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DUELS WITH POSSIBLY ASYMMETRIC GOALS

1. Introduction. We generalize slightly the usual formulation of the family of zero-sum, two-person games called *duels*. Player I (II) has m (n) bullets. He may fire at any times during $[0, 1]$. If Player I (II) shoots at time t, he hits with probability $P_1(t)$ ($P_2(t)$). The probabilities P_i are called *accuracy functions* and are assumed continuous and non-decreasing with $P_i(0) = 0$, $P_i(1) = 1$. If either player hits, the game immediately ends. The pay-off is μ ($-\varrho$) if only Player I (II) hits and it is $\mu-\varrho$ if both hit. Otherwise the pay-off is 0.

The mild generalization here is in going beyond the usual case of $\mu = \varrho = 1$.

A bullet is *noisy* (*silent*) if the opponent of the shooter knows (does not know) that the bullet has been fired. Such information is instantaneous. A player is *noisy* (*silent*) if all his bullets are noisy (silent). A duel is *noisy* (*silent*) if both players are noisy (silent).

For $\mu = \varrho = 1$, general solutions exist for the silent duel [3], the noisy duel [1] and the silent vs. noisy case (Player I silent, Player II noisy) with $n = 1$ [4]. A formulation of a non-discrete firing duel and its solution for $\mu = \varrho = 1$ has been provided by Lang and Kimeldorf in [2].

In this paper these results are all given in the more general context of arbitrary $\mu, \varrho \geqslant 0$. The results in the more general context are generally the same, but in the silent vs. noisy case we must deal separately with $\varrho = 0$. Proofs are not presented when they are identical with those of previous papers.

Sections 2 and 3 present the noisy and silent cases. Section 4 contains silent vs. noisy duels and is divided into three subsections dealing, respectively, with the cases $n = 1$, $\varrho > 0$; $m = 1$, $\varrho = 0$; and $n = 1$, $\varrho = 0$. The solution of the first of these cases is a straightforward generalization of Styszyński's [4] result. The second reduces to the solution of the noisy duel. For the third case, the solution is, in a sense, partially Styszyński's and partially the solution of the noisy duel. Section 5 deals with the non-discrete firing case.

2. The noisy case. If there exists a $t \in (0, 1)$ such that $P_1(t) = 0$ and $P_2(t) = 1$ or $P_1(t) = 1$ and $P_2(t) = 0$, the solution of the game is obvious. If this is not the case, Fox and Kimeldorf ([1], Section 2) show that there exist $t_{ij} \in (0, 1)$ satisfying

$$(2.1) \qquad \prod_{i=1}^{m} [1-P_1(t_{in})] + \prod_{j=1}^{n} [1-P_2(t_{mj})] = 1$$

for each $m, n = 1, 2, \ldots$ Furthermore, $0 < P_1(t_{ij}), P_2(t_{ij}) < 1$ and $t_{ij} < \min(t_{i-1,j}, t_{i,j-1})$, where $t_{0j} = t_{i0} = 1$.

Multiply both sides of (2.1) by $\mu + \varrho$ and rearrange to obtain

$$(2.2) \qquad \mu - (\mu+\varrho)\prod_{i=1}^{m} [1-P_1(t_{in})] = (\mu+\varrho)\prod_{j=1}^{n} [1-P_2(t_{mj})] - \varrho$$

and denote the common value of the two sides of (2.2) by v_{mn}. It is then easy to verify the recursive relations

$$v_{mn} = \mu P_1(t_{mn}) + [1-P_1(t_{mn})]v_{m-1,n} = -\varrho P_2(t_{mn}) + [1-P_2(t_{mn})]v_{m,n-1},$$

where $v_{0n} = -\varrho$ and $v_{m0} = \mu$ for $m, n = 1, 2, \ldots$

The proof of the following theorem is as in [1], Section 3.

THEOREM 1. *The noisy duel has the value v_{mn}.*

The ε-good strategies used to prove Theorem 1 are randomized. Player I chooses a time t for his first shot according to an arbitrary probability density on $(t_{mn}, t_{mn}+\delta)$, where δ is chosen to satisfy

(i) $P_2(t_{mn}+\delta) < P_2(t_{mn}) + \varepsilon/3$;

(ii) $t_{mn}+\delta < \min(t_{m-1,n}, t_{m,n-1})$.

