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On categories of structures and classes of algebras

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CONTENTS

1. Introduction and preliminaries

Primitive and related classes of universal algebras can be characterized in terms of categories, i.e. necessary and sufficient conditions can be found for a given category to be isomorphic to a primitive (or quasiprimitive, ...) class of algebras of a type; see e.g. Isbell [2] and Felscher [1]. Another problem is to characterize such classes in terms of the theory of categories of structures; see Malcev ([3]–[5]). We shall go in this second direction. The notion of category is not used in this paper and we shall be dealing only with categories of structures (these will be called scategories here) in the sense of Malcev. Instead of isomorphism of categories, another notion must be introduced: namely, that of the equivalence of scategories. All the notions are introduced below and the paper is written so that no special knowledge is supposed.

Malcev [4] characterized I, S, P-closed classes of algebras (classes closed under formation of isomorphic images, subalgebras and Cartesian products); from this he got easily (see [5]) a characterization of primitive classes. In the present paper we shall give a characterization of S-closed, S, P-closed and I, S, P-closed classes of quasi-algebras (Section 14). We give also in Section 14 a characterization of arbitrary sets of quasi-algebras. The problem of characterizing arbitrary classes of quasi-algebras remains open.

I have also included (in Sections 8, 10, 11 and 15) some non-original theorems: they are stated in Malcev's papers without proofs. These theorems are only generalized to the infinitary case. In Theorem 29, condition (iii), we give another characterization of S, P-closed classes of algebras.

We shall work in the Gödel–Bernays set theory. Let us clear up some notation.

The ordered pair of a and b is denoted by $\langle a, b\rangle$. Classes of ordered pairs are called relations. Mapping is a relation satisfying a well-known condition. If φ is a mapping, then its domain is denoted by $\mathscr{D}(\varphi)$ and its range by $W(\varphi)$; these may be proper classes. The value of φ at a point x is denoted by $\varphi(x)$. By $\varphi{\restriction}A$ we denote the restriction of φ to A, and we put $\varphi''A = W(\varphi{\restriction}A)$. If $B \subseteq W(\varphi)$, then $\varphi^{-1''}B$ denotes the class of all $x \in \mathscr{D}(\varphi)$ such that $\varphi(x) \in B$. If φ is a mapping of A into B and ψ a mapping of B into C, then the composition is denoted by $\psi \circ \varphi$; it is a mapping of A into C. If A is a class, then id_A is the identical mapping of A onto itself. If A and M are sets, then A^M is the set of all mappings of M into A. If φ is a mapping of A into B, then $\mathrm{Ker}\,\varphi$ (the *kernel* of φ) is the relation in A which is defined by $\langle x, y\rangle \in \mathrm{Ker}\,\varphi$ if and only if $\varphi(x) = \varphi(y)$; it is evidently an equivalence relation in A.

An ordinal number α is identified with the set of all ordinal numbers smaller than α. Hence, 0 is the empty set. If A is a class, then 0 is the only mapping of 0 into A. If $A \neq 0$, then there does not exist any mapping of A into 0.

An ordinal number α is called cardinal if there does not exist a one-to-one mapping of α onto any ordinal number $\beta \in \alpha$. The cardinality $\mathrm{Card}(A)$ of a set A is that cardinal number α, for which there exists a one-to-one mapping of A onto α.

An ordinal number ϑ is called regular if it is infinite and the following holds: if $\alpha \in \vartheta$, and f is a mapping of α into ϑ, then $W(f)$ is not a confinal subset of ϑ. Every regular number is cardinal and every infinite cardinal number with non-limitting index is regular. An infinite cardinal number ϑ is regular if and only if, whenever $(A_\iota)_{\iota\in T}$ is a family such that $\mathrm{Card}\, T < \vartheta$ and $\mathrm{Card}\, A_\iota < \vartheta$ for all $\iota \in T$, then $\mathrm{Card}(\bigcup_{\iota\in T} A_\iota) < \vartheta$.

We shall use the axiom of choice and the following weaker form of Fundierungsaxiom: If a class A and an equivalence relation R in A are given, then there exists a class B and a mapping φ of A onto B such that $R = \mathrm{Ker}\,\varphi$.

