\gamma_{p-}(p)$ and $\gamma_{p+}(p)$ of the function γ_p exist at p, and $\gamma_{p-}(p) \neq \gamma_{p+}(p)$ for $p \in \boldsymbol{R}$. If C is the smallest differential structure on $\boldsymbol{R}$ containing C_0, then for every $p \in \boldsymbol{R}$ the tangent space $(\boldsymbol{R}, C)_p$ to $(\boldsymbol{R}, C)$ at p is two-dimensional.*

Proof. Let $p \in \boldsymbol{R}$. For every $\alpha \in C$ there exist a neighbourhood $A \in \tau_C$ of p and a function $\psi \in E_2$ such that (1) holds. Hence it follows that the equalities $e_1(p)(\gamma_p) = \gamma_{p-}(p)$ and $e_2(p)(\gamma_p) = \gamma_{p+}(p)$ well define two vectors $e_1(p)$ and $e_2(p)$ tangent to $(\boldsymbol{R}, C)$ at p. To prove that $e_1(p)$ and $e_2(p)$ are linearly independent, we take $a_1, a_2 \in \boldsymbol{R}$ such that

$$a_1 e_1(p) + a_2 e_2(p) = 0. \tag{2}$$

There exists, by Lemma 2, a function $\alpha \in C$ being one-to-one on some neighbourhood A of p such that both $\alpha|A$ and $(\alpha|A)^{-1}$ are infinitely differentiable. From (2) it follows that

$$a_1 e_1(p)(\alpha) + a_2 e_2(p)(\alpha) = 0 \quad \text{and} \quad a_1 e_1(p)(\gamma_p) + a_2 e_2(p)(\gamma_p) = 0.$$

We have $e_1(p)(\alpha) = e_2(p)(\alpha) = \alpha'(p) \neq 0$. Thus $a_1 + a_2 = 0$ and $a_1 \gamma_{p-}(p) + a_2 \gamma_{p+}(p) = 0$. Hence $a_1 = a_2 = 0$. Consequently, $\dim(\boldsymbol{R}, C)_p \geqslant 2$.

Now, for every $\alpha \in C$ we set $e_{ip}(\alpha) = \psi_{|i}(p, \gamma_p(p))$, where ψ satisfies (1) and $\psi_{|i}$ denotes the partial derivative of ψ with respect to i-th variable. It is easy to check that e_{1p}, e_{2p} are vectors tangent to $(\boldsymbol{R}, C)$ at p, and for every vector v of $(\boldsymbol{R}, C)_p$ we have $v = v(\mathrm{id}_R) e_{1p} + v(\gamma_p) e_{2p}$. Then the vector space $(\boldsymbol{R}, C)_p$ is two-dimensional.

EXAMPLE 1. Let C be the smallest differential structure on $\boldsymbol{R}$ containing the set of all functions $x \mapsto |x-p|$, where $p \in \boldsymbol{R}$. By Theorem 1 we have $\dim(\boldsymbol{R}, C)_p = 2$ for $p \in \boldsymbol{R}$.

3. Arcwise tangent space. Let I be an open interval of $\boldsymbol{R}$ and (M, C) be a differential space. We consider I together with its natural differential structure E_I. Every smooth mapping $\gamma\colon (I, E_I) \to (M, C)$ will be called a *smooth run* in (M, C). If $t_0 \in I$ and $\gamma(t_0) = p$, we say that the run γ is *passing through* p at the moment t_0. We define the velocity $\dot{\gamma}(t_0)$ of γ at the moment t_0 as the vector of $(M, C)_{\gamma(t_0)}$ such that $\dot{\gamma}(t_0)(\alpha) = (\alpha \circ \gamma)'(t_0)$ for $\alpha \in C$, where $'$ denotes the derivative. The subspace of $(M, C)_p$ generated by all velocities $\dot{\gamma}(t_0)$ such that the run γ is passing through p at the moment t_0

will be called the *arcwise tangent space* to (M, C) at p and denoted by $(M, C)_p^{\smile}$.

If (M, C) is a differentiable manifold, then $(M, C)_p^{\smile} = (M, C)_p$ and $\dim (M, C)_p = \dim (M, \tau_C)$.

The example that follows show that the differential dimension, $\dim (M, C)_p$, of a differential space at the point can differ from its topological dimension.

EXAMPLE 2. Let $C_{p,q}$ be the least differential structure on $\boldsymbol{R}$ containing the function $x \mapsto (x-p)^q$, where p, q are constants and $q > 1$. Then, for any run γ in $(\boldsymbol{R}, C_{p,q})$ passing through p at t_0 we have the function $t \mapsto (\gamma(t)-p)^q$, differentiable at t_0. Hence

$$\dot{\gamma}(t_0)\big(x \mapsto (x-p)^q\big) = q \cdot \big(\gamma(t_0)-p\big)^{q-1} \cdot \gamma'(t_0) = 0.$$

From the fact that $\tau_{C_{p,q}}$ is the natural topology of $\boldsymbol{R}$ it follows that $\dot{\gamma}(t_0)(\alpha) = 0$ for α of $C_{p,q}$. Then $\dot{\gamma}(t_0) = 0$. Therefore,

$$\dim (\boldsymbol{R}, C_{p,q})_p^{\smile} = 0;$$

however, the topological dimension of the space considered is equal to 1.