He will shoot at time t unless Player II shoots first. If both players survive, he will then choose an $(\varepsilon/3)$-good strategy in the resulting game (available as a consequence of (ii) and an inductive hypothesis). This makes the probability of simultaneous firing equal to 0.

Set

$$b_{mn} = \mu P_1(t_{mn}) - \varrho P_2(t_{mn}) + [1-P_1(t_{mn})][1-P_2(t_{mn})]v_{m-1,n-1},$$

the result, to within ε, of simultaneous firing at time t_{mn} followed by ε-good strategies in the resulting game if both players survive. As in [1], Section 4, if $b_{mn} \geqslant v_{mn}$ ($b_{mn} \leqslant v_{mn}$), Player I (II) has an ε-good strategy in which, unless his opponent shoots earlier, he shoots his first bullet at time t_{mn}. In this case we say that t_{mn} is a *good first shot time.*

Example 2.1. Let $P_1(t) = P_2(t) = t$. Then

$$t_{mn} = 1/(m+n), \quad v_{mn} = (m\mu - n\varrho)/(m+n) \text{ and } b_{mn} = v_{m-1,n-1}/(m+n)^2.$$

Hence $b_{mn} \geqslant v_{mn}$ iff $v_{m-1,n-1} \geqslant 0$, that is, $(m-1)\mu \geqslant (n-1)\varrho$. A player whose opponent has one bullet always has a good first shot time t_{mn}.

Example 2.2. Let $\varrho = 0$. It is easy to see that $P_2(t_{m-1,n-1}) > P_2(t_{mn})$. Hence

$$(2.3) \qquad \begin{aligned} b_{mn} &\geqslant \mu P_1(t_{mn}) + [1-P_1(t_{mn})][1-P_2(t_{m-1,n-1})]v_{m-1,n-1} \\ &= \mu P_1(t_{mn}) + [1-P_1(t_{mn})]v_{m,n-1} = v_{mn}, \end{aligned}$$

so that Player I has a good first shot time t_{mn}. In this case, Player I has a good strategy. Equality results in (2.3) if and only if $v_{m-1,n-1} = 0$ which, for $\mu > 0$, the only interesting case, requires $m = 1$.

3. The silent case. In the silent case we proceed as in [3]. Assume that P_i are differentiable. Set

$$r(z_k) = \begin{cases} \mu P_1(x_i) & \text{if } z_k = x_i, \\ -\varrho P_2(y_j) & \text{if } z_k = y_k, \end{cases}$$

and define $s(z_k)$ and $\psi(z_l, \ldots, z_k)$ as in [3]. Then $M(x, y) = \psi(z_1, \ldots, z_{m+n})$. The lemmas of Restrepo ([3], Section 4) are identically proved. Note that m and n are interchanged here.

We modify Restrepo ([3], Section 5) by putting

$$\varphi(\bar{D}^{k-1}) = \mu D_k + (1-D_k)\varphi(\bar{D}^k).$$

Then the formula in Lemma 4 takes the form

$$R(y_1, \ldots, y_{m-1}, y_m) - R(y_1, \ldots, y_{m-1}) = \prod_{i=1}^{k-1}(1-D_i)\prod_{j=1}^{m-1}[1-P_2(y_j)][1-P_2(y_m)] \times \times \Big\{(\mu+\varrho)\int_{y_m}^{a_{k+1}} P_1(x_k)\,dF_k(x_k) + (1-D_k)[\varrho+\varphi(\bar{D}^k)]\Big\}.$$

The remainder of the proof is as in the remaining sections of [3]. In Section 6 the factor 2 appearing in the definitions of the constants h_{ij} and γ_{ij} becomes $\mu+\varrho$.