2. Pre-scategories and scategories

By a *pre–scategory* we mean a triple $\mathcal{O}$, $\mathcal{U}$, $\mathcal{H}$ of classes such that $\mathcal{U}$ is a mapping of $\mathcal{O}$ into the class of all sets and $\mathcal{H}$ is a mapping assigning to every ordered pair $\boldsymbol{A}, \boldsymbol{B}$ of elements of $\mathcal{O}$ a set of mappings of $\mathcal{U}(\boldsymbol{A})$ into $\mathcal{U}(\boldsymbol{B})$.

Let a pre-scategory $\mathfrak{A}$ be given. The first member of this triple will be denoted by $\mathfrak{A}^{\mathcal{O}}$, the second by $\mathcal{U}_{\mathfrak{A}}$ and the third by $\mathcal{H}_{\mathfrak{A}}$. Elements of $\mathfrak{A}^{\mathcal{O}}$ are called $\mathfrak{A}$-*structures* (or briefly *structures*, if it is clear from context which pre-scategory $\mathfrak{A}$ is considered) and they will be denoted by $\boldsymbol{A}$, $\boldsymbol{B}, \ldots, \boldsymbol{Z}, \bar{\boldsymbol{A}}, \boldsymbol{A}_\iota, \ldots$ If $\boldsymbol{A}$ is an $\mathfrak{A}$-structure, then $\mathcal{U}_{\mathfrak{A}}(\boldsymbol{A})$ is called the *underlying set* of $\boldsymbol{A}$ (with respect to $\mathfrak{A}$). Let us make a convention: if a structure is denoted by $\boldsymbol{A}$ (or $\boldsymbol{B}$, or $\boldsymbol{A}_\iota, \ldots$ resp.), then its underlying set is denoted by A (or B, or $A_\iota, \ldots$ resp.). If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\mathfrak{A}$-structures, then the elements of $\mathcal{H}_{\mathfrak{A}}(\boldsymbol{A}, \boldsymbol{B})$ are called $\mathfrak{A}$-*morphisms* of $\boldsymbol{A}$ into $\boldsymbol{B}$; we shall write "$\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$" or briefly "$\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$" instead of "$\varphi \in \mathcal{H}_{\mathfrak{A}}(\boldsymbol{A}, \boldsymbol{B})$".

If $\boldsymbol{A}$ is an $\mathfrak{A}$-structure, then Card A is called the *order* of $\boldsymbol{A}$. Structures of order 0 and 1 are called *trivial*. A pre-scategory is called trivial if it contains only trivial structures.

A pre-scategory $\mathfrak{A}$ is called *scategory* if it satisfies the following three conditions:

(1) if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$, then $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{A}$;

(2) if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathfrak{A}^{\mathcal{O}}$, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\psi\colon \boldsymbol{B} \to \boldsymbol{C}$, then $\psi \circ \varphi\colon \boldsymbol{A} \to \boldsymbol{C}$;

(3) if $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, $A = B$, $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\mathrm{id}_A\colon \boldsymbol{B} \to \boldsymbol{A}$, then $\boldsymbol{A} = \boldsymbol{B}$.

A pre-scategory $\mathfrak{B}$ is called *subpre-scategory of a pre-scategory* $\mathfrak{A}$ if it satisfies the following three conditions:

(4) $\mathfrak{B}^{\mathcal{O}} \subseteq \mathfrak{A}^{\mathcal{O}}$;

(5) $\mathscr{U}_{\mathfrak{B}}(\boldsymbol{A}) = \mathscr{U}_{\mathfrak{A}}(\boldsymbol{A})$ for all $\boldsymbol{A} \in \mathfrak{B}^{\mathcal{O}}$;

(6) $\mathscr{H}_{\mathfrak{B}}(\boldsymbol{A}, \boldsymbol{B}) = \mathscr{H}_{\mathfrak{A}}(\boldsymbol{A}, \boldsymbol{B})$ for all $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{B}^{\mathcal{O}}$.

