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A MONOTONICITY CRITERION FOR ARBITRARY FUNCTIONS

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Let f be a real-valued function of a real variable and let $\underline{D}_+f(x)$ denote the right-hand lower derivate of f at x. Ważewski [3] proved that a necessary and sufficient condition for a continuous function f to be non-increasing is $mf(W) = 0$, where $W = \{x\colon \underline{D}_+f(x) > 0\}$ and m is the Lebesgue measure. This theorem can be regarded as a consequence of Zygmund's theorem ([2], p. 203) and a theorem which says $mf(Z) = 0$, where $Z = \{x\colon \underline{D}_+f(x) = 0\}$ ([2], p. 272). However, the latter theorem is quite profound while Ważewski's proof for his theorem is very elementary and elegant. The only disadvantage of his proof is that it calls for continuity of the function f several times and this condition is too strong for the result. In the present paper, the above-mentioned disadvantage is overcome and a criterion for arbitrary functions is obtained by an elementary method.

Although the proof for the following lemma parallels Ważewski's, we have to make some non-trivial modifications.

LEMMA. *Let f be a function defined on the closed interval $[a, b]$ such that*

(i) $\liminf_{h\to 0+} f(x-h) \geqslant f(x) \geqslant \liminf_{h\to 0+} f(x+h)$ *for all $x \in [a, b]$ (it is understood that only one of the inequalities is considered when $x = a$ or $x = b$),*

(ii) $f(a) = A < B = f(b)$.

Then $m_e f(W) > 0$, where $W = \{x \in [a, b)\colon \underline{D}_+f(x) > 0\}$ and m_e is the Lebesgue outer measure.

Proof. For each $y \in [A, B]$, we define $E(y) = \{x \in [a, b]\colon f(x) \leqslant y\}$. Clearly, $a \in E(y)$ and there exists $\sup E(y)$ in $[a, b]$ for each $y \in [A, B]$. Setting $\varrho(y) = \sup E(y)$, we get a function defined on $[A, B]$ with values in $[a, b]$. We break the proof into several steps and omit the reasoning for the easy ones.

(I) If $x \in [a, b]$, $f(x) \in [A, B]$ and $\xi = \varrho(f(x))$, then $a \leqslant x \leqslant \xi \leqslant b$.

(II) If $y \in [A, B]$ and $\xi = \varrho(y)$, then $y = f(\xi)$.

The assertion is obvious if $y = B$, for in this case we have $\xi = b$. Now we assume that $y < B$. Since

$$\liminf_{h\to 0+} f(b-h) \geqslant f(b) = B,$$

we see that $\xi < b$. If $f(\xi) < y$, by the second inequality of (i), there exists an x with $\xi < x < b$ and $f(x) < y$. This is contradictory to the equality $\xi = \sup E(y)$. Therefore $f(\xi) \geqslant y$. Since $y \in [A, B]$ and $f(a) = A$, it is trivial that $y \geqslant f(\xi)$ if $\xi = a$. If $\xi > a$, the first inequality of (i) implies that $f(\xi) \leqslant y$. Thus $f(\xi) = y$ in any case.

(III) ϱ is strictly increasing and $a < \varrho(y) < b$ if $A < y < B$.

Let $A \leqslant y_1 < y_2 \leqslant B$, $\xi_1 = \varrho(y_1)$ and $\xi_2 = \varrho(y_2)$ be given. It is obvious that $E(y_1) \subset E(y_2)$, hence $\sup E(y_1) \leqslant \sup E(y_2)$, that is, $\xi_1 \leqslant \xi_2$. By (II), $y_1 = f(\xi_1)$ and $y_2 = f(\xi_2)$. thus $y_1 \neq y_2$ implies $\xi_1 \neq \xi_2$. Consequently, $\xi_1 < \xi_2$. Clearly, $a \leqslant \varrho(A)$ and $b = \varrho(B)$. Therefore, we have $a < \varrho(y) < b$ if $A < y < B$.

(IV) $m_e(U) > 0$, where $U = \{y \in (A, B)\colon 0 \leqslant \varrho'(y) < +\infty\}$.

It follows from (III) that $(A, B) - U$ is a null set, hence $m_e(U) = m(U) = B - A > 0$.

(V) $f(V) = U$, where $V = \varrho(U)$.

(VI) $\underline{D}_+ f(x) > 0$ whenever $x \in V$.