Thus, we find that there exist constants $a_1 < a_2 < \ldots < a_m < 1$, $b_1 = a_1 < b_2 < \ldots < b_n = 1$, h_i $(i = 1, \ldots, m)$, k_j $(j = 1, \ldots, n)$, a $(0 \leqslant a < 1)$ and β $(0 \leqslant \beta < 1)$ such that, setting

$$f^*(t) = \prod_{b_j>t}[1-P_2(b_j)]\frac{P_2'(t)}{P_2^2(t)P_1(t)} \tag{3.1}$$

and

$$g^*(t) = \prod_{a_i>t}[1-P_1(a_i)]\frac{P_1'(t)}{P_1^2(t)P_2(t)}, \tag{3.1'}$$

we have

$$h_m\int_{a_m}^{1} f^*(t)\,dt + a = 1, \tag{3.2}$$

$$(3.2') \qquad k_n \int_{b_m}^{1} g^*(t)\,dt + \beta = 1,$$

$$(3.3) \qquad h_i \int_{a_i}^{a_{i+1}} f^*(t)\,dt = 1 \quad (i = 1, \ldots, m-1),$$

$$(3.3') \qquad k_j \int_{b_j}^{b_{j+1}} g^*(t)\,dt = 1 \quad (j = 1, \ldots, n-1),$$

$$(3.4) \qquad \int_{a_m}^{1} [\varrho + \mu a - \varrho(1-a)P_1(t)] f^*(t)\,dt = (\mu+\varrho)(1-a),$$

$$(3.4') \qquad \int_{b_n}^{1} [\mu + \varrho\beta - \mu(1-\beta)P_2(t)] g^*(t)\,dt = (\mu+\varrho)(1-\beta),$$

$$(3.5) \qquad \int_{a_i}^{a_{i+1}} [1-P_1(t)] f^*(t)\,dt = 1/h_{i+1} \quad (i = 1, \ldots, m-1),$$

$$(3.5') \qquad \int_{b_j}^{b_{j+1}} [1-P_2(t)] g^*(t)\,dt = 1/k_{j+1}. \quad (j = 1, \ldots, n-1),$$

$$(3.6) \qquad a\beta = 0.$$

Then the following theorem is obtained.

THEOREM 2. *The silent duel with the P_i differentiable has a value and both players have good strategies. Player I's good strategy requires firing the i-th bullet* ($i = 1, \ldots, m-1$) *at a time chosen in the interval* (a_i, a_{i+1}) *according to the density $k_i f^*$. His m-th bullet is fired at a time chosen in the interval* $(a_m, 1]$ *according to the distribution function F_m given by $F'_m(t) = k_m f^*(t)$ on* $(a_m, 1)$. *The a_i and k_i are chosen to satisfy* (3.1)-(3.5). *Player II's good strategy is similarly defined using* (3.1')-(3.5'). *Furthermore,* (3.6) *is satisfied.*

Example 3.1. Let $m = n = 1$, $P_1(t) = P_2(t) = t$. If $\mu = \varrho$, it is easy to see that $h_1 = k_1 = 1/2$, $a = \beta = 0$, $a_1 = 1/3$ and the value is 0.

Otherwise, try $a = 0$. Then, (3.2) yields $a_1^{-2} - 1 = 2/h_1$. This with (3.4) results in $\varrho h_1 \sqrt{1+2/h_1} = \varrho - \mu h_1$ which has solutions

$$h_1 = [\varrho(\mu+\varrho) \pm \varrho\sqrt{2\varrho(\mu+\varrho)}]/(\mu^2 - \varrho^2).$$

However, since $\varrho - \mu h_1 > 0$, this solution requires $\mu > \varrho$ which, in turn, requires the use of the negative sign.

By (3.2') we see that $\beta = 1 - k_1/h_1$ which with (3.4') and the previous results yields

$$k_1 = [2\varrho - \sqrt{2\varrho(\mu+\varrho)}]/[2(\varrho-\mu)].$$

Finally, the value is

$$v = -k_1 \int_{a_1}^{1} [\varrho t - \mu(1-t)]t^{-3}dt = -k_1[(\mu+\varrho)(a_1^{-1}-1) - \mu(a_1^{-2}-1)].$$

Similarly, we find that $\beta = 0$ requires $\mu > \varrho$ and the constants can all be found by symmetry.

4. The silent vs. noisy case. In the silent vs. noisy case we must consider separately $\varrho > 0$ and $\varrho = 0$. For $\varrho > 0$ we deal only with the case $n = 1$. For $\varrho = 0$ we consider separately the cases $m = 1$ and $n = 1$.

These cases are discussed in the three subsections following. Unfortunately, the silent vs. noisy duel has not been solved yet in greater generality.