Subpre-scategories of a given pre-scategory $\mathfrak{A}$ are in an obvious one-to-one correspondence with subclasses of $\mathfrak{A}^{\mathcal{O}}$; if such a subclass is given, we shall speak of a corresponding subpre-scategory. Subpre-scategories which actually are scategories are called *subscategories*. Evidently, every subpre-scategory of a scategory is a subscategory.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two pre-scategories. A mapping ε of $\mathfrak{A}^{\mathcal{O}}$ into $\mathfrak{B}^{\mathcal{O}}$ is called *sfunctor of* $\mathfrak{A}$ *into* $\mathfrak{B}$ if it satisfies the following two conditions:

(7) if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$, then $\boldsymbol{A}$ and $\varepsilon(\boldsymbol{A})$ have the same underlying sets;

(8) if $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, then every $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is a $\mathfrak{B}$-morphism of $\varepsilon(\boldsymbol{A})$ into $\varepsilon(\boldsymbol{B})$.

If, moreover, $W(\varepsilon) = \mathfrak{B}^{\mathcal{O}}$, then ε is called *sfunctor of* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

A sfunctor ε of $\mathfrak{A}$ into $\mathfrak{B}$ is called an *equivalence of* $\mathfrak{A}$ *onto* $\mathfrak{B}$ if there exists a sfunctor $\bar{\varepsilon}$ of $\mathfrak{B}$ into $\mathfrak{A}$ such that $\bar{\varepsilon} \circ \varepsilon = \mathrm{id}_{\mathfrak{A}^{\mathcal{O}}}$ and $\varepsilon \circ \bar{\varepsilon} = \mathrm{id}_{\mathfrak{B}^{\mathcal{O}}}$. It is evidently a one-to-one mapping of $\mathfrak{A}^{\mathcal{O}}$ onto $\mathfrak{B}^{\mathcal{O}}$.

Pre-scategories $\mathfrak{A}$ and $\mathfrak{B}$ are called *equivalent* if there exists an equivalence of $\mathfrak{A}$ onto $\mathfrak{B}$.

THEOREM 1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two scategories. A mapping* ε *of* $\mathfrak{A}^{\mathcal{O}}$ *into* $\mathfrak{B}^{\mathcal{O}}$ *is an equivalence of* $\mathfrak{A}$ *onto a subscategory of* $\mathfrak{B}$ *if and only if the following holds: if* $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, *then a mapping of* A *into* B *is an* $\mathfrak{A}$*-morphism of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$ *if and only if it is a* $\mathfrak{B}$*-morphism of* $\varepsilon(\boldsymbol{A})$ *into* $\varepsilon(\boldsymbol{B})$.

The proof is easy.

3. Scategorization

Let $\mathfrak{A}$ be a pre-scategory. A scategory $\mathfrak{B}$ is called a *scategorization of* $\mathfrak{A}$ *with respect to* ε if ε is a sfunctor of $\mathfrak{A}$ onto $\mathfrak{B}$ and for every sfunctor $\bar{\varepsilon}$ of $\mathfrak{A}$ into a scategory $\overline{\mathfrak{B}}$ there exists a sfunctor ε' of $\mathfrak{B}$ into $\overline{\mathfrak{B}}$ such that $\bar{\varepsilon} = \varepsilon' \circ \varepsilon$. $\mathfrak{B}$ is called a *scategorization of* $\mathfrak{A}$ if it is a scategorization with respect to some ε.

THEOREM 2. (i) *Every pre-scategory has a scategorization.*

(ii) *If $\mathfrak{B}_1$ is a scategorization of $\mathfrak{A}$ with respect to ε_1 and $\mathfrak{B}_2$ a scategorization of $\mathfrak{A}$ with respect to ε_2, then there exists exactly one equivalence ε of $\mathfrak{B}_1$ onto $\mathfrak{B}_2$ such that $\varepsilon_2 = \varepsilon \circ \varepsilon_1$.*