Let $x_0 \in V$ be given; then there is a $y_0 \in U$ such that $x_0 = \varrho(y_0)$. Since $y_0 \in U$, we have $A < y_0 < B$. By (III), $a < x_0 < b$. By (II), $f(x_0) = y_0$. There exists a sequence $\{x_n\}$ in (x_0, b) such that $x_n \to x_0$ and

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \underline{D}_+ f(x_0).$$

Let $y_n = f(x_n)$ for each n. We have $x_n > x_0 = \sup E(y_0)$, and hence $y_n > y_0$ for each n. The sequence $\{y_n\}$ either diverges to $+\infty$ or has a subsequence converging to a number not less than y_0. Thus we may assume without loss of generality that there exists $\lim_n y_n \geqslant y_0$. In case $\lim_n y_n > y_0$, we easily see that

$$\underline{D}_+ f(x_0) = \lim_n \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_n \frac{y_n - y_0}{x_n - x_0} = +\infty > 0.$$

If $\lim_n y_n = y_0$, then there exists an n_0 such that $y_n \in (A, B)$ for all $n > n_0$. Thus for each $n > n_0$ there exists $\xi_n = \varrho(y_n)$. By (I) and the inequality $x_n > x_0$, we have $x_0 < x_n \leqslant \xi_n$ for all $n > n_0$. It follows that, for $n > n_0$,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{y_n - y_0}{x_n - x_0} \geqslant \frac{y_n - y_0}{\xi_n - x_0} = \frac{y_n - y_0}{\varrho(y_n) - \varrho(y_0)}.$$

But the last term has a positive limit (it is either $1/\varrho'(y_0)$ or $+\infty$) since $y_0 \in U$. Consequently, $\underline{D}_+f(x_0) > 0$.

(VII) Our conclusion $m_e f(W) > 0$ follows from (VI), (V) and (IV).

From now on f is defined on an interval which is not necessarily bounded and the same convention as stated in (i) of the Lemma is observed whenever similar inequalities appear and the domain of f contains an end-point.

THEOREM. *f is non-increasing if and only if*

(i) $\liminf\limits_{h\to 0+} f(x-h) \geqslant f(x) \geqslant \liminf\limits_{h\to 0+} f(x+h)$ *for every* x *and*

(ii) $mf(W) = 0$, *where* $W = \{x: \underline{D}_+f(x) > 0\}$.

Proof. The "if" part follows immediately from the above lemma and the "only if" part is trivial.

Remark. Applying the Theorem to $-f(x)$, $f(-x)$ and $-f(-x)$, we get the following similar results:

(1) *f is non-decreasing if and only if*

(i) $\limsup\limits_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup\limits_{h\to 0+} f(x+h)$ *for every* x *and*

(ii) $mf(\{x: \overline{D}_+f(x) < 0\}) = 0$, *where* $\overline{D}_+f(x)$ *is the right-hand upper derivate of f at x.*

(2) *f is non-decreasing if and only if*

(i) $\liminf\limits_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \liminf\limits_{h\to 0+} f(x+h)$ *for every* x *and*

(ii) $mf(\{x: \overline{D}_-f(x) < 0\}) = 0$, *where* $\overline{D}_-f(x)$ *is the left-hand upper derivate of f at x.*

(3) *f is non-increasing if and only if*

(i) $\limsup\limits_{h\to 0+} f(x-h) \geqslant f(x) \geqslant \limsup\limits_{h\to 0+} f(x+h)$ *for every* x *and*

(ii) $mf(\{x: \underline{D}_-f(x) > 0\}) = 0$, *where* $\underline{D}_-f(x)$ *is the left-hand lower derivate of f at x.*

Next we shall state some immediate consequences of our theorem as corollaries. It is evident that each of the results in this remark has similar corollaries but we do not state them explicitly here.

COROLLARY 1. *If f satisfies condition* (i) *of the Theorem and* $\underline{D}_+f(x) \leqslant 0$ *except on an at most countable set, then f is non-increasing.*

COROLLARY 2. *If f satisfies condition* (i) *of the Theorem,* $\underline{D}_+f(x) \leqslant 0$ *almost everywhere and if f fulfils Lusin's condition* (N) *(that is, if* $mf(H) = 0$ *whenever H is a subset of the domain with* $m(H) = 0$; [2], p. 224), *then f is non-increasing.*

COROLLARY 3. *If f satisfies condition* (i) *of the Theorem, then the set* $\{x: \underline{D}_+f(x) > 0\}$ *is either empty or has the power of the continuum.*

The reader can compare Corollaries 1 and 3 with a theorem of Dini [2], p. 204, and a result of Manna [1], p. 79, respectively.

REFERENCES

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