4.1. $n = 1$, $\varrho > 0$. In this case, the method is that of Styszyński [4]. We assume that P_1 and P_2 are differentiable and their derivatives are positive on $(0, 1)$. The pay-off function is

$$M(x, y) = \begin{cases} \mu - [\mu + \varrho P_2(y)] \prod_{i=1}^{m} [1 - P_1(x_i)] & \text{if } x_m < y, \\ \mu - (\mu+\varrho)P_2(y) \prod_{i=1}^{k} [1 - P_1(x_i)] & \text{if } x_k < y < x_{k+1} \\ & (k = 1, \ldots, m-1), \\ \mu - (\mu+\varrho)P_2(y) & \text{if } y < x_1. \end{cases} \tag{4.1}$$

We will not consider $y = x_i$ for any $i = 1, \ldots, m$, since this will occur with probability 0.

The equation and solution for the probability density f_i $(i = 1, \ldots \ldots, m-1)$ for the time of firing of the i-th bullet by Player I is exactly as in [4]. However, for f_m we have

$$-\frac{[\mu + \varrho P_1(y)] f_m(y)}{\varrho - \varrho \int_{a_m}^{y} P_1(t) f_m(t)\,dt + \mu \int_{y}^{1} f_m(t)\,dt} = \frac{P_2'(y)[\mu + \varrho P_1(y)]}{[\mu + \varrho P_2(y)][1 - P_1(y)] - (\mu+\varrho)P_2(y)} \tag{4.2}$$

on $[a_m, 1)$. Since

$$f_m(a_m) = \frac{(\mu+\varrho)P_2'(a_m)}{(\mu+\varrho)P_2(a_m) - [\mu + \varrho P_2(a_m)][1 - P_1(a_m)]},$$

the solution of (4.2) we obtain is

$$f_m(t) = (\mu+\varrho)P_2'(t)K(t)\exp\Big\{-\int_{a_m}^{t} [\mu + \varrho P_1(u)]P_2'(u)K(u)\,du\Big\}, \tag{4.3}$$

where

$$K(t) = 1/\{(\mu+\varrho)P_2(t) - [\mu+\varrho P_2(t)][1-P_1(t)]\}.$$

The form of Player II's strategy is an absolutely continuous part with density g on $(a_1, 1)$ and probability β of firing at time 1. Here

$$\int_{a_1}^{1} g(t)\,dt = 1-\beta.$$

For $t \in (a_m, 1)$ we obtain

$$(4.4) \qquad -\frac{[\mu+\varrho P_2(t)]g(t)}{(\mu+\varrho)\beta + \int_t^1 [\mu+\varrho P_2(y)]g(y)\,dy} = \frac{P_1'(t)[\mu+\varrho P_2(t)]}{[\mu+\varrho P_2(t)][1-P_1(t)] - (\mu+\varrho)P_2(t)}.$$

The solution of (4.4) is of the form

$$(4.5) \qquad g(t) = (\mu+\varrho)\beta P_1'(t)K(t)\exp\left\{\int_t^1 [\mu+\varrho P_2(u)]P_1'(u)K(u)\,du\right\}.$$

The equation and solution for g on (a_i, a_{i+1}) $(i = 1, \ldots, m-1)$ is exactly as in [4] as is the proof of the existence of solutions for $a_1, \ldots, a_m$ and β.

Thus we have the following

THEOREM 3. *Consider the silent vs. noisy case with $\varrho > 0$ and $n = 1$. Assume that P_1 and P_2 are differentiable and strictly increasing. The game has a value and the players have good strategies. Player I's good strategy requires firing his i-th bullet $(i = 1, \ldots, m-1)$ on an interval (a_i, a_{i+1}) using the density proportional to that in* (3.1) *while his m-th bullet is fired on the interval $(a_m, 1)$ using density* (4.3). *Player II's good strategy requires firing his bullet at time* 1 *with probability $\beta > 0$ and, otherwise, on the interval $(a_1, 1)$ with density g which, on each (a_i, a_{i+1}) $(i = 1, \ldots, m-1)$, is proportional to that in* (3.1) *with P_1 and P_2 interchanged. On $(a_m, 1)$ the density g is given by* (4.5). *The value of the game is $\mu-(\mu+\varrho)P_2(a_1)$.*