(iii) *Let $\mathfrak{B}$ be a scategorization of $\mathfrak{A}$ with respect to ε. If $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, then a mapping φ of A into B is a $\mathfrak{B}$-morphism of $\varepsilon(\boldsymbol{A})$ into $\varepsilon(\boldsymbol{B})$ if and only if either*

$$\boldsymbol{A} = \boldsymbol{B} \quad \textit{and} \quad \varphi = \mathrm{id}_A$$

or there exists a finite sequence

$$\varphi_1\colon \boldsymbol{A}_0 \to \boldsymbol{A}_1, \ \ldots, \ \varphi_n\colon \boldsymbol{A}_{n-1} \to \boldsymbol{A}_n \quad (n \geqslant 1) \tag{9}$$

of $\mathfrak{A}$-structures and $\mathfrak{A}$-morphisms such that

$$\boldsymbol{A} = \boldsymbol{A}_0, \quad \boldsymbol{B} = \boldsymbol{A}_n \quad \textit{and} \quad \varphi = \varphi_n \circ \ldots \circ \varphi_1. \tag{10}$$

(iv) *Let $\mathfrak{B}$ be a scategorization of $\mathfrak{A}$ with respect to ε and let $\mathfrak{A}$ fulfil* (1) *and* (2). *If $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, then a mapping of A into B is a $\mathfrak{B}$-morphism of $\varepsilon(\boldsymbol{A})$ into $\varepsilon(\boldsymbol{B})$ if and only if it is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

Proof. Let $\mathfrak{A}$ be a pre-scategory. Let us define a pre-scategory $\overline{\mathfrak{A}}$ in the following way: $\overline{\mathfrak{A}}^{\mathcal{O}} = \mathfrak{A}^{\mathcal{O}}$; if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$, then $\mathscr{U}_{\overline{\mathfrak{A}}}(\boldsymbol{A}) = \mathscr{U}_{\mathfrak{A}}(\boldsymbol{A})$; if $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, then a mapping φ of A into B is an $\overline{\mathfrak{A}}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ if and only if either $\boldsymbol{A} = \boldsymbol{B}$ and $\varphi = \mathrm{id}_A$ or there exists a finite sequence (9) of $\mathfrak{A}$-structures and $\mathfrak{A}$-morphisms such that (10) holds. We define a binary relation R in $\mathfrak{A}^{\mathcal{O}}$ by: $\langle \boldsymbol{A}, \boldsymbol{B}\rangle \in R$ if and only if $A = B$, $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\overline{\mathfrak{A}}$ and $\mathrm{id}_A\colon \boldsymbol{B} \to \boldsymbol{A}$ in $\overline{\mathfrak{A}}$. It is evidently an equivalence relation in $\mathfrak{A}^{\mathcal{O}}$; from the Fundierungsaxiom it follows that there exists a class $\mathscr{B}$ and a mapping ε of $\mathfrak{A}^{\mathcal{O}}$ onto $\mathscr{B}$ such that R is the kernel of ε. It is evidently possible to define a pre-scategory $\mathfrak{B}$ in this way: $\mathfrak{B}^{\mathcal{O}} = \mathscr{B}$; the underlying set of a $\mathfrak{B}$-structure $\varepsilon(\boldsymbol{A})$ is the set A; a mapping of A into B is a $\mathfrak{B}$-morphism of $\varepsilon(\boldsymbol{A})$ into $\varepsilon(\boldsymbol{B})$ if and only if it is an $\overline{\mathfrak{A}}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. It can be easily proved that $\mathfrak{B}$ is a scategory and that it is a scategorization of $\mathfrak{A}$ with respect to ε. Everything else is evident.

4. Abstract scategories

Let a scategory $\mathfrak{A}$ and $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$ be given. A morphism φ of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called *isomorphism of* $\boldsymbol{A}$ *onto* $\boldsymbol{B}$ if there exists a morphism ψ of $\boldsymbol{B}$ into $\boldsymbol{A}$ such that $\psi \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \psi = \mathrm{id}_B$. It is evidently a one-to-one mapping of A onto B and ψ is the inverse mapping. We write $\varphi\colon \boldsymbol{A} \simeq \boldsymbol{B}$ if φ is an isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$. If there exists an isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$, then we write $\boldsymbol{A} \simeq \boldsymbol{B}$ and say that $\boldsymbol{A}$ is *isomorphic to* $\boldsymbol{B}$.

A scategory $\mathfrak{A}$ is called *abstract* if the following holds: if an $\mathfrak{A}$-structure $\boldsymbol{A}$ and a one-to-one mapping φ of A onto a set B is given, then there exists an $\mathfrak{A}$-structure $\boldsymbol{B}$ with the underlying set B such that $\varphi\colon \boldsymbol{A} \simeq \boldsymbol{B}$. (It follows from (3) that this $\boldsymbol{B}$ is uniquely determined.)