LEMMA 3. *Let $p \in \boldsymbol{R}$ and let C be a differential structure on $\boldsymbol{R}$ such that*

(C_p) every open interval with center p contains a neighbourhood of p open in τ_C.

If a function λ of class C is such that Dini's derivatives: $\lambda_-(p)$, $\lambda^-(p)$, $\lambda_+(p)$ and $\lambda^+(p)$ satisfy the condition

(iv) $\lambda^-(p) < 0 < \lambda_+(p)$ *or* $\lambda^+(p) < 0 < \lambda_-(p)$,

then every smooth run γ in $(\boldsymbol{R}, C)$ passing through p at the moment t_0 satisfies the condition

$$\dot{\gamma}(t_0)(\lambda) = 0. \tag{3}$$

Proof. We may assume $t_0 = 0$. Suppose that $\lambda \in C$ satisfies (iv). Of course, $-\lambda \in C$ and $(-\lambda)_-(p) = -\lambda^-(p)$, $(-\lambda)_+(p) = -\lambda^+(p)$. Therefore, we may assume that, e.g., the first of inequalities in (iv) is satisfied. Let $c > 0$ and $\lambda^-(p) < -c < 0 < c < \lambda_+(p)$. Hence there exists $\delta > 0$ such that

$$\big(\lambda(x)-\lambda(p)\big)/(x-p) < -c \quad \text{for } x \in (p-\delta; p)$$

and

$$\big(\lambda(x)-\lambda(p)\big)/(x-p) > c \quad \text{for } x \in (p; p+\delta).$$

Thus

$$\big(\lambda(x)-\lambda(p)\big)/|x-p| > c, \quad \text{when} \quad 0 < |x-p| < \delta. \tag{4}$$

Let us take a smooth run in $(\boldsymbol{R}, C)$ passing through p at the moment 0. First, we consider the case when there exists $\eta > 0$ such that $\gamma(t) = \gamma(0)$ for $t \in (-\eta; 0)$ or $\gamma(t) = \gamma(0)$ for $t \in (0; \eta)$. Then, $\lambda(\gamma(t)) = \lambda(\gamma(0))$ for

$t \in (-\eta; 0) \cup (0; \eta)$. Hence, $(\lambda \circ \gamma)'(0) = 0$. Now, in the opposite case, for every $\eta > 0$ we have non-empty sets

$$A_\eta^- = \{t; t \in (-\eta; 0) \text{ and } \gamma(t) \neq \gamma(0)\},$$

$$A_\eta^+ = \{t; t \in (0; \eta) \text{ and } \gamma(t) \neq \gamma(0)\}.$$

According to (C_p) the interval $(p-\delta; p+\delta)$ contains a neighbourhood of p open in τ_C. By the continuity of γ, there exists $\eta > 0$ such that

$$(-\eta; \eta) \subset \gamma^{-1}[(p-\delta; p+\delta)].$$

Thus $A_\eta^- \cup A_\eta^+ \subset \gamma^{-1}[(p-\delta; p+\delta)]$. In other words, $|\gamma(t)-p| < \delta$ for $t \in A_\eta^- \cup A_\eta^+$. Hence, by (4), we have

$$\frac{\lambda(\gamma(t))-\lambda(\gamma(0))}{|\gamma(t)-\gamma(0)|} > c \quad \text{for } t \in A_\eta^- \cup A_\eta^+. \tag{5}$$

Suppose that $(\lambda \circ \gamma)'(0) > 0$. Then there exists $\eta' \in (0; \eta)$ such that

$$\big(\lambda(\gamma(t))-\lambda(\gamma(0))\big)/t > 0 \quad \text{when } 0 < |t| < \eta'.$$

This implies

$$\frac{\lambda(\gamma(t))-\lambda(\gamma(0))}{|\gamma(t)-\gamma(0)|} \cdot \frac{|\gamma(t)-\gamma(0)|}{t} > 0 \quad \text{for } t \in A_{\eta'}^- \cup A_{\eta'}^+.$$

Therefore, by (5), we get $|\gamma(t)-\gamma(0)|/t > 0$ for $t \in A_{\eta'}^- \cup A_{\eta'}^+$, which is impossible because the sets $A_{\eta'}^-$ and $A_{\eta'}^+$ are non-empty. Similarly we show that the assumption $(\lambda \circ \gamma)'(0) < 0$ leads to a contradiction. Hence $(\lambda \circ \gamma)'(0) = 0$. Thus (3) holds. This completes the proof.