Example 4.1. Let $m = 1$, $P_1(t) = P_2(t) = t$. Then (4.3) yields

$$f_1(t) = (\mu+\varrho)(\varrho a_1^2 + 2\mu a_1 - \mu)^{1/2}(\varrho t^2 + 2\mu t - \mu)^{-3/2}$$

so that we solve

$$\int_{a_1}^{1} f_1(t)\,dt = 1$$

to obtain

$$a_1 = [(\mu+\varrho)^{1/2}(\mu+2\varrho)^{1/2} - (\mu+\varrho)]/\varrho.$$

Furthermore, (4.4) yields

$$g(t) = \beta(\mu+\varrho)^{3/2}(\varrho t^2+2\mu t-\mu)^{-3/2},$$

and solving

$$\int_{a_1}^{1} g(t)\,dt = 1-\beta$$

we obtain

$$\beta = \varrho a_1/[\mu+\varrho(1+a_1)] = 1-(\mu+\varrho)^{1/2}/(\mu+2\varrho)^{1/2}.$$

Finally, the value of the game is $\mu-(\mu+\varrho)a_1$.

4.2. $m = 1$, $\varrho = 0$. Let t_{ij} be as defined by (2.1) and let v_{1n} be as defined by (2.2). It is easy to see that the solution is that of the noisy case. This includes the fact, noted in Example 2.2, that both players have a good first shot time t_{1n}. Thus Player II should shoot his j-th bullet at time $t_{1,n-j+1}$. Player I should fire at the first of the $t_{1,n-j+1}$ $(j = 1, \ldots, n+1)$ for which it is not true that Player II has previously fired j bullets.

4.3. $n = 1$, $\varrho = 0$. Assume that P_1 and P_2 are strictly increasing and differentiable. Let t_{11} be as defined in (2.1), i.e., $P_1(t_{11})+P_2(t_{11}) = 1$. Let $x = (x_1, \ldots, x_m)$ be a strategy for Player I with $x_m = t_{11}$. Then

$$(4.6)\qquad M(x, y) = \begin{cases} \mu[1-P_2(y)] & \text{if } y < x_1, \\ \mu\Big\{1-P_2(y)\prod_{j=1}^{i}[1-P_1(x_j)]\Big\} & \text{if } x_i \leqslant y < x_{i+1} \\ & (i = 1, \ldots, m-1), \\ \mu\Big\{1-\prod_{j=1}^{m}[1-P_1(x_j)]\Big\} & \text{if } y \geqslant t_{11}. \end{cases}$$

For $i = 1, \ldots, m-1$ let f_i be the density of x_i and assume that the support of f_i is (a_i, a_{i+1}), where $a_m = t_{11}$. We consider the randomized strategy F for which x_i $(i = 1, \ldots, m-1)$ is chosen according to f_i and $x_m = t_{11}$. Then (4.6) yields

$$(4.7)\qquad M(F, y) = $$

$$= \begin{cases} \mu\Big\{1-P_2(y)\prod_{j=1}^{i-1}\int_{a_j}^{a_{j+1}}[1-P_1(x_j)]f_j(x_j)\,dx_j\Big[1-\int_{a_1}^{y}P_1(x_j)f_i(x_i)\,dx_i\Big]\Big\} \\ \qquad\qquad \text{if } a_i < y < a_{i+1} \ (i = 1, \ldots, m-1), \\ \mu\Big\{1-\prod_{j=1}^{m}\int_{a_j}^{a_{j+1}}[1-P_1(x_j)]f_j(x_j)\,dx_j\Big\} \qquad \text{if } y \geqslant t_{11}. \end{cases}$$

Assume that F is a good strategy, so $M(F, y) = v$ for $a_1 < y < t_{11}$. Differentiating in (4.7) with respect to $y \in (a_1, t_{11})$ yields, as in [4],

$$(4.8) \quad f_i(x_i) = \frac{P_2(a_i)P_2'(x_i)}{P_1(x_i)P_2^2(x_i)} \quad \text{if } a_i < x_i < a_{i+1} \ (i = 1, \ldots, m-1).$$