A scategory $\mathfrak{A}$ is called a *skeleton of a scategory* $\mathfrak{B}$ if it is a subscategory of $\mathfrak{B}$ and every $\mathfrak{B}$-structure is in $\mathfrak{B}$ isomorphic to an $\mathfrak{A}$-structure.

A scategory $\mathfrak{B}$ is called an *abstraction of a scategory* $\mathfrak{A}$ if it is abstract and contains a skeleton which is equivalent to $\mathfrak{A}$.

THEOREM 3. (i) *Every scategory has an abstraction.*

(ii) *Every two abstractions of a given scategory are equivalent scategories. If* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are two abstract scategories,* $\mathfrak{A}_1$ *a skeleton of* $\mathfrak{B}_1$, $\mathfrak{A}_2$ *a skeleton of* $\mathfrak{B}_2$ *and* ε *an equivalence of* $\mathfrak{A}_1$ *onto* $\mathfrak{A}_2$, *then* ε *can be extended to an equivalence of* $\mathfrak{B}_1$ *onto* $\mathfrak{B}_2$ *in exactly one way.*

Proof. Let $\mathfrak{A}$ be a given scategory. Let $\mathscr{B}$ be the class of all ordered pairs $\langle \boldsymbol{A}, f\rangle$ such that $\boldsymbol{A}$ is an $\mathfrak{A}$-structure and f a one-to-one mapping such that $A = \mathscr{D}(f)$. Let us define a pre-scategory $\overline{\mathfrak{B}}$ in this way: $\overline{\mathfrak{B}}^{\mathscr{O}} = \mathscr{B}$; $\mathscr{U}_{\overline{\mathfrak{B}}}(\langle \boldsymbol{A}, f\rangle) = W(f)$; a mapping φ of $W(f)$ into $W(g)$ is a $\overline{\mathfrak{B}}$-morphism of $\langle \boldsymbol{A}, f\rangle$ into $\langle \boldsymbol{B}, g\rangle$ if and only if $g^{-1} \circ \varphi \circ f$ is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. This pre-scategory $\overline{\mathfrak{B}}$ fulfils (1) and (2). Let $\mathfrak{B}$ be its scategorization with respect to $\bar{\varepsilon}$. It is easy to prove that $\mathfrak{B}$ is abstract and that the mapping ε of $\mathfrak{A}^{\mathscr{O}}$ into $\mathfrak{B}^{\mathscr{O}}$ defined by $\varepsilon(\boldsymbol{A}) = \bar{\varepsilon}(\langle \boldsymbol{A}, \mathrm{id}_A\rangle)$ is an equivalence of $\mathfrak{A}$ onto a skeleton of $\mathfrak{B}$. Everything else is evident.

5. Substructures and products of structures

Let $\mathfrak{A}$ be a scategory. An $\mathfrak{A}$-structure $\boldsymbol{A}$ is called *substructure* (or: $\mathfrak{A}$-*substructure*) *of an* $\mathfrak{A}$-*structure* $\boldsymbol{B}$ if the following conditions have place:

(11) $A \subseteq B$;

(12) $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$;

(13) if $\boldsymbol{D} \in \mathfrak{A}^{\mathscr{O}}$, $\varphi\colon \boldsymbol{D} \to \boldsymbol{B}$ and $W(\varphi) \subseteq A$, then $\varphi\colon \boldsymbol{D} \to \boldsymbol{A}$.

We also say that $\boldsymbol{B}$ is an *overstructure* of $\boldsymbol{A}$.

Remark. Malcev [3] defines: a substructure $\boldsymbol{B}$ of $\boldsymbol{A}$ is called *strong* if every $\mathfrak{A}$-morphism into $\boldsymbol{B}$ is an $\mathfrak{A}$-morphism into $\boldsymbol{A}$. Every substructure is evidently strong.