LEMMA 4. *If $p \in \boldsymbol{R}$ and C is a set of real functions on $\boldsymbol{R}$ such that*

(a) *condition (C_p) holds,*

(b) *there exists a function $\lambda \in C$ such that* (iv) *holds,*

(c) *Dini's derivatives at the point p of every function $\lambda \in C$ for which* (iv) *does not hold are finite, then every smooth run γ in $(\boldsymbol{R}, (\mathrm{sc}\, C)_{\boldsymbol{R}})$ passing through p at the moment t_0 satisfies condition* (3) *for $\lambda \in C$.*

Proof. Let $\lambda \in C$. In case (iv) we get (3) by Lemma 3. Therefore, we may take α of class C with Dini's derivatives being finite at p. Then there exist positive numbers c_1 and η_1 such that

$$\left|\frac{\alpha(x)-\alpha(p)}{x-p}\right| < c_1 \quad \text{when } 0 < |x-p| < \eta_1. \tag{6}$$

From (b) it follows that for some λ, where $\lambda \in C$ or $-\lambda \in C$, and for some positive numbers c_2 and η_2 we have

$$\frac{\lambda(x)-\lambda(p)}{|x-p|} > c_2 \quad \text{when } 0 < |x-p| < \eta_2. \tag{7}$$

Condition (C_p) yields the existence of a $U \in \tau_C$ such that $p \in U \subset (p-\eta; p+\eta)$, $\eta = \min\{\eta_1, \eta_2\}$. The continuity of γ yields the existence of $\delta > 0$ such that $\gamma(t) \in U$ for $t \in (t_0-\delta; t_0+\delta)$. Thus we get $|\gamma(t)-p| < \eta$ for $|t-t_0| < \delta$. Assume that $0 < |t-t_0| < \delta$ and $\gamma(t) \neq p$. Then

$$\left|\frac{\alpha(\gamma(t))-\alpha(\gamma(t_0))}{t-t_0}\right| = \left|\frac{\alpha(\gamma(t))-\alpha(p)}{\gamma(t)-p}\right| \cdot \left|\frac{\gamma(t)-p}{\lambda(\gamma(t))-\lambda(p)}\right| \cdot \left|\frac{\lambda(\gamma(t))-\lambda(p)}{t-t_0}\right|.$$

Hence, by (6) and (7), we have

$$\left|\frac{\alpha(\gamma(t))-\alpha(\gamma(t_0))}{t-t_0}\right| \leqslant \frac{c_1}{c_2} \left|\frac{\lambda(\gamma(t))-\lambda(\gamma(t_0))}{t-t_0}\right|. \tag{8}$$

Of course, this inequality is also true when $\gamma(t) = p$, as well as when λ is replaced by $-\lambda$. We have obtained inequality (8), where λ is some function of C satisfying (iv). Thus by Lemma 3, we obtain $(\alpha \circ \gamma)'(t_0) = 0$. In other words,

$$\dot{\gamma}(t_0)(\alpha) = 0. \tag{9}$$

THEOREM 2. *Let $p \in \boldsymbol{R}$ and let C be a set of real functions on $\boldsymbol{R}$ fulfilling condition (C_p). If there exists $\lambda \in C$ such that* (iv) *holds and Dini's derivatives of each function of C for which* (iv) *does not hold are finite, then $\boldsymbol{R}$ together with the least differential structure containing C constitute a differential space whose arcwise tangent space at p is the zero space.*

Proof. Let γ be a smooth run on $(\boldsymbol{R}, (\mathrm{sc}\, C)_{\boldsymbol{R}})$ passing through p at the moment t_0 and let $\alpha \in \mathrm{sc}\, C$. Thus $\alpha = \omega(\alpha_1(\cdot), \ldots, \alpha_s(\cdot))$, where $\alpha_1, \ldots, \alpha_s \in C$ and $\omega \in E_s$. Hence, by Lemma 4, we get

$$\dot{\gamma}(t_0)(\alpha) = \sum_{i=1}^{s} \omega_{|i}(\alpha_1(p), \ldots, \alpha_s(p)) \cdot \dot{\gamma}(p_0)(\alpha_i) = 0.$$

Now, let us take $\beta \in (\mathrm{sc}\, C)_{\boldsymbol{R}}$. Then we have $\beta | U = \alpha | U$, where $p \in U \in \tau_C$, $\alpha \in \mathrm{sc}\, C$. Thus, $\dot{\gamma}(t_0)(\beta) = \dot{\gamma}(t_0)(\alpha) = 0$, and so $\dot{\gamma}(t_0) = 0$. In other words,

$$(\boldsymbol{R}, (\mathrm{sc}\, C)_{\boldsymbol{R}})_p^{\cup} = 0.$$

This completes the proof.

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