Let G assign the probability δ to t_{11} and the probability $1-\delta$ according to the density g on (a_1, t_{11}). Here

$$\int_{a_1}^{t_{11}} g(y)\,dy = 1-\delta.$$

Then, by (4.4),

$$(4.9) \quad M(x, G) = \mu\Big\{1 - \int_{a_1}^{x_1} P_2(y)g(y)\,dy - \\ - \sum_{i=1}^{m-1} \prod_{j=1}^{i} [1-P_1(x_j)] \int_{x_i}^{x_{i+1}} P_2(y)g(y)\,dy - \delta \prod_{j=1}^{m} [1-P_1(x_j)]\Big\}$$

for x in the support of F. But, for such x, if G is a good strategy, then $M(x, G) = v$. Differentiating in (4.9) successively with respect to $x_{m-1}, \ldots$ $\ldots, x_1$ yields, as in [4],

$$(4.10) \quad g(y) = \frac{l_i P_1'(y)}{P_2(y)P_1^2(y)} \quad \text{if } a_i < y < a_{i+1} \ (i = 1, \ldots, m-1),$$

where $l_{m-1} = \delta P_1(t_{11})P_2(t_{11})$ and $l_{i-1} = l_i[1-P_1(a_i)]$.

Substituting (4.8) into (4.7) and using $M(F, t_{11}) = v$ we get

$$v = \mu[1-P_2(a_1)].$$

The same result is obtained from $M(F, y) = v$ if $a_1 < y < t_{11}$. Substituting (4.10) into (4.9), using $M(x, G) = v$ and setting $x_i = a_i$ $(i = 2, \ldots, m-1)$ we obtain

$$v = \mu\{1 - l_1[1-P_1(a_1)]/P_1(a_1)\}.$$

The existence and uniqueness of the a_i $(i = 1, \ldots, m-1)$ follow as in [4].

Then, by using

$$l_i = l_{m-1} \prod_{j=i+1}^{m-1} [1-P_1(a_j)] = \delta P_1(t_{11})P_2(t_{11}) \prod_{j=i+1}^{m-1} [1-P_1(a_j)],$$

the normalizing equation

$$\int_{a_1}^{t_{11}} g(y)\,dy + \delta = 1$$

takes the form

$$\delta\left\{P_1(t_{11})P_2(t_{11})\sum_{i=1}^{m-1}T_i\prod_{j=i+1}^{m-1}[1-P_1(a_j)]+1\right\}=1,$$

where

$$T_i=\int_{a_i}^{a_{i+1}}\frac{P_1'(y)}{P_2(y)P_1^2(y)}\,dy,$$

so the remaining constants, $\delta, l_1, \ldots, l_{m-1}$, are uniquely determined.

This proves the following theorem:

THEOREM 4. *Consider the silent vs. noisy duel with $\varrho = 0$ and $n = 1$. Assume that P_1 and P_2 are strictly increasing and differentiable. The game has a value and the players have good strategies. Player I's good strategy requires firing his i-th bullet ($i = 1, \ldots, m-1$) on an interval (a_i, a_{i+1}) using density (4.8), where $a_m = t_{11}$ as defined by (2.1). The m-th bullet is fired at time t_{11}. Player II's good strategy requires firing his bullet at time t_{11} with probability $\delta > 0$ and, otherwise, on the interval (a_1, t_{11}) using density (4.10). The value of the game is $\mu[1-P_2(a_1)]$.*

It is interesting to note, when $\varrho = 0$, that (4.3) yields

$$(4.11)\quad f_m(t)=\frac{P_2'(t)}{P_1(t)+P_2(t)-1}\exp\left\{-\frac{1}{\mu}\int_{a_m}^{t}\frac{[1+P_1(u)]P_2'(u)}{P_1(u)+P_2(u)-1}\,du\right\}.$$

However,

$$\lim_{\varrho\to 0} a_m = t_{11}.$$

But (4.11) with $a_m = t_{11}$ is not a density. In fact, the distribution of x_m is converging as $\varrho\to 0$ to degenerate at t_{11}. A similar remark applies to g on $(t_{11}, 1)$ as given by (4.5). Furthermore, β converges to 0 as $\varrho\to 0$. It is the probability assigned by g on $(a_m, 1)$ which tends to δ.