THEOREM 4. *If $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$ and $\boldsymbol{B}$ a substructure of $\boldsymbol{C}$, then $\boldsymbol{A}$ is a substructure of $\boldsymbol{C}$.*

Proof. Evidently $A \subseteq C$. We have $\mathrm{id}_A = \mathrm{id}_B \circ \mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{C}$. Let $\varphi\colon \boldsymbol{D} \to \boldsymbol{C}$ and $W(\varphi) \subseteq A$. Since $\boldsymbol{B}$ is a substructure of $\boldsymbol{C}$, we get $\varphi\colon \boldsymbol{D} \to \boldsymbol{B}$; since $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, we get $\varphi\colon \boldsymbol{D} \to \boldsymbol{A}$.

THEOREM 5. *Let $\boldsymbol{A}$ be a substructure of $\boldsymbol{B}$ and $\boldsymbol{D} \in \mathfrak{A}^{\mathcal{O}}$. If $\varphi\colon \boldsymbol{D} \to \boldsymbol{A}$, then also $\varphi\colon \boldsymbol{D} \to \boldsymbol{B}$. If $\varphi\colon \boldsymbol{B} \to \boldsymbol{D}$, then $\varphi \restriction A\colon \boldsymbol{A} \to \boldsymbol{D}$.*

Proof. In the first case $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$, hence $\varphi = \mathrm{id}_A \circ \varphi\colon \boldsymbol{D} \to \boldsymbol{B}$. In the second case $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\varphi\colon \boldsymbol{B} \to \boldsymbol{D}$, hence $\varphi \restriction A = \varphi \circ \mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{D}$.

THEOREM 6. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be substructures of $\boldsymbol{C}$ and $A \subseteq B$. Then $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$.*

Proof. $A \subseteq B$ is supposed. Since $\boldsymbol{A}$ is a substructure of $\boldsymbol{C}$, we have $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{C}$; since $\boldsymbol{B}$ is a substructure of $\boldsymbol{C}$, we get $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$. Let $\boldsymbol{D} \in \mathfrak{A}^{\mathcal{O}}$, $\varphi\colon \boldsymbol{D} \to \boldsymbol{B}$ and $W(\varphi) \subseteq A$. By Theorem 5 we have $\varphi\colon \boldsymbol{D} \to \boldsymbol{C}$; since $\boldsymbol{A}$ is a substructure of $\boldsymbol{C}$, we get $\varphi\colon \boldsymbol{D} \to \boldsymbol{A}$.

THEOREM 7. *Let $\mathfrak{A}$ be an abstract scategory and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathfrak{A}^{\mathcal{O}}$; let $\boldsymbol{A}$ be isomorphic to $\boldsymbol{B}$ and $\boldsymbol{B}$ a substructure of $\boldsymbol{C}$. Then there exists an $\mathfrak{A}$-structure $\boldsymbol{D}$ such that $\boldsymbol{A}$ is a substructure of $\boldsymbol{D}$ and $\boldsymbol{D}$ is isomorphic to $\boldsymbol{C}$; a given isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ may be extended to an isomorphism of $\boldsymbol{D}$ onto $\boldsymbol{C}$.*

Proof. Let $\varphi\colon \boldsymbol{A} \simeq \boldsymbol{B}$. This φ can be extended to a one-to-one mapping ψ of a set D onto C; we may e.g. take the first ordinal number α such that $\langle m, \alpha\rangle$ does not belong to A for any m and define $\psi^{-1}(x) = \varphi^{-1}(x)$ for $x \in B$ and $\psi^{-1}(x) = \langle x, \alpha\rangle$ for $x \in C - B$. As $\mathfrak{A}$ is abstract, there exists an $\mathfrak{A}$-structure $\boldsymbol{D}$ with the underlying set D such that $\psi^{-1}\colon \boldsymbol{C} \simeq \boldsymbol{D}$, i. e. $\psi\colon \boldsymbol{D} \simeq \boldsymbol{C}$. It remains to prove that $\boldsymbol{A}$ is a substructure of $\boldsymbol{D}$. Clearly $A \subseteq D$. We have $\mathrm{id}_A = \psi^{-1} \circ \varphi\colon \boldsymbol{A} \to \boldsymbol{D}$. Let $\boldsymbol{H} \in \mathfrak{A}^{\mathcal{O}}$, $\chi\colon \boldsymbol{H} \to \boldsymbol{D}$ and $W(\chi) \subseteq A$. We have $\psi \circ \chi\colon \boldsymbol{H} \to \boldsymbol{C}$ and $W(\psi \circ \chi) \subseteq B$; $\boldsymbol{B}$ is a substructure of $\boldsymbol{C}$, so $\psi \circ \chi\colon \boldsymbol{H} \to \boldsymbol{B}$. Hence $\chi = \varphi^{-1} \circ \psi \circ \chi\colon \boldsymbol{H} \to \boldsymbol{A}$.