Example 4.2. Let $m = 2$ and $P_1(t) = P_2(t) = t$ so that $t_{11} = 1/2$ and $l_1 = \delta/4$. Since $f_1(t) = a_1 t^{-3}$, we obtain $a_1 = (\sqrt{5}-1)/4$. Also $g(t) = \delta t^{-3}/4$ for $a_1 < t < 1/2$ yields $\delta = 1-\sqrt{5}/5$. Finally, the value is

$$\mu(1-a_1)=\mu(5-\sqrt{5})/4.$$

5. Non-discrete firing case. The formulation and theorem of this section follow the pattern set by Lang and Kimeldorf in [2]. See their paper for a full motivation.

For $i = 1, 2$ assume that we are given numbers $N_i \geqslant 0$ (ammunition available to Player i) and strictly increasing, absolutely continuous functions A_i on $[0, 1]$ satisfying $A_i(0) = 0$ and $A_i(1-) = \infty$. The A_i are called *modified accuracy functions*. A pure strategy for Player i is a measure

λ_i on $(0, 1)$ for which $\lambda_i(0, 1) \leqslant N_i$. Set

$$Q_i(t) = 1 - \exp\Big\{-\int_{(0,t]} A_i d\lambda_i\Big\}.$$

The pay-off function is

$$M(\lambda_1, \lambda_2) = \mu \int_{(0,1)} (1-Q_2)dQ_1 - \varrho \int_{(0,1)} (1-Q_1)dQ_2. \tag{5.1}$$

There is an implicit assumption in this formulation that the duel is silent. However, any randomized strategy is clearly equivalent to a non--randomized strategy. Hence, the good strategies, the existence of which can be demonstrated, are pure. By the spy-proof property of good strategies, we will then have the solution for the noisy case and for all mixed information cases.

Let

$$f_{\tau,a}(t) = \begin{cases} \dfrac{A_2'(t)}{A_2(t)[A_1(t)+A_2(t)/\tau]} & \text{if } a < t < 1, \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

and

$$g_{\tau,a}(t) = \begin{cases} \dfrac{A_1'(t)}{A_1(t)[\tau A_1(t)+A_2(t)]} & \text{if } a < t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Lang and Kimeldorf [2] proved the existence of $\tau > 0$ and $a \in (0, 1)$ such that

$$\int_a^1 f_{\tau,a}(t)\,dt = N_1 \quad \text{and} \quad \int_a^1 g_{\tau,a}(t)\,dt = N_2. \tag{5.4}$$

Let f_0 and g_0 be the functions defined in (5.2) and (5.3), respectively, with a and τ chosen to satisfy (5.4). We can prove that f_0 and g_0 are densities (normalized to give total masses N_1 and N_2, respectively) with respect to Lebesgue measure of good strategies for Players I and II, respectively.

The proof of the following theorem proceeds exactly as in [2].

THEOREM 5. *The densities f_0 and g_0 are good strategies for Players* I *and* II, *respectively. The value of the game is*

$$\frac{\tau\mu A_1(a) - \varrho A_2(a)}{\tau A_1(a) + A_2(a)}.$$

Example 5.1. Let $A_1(t) = A_2(t) = -\log(1-t)$. This corresponds to setting $P_1(t) = P_2(t) = t$ in usual duels (see [2]). Then

$$f_{\tau,a}(t) = [(1-t)(1+1/\tau)\log^2(1-t)]^{-1}$$

and

$$g_{\tau,a}(t) = [(1-t)(1+\tau)\log^2(1-t)]^{-1}.$$

Solving (5.4) we get $\tau = N_1/N_2$ and $a = 1-\exp[-1/(N_1+N_2)]$, so the value is $(N_1\mu - N_2\varrho)/(N_1+N_2)$. Compare this with the expression for v_{mn} in Example 2.1.

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POJEDYNKI Z MOŻLIWOŚCIĄ ASYMETRYCZNYCH WYPŁAT

STRESZCZENIE

W pracy uogólnia się standardowy model pojedynku na przypadek niesymetrycznych wypłat. Przedstawione są wyniki dla pojedynku głośnego, cichego, cicho-głośnego i niedyskretnego. Ponieważ użyte metody są podobne do metod używanych w przypadku symetrycznym, dowody twierdzeń są na ogół opuszczone.