Let $\boldsymbol{A}$ be an $\mathfrak{A}$-structure. For every set $M \subseteq A$ there exists evidently at most one substructure of $\boldsymbol{A}$ with the underlying set M; if such a substructure exists, then M is called a *set underlying in* $\boldsymbol{A}$ (or: a *set* $\mathfrak{A}$-*underlying in* $\boldsymbol{A}$). If M is underlying in $\boldsymbol{A}$, then we shall speak of the corresponding substructure of $\boldsymbol{A}$.

A set $M \subseteq A$ is called $\mathfrak{A}$-*generating in* $\boldsymbol{A}$ if there exists no set B underlying in $\boldsymbol{A}$ such that $M \subseteq B \subset A$.

Let a scategory $\mathfrak{A}$ and a family $(\boldsymbol{A}_t)_{t \in T}$ (T being a set) of $\mathfrak{A}$-structures be given. An $\mathfrak{A}$-structure $\boldsymbol{A}$ and a family $\pi_t\colon \boldsymbol{A} \to \boldsymbol{A}_t$ ($t \in T$) of $\mathfrak{A}$-morphisms is called the *direct product of* $(\boldsymbol{A}_t)_{t\in T}$ *in* $\mathfrak{A}$ if for every $\mathfrak{A}$-structure $\boldsymbol{B}$ and every family φ_t: $\boldsymbol{B} \to \boldsymbol{A}_t$ $(t \in T)$ of $\mathfrak{A}$-morphisms there exists exactly one φ: $\boldsymbol{B} \to \boldsymbol{A}$ in $\mathfrak{A}$ such that $\varphi_t = \pi_t \circ \varphi$ for all $t \in T$.

If π_t: $\boldsymbol{A} \to \boldsymbol{A}_t$ $(t \in T)$ and $\bar{\pi}_t$: $\bar{\boldsymbol{A}} \to \boldsymbol{A}_t$ $(t \in T)$ are two direct products in $\mathfrak{A}$ of a family $(\boldsymbol{A}_t)_{t\in T}$, then evidently $\boldsymbol{A} \simeq \bar{\boldsymbol{A}}$; there exists exactly one isomorphism φ: $\boldsymbol{A} \simeq \bar{\boldsymbol{A}}$ such that $\bar{\pi}_t = \pi_t \circ \varphi$ for all $t \in T$.

If $(A_t)_{t\in T}$ is a family of sets, then $\bigtimes_{t\in T} A_t$ denotes the set of all mappings f such that $\mathscr{D}(f) = T$ and $f(t) \in A_t$ for all $t \in T$. If $t \in T$, then the mapping φ of A into A_t, defined by $\varphi(f) = f(t)$ for all $f \in A$, is denoted by pr_t^A (or just pr_t).

Let $(\boldsymbol{A}_t)_{t\in T}$ be a family of $\mathfrak{A}$-structures; put $A = \bigtimes_{t\in T} A_t$. An $\mathfrak{A}$-structure $\boldsymbol{A}$ with the underlying set A is called the *Cartesian product* (in $\mathfrak{A}$) *of* $(\boldsymbol{A}_t)_{t\in T}$ if pr_t is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{A}_t$ for all $t \in T$ and the family pr_t: $\boldsymbol{A} \to \boldsymbol{A}_t$ $(t \in T)$ is a direct product of $(\boldsymbol{A}_t)_{t\in T}$. Every family has at most one Cartesian product.

A scategory is called *Cartesian* if every family of its structures has the Cartesian product.

THEOREM 8. *Let an abstract scategory* $\mathfrak{A}$ *and a family* $(\boldsymbol{A}_t)_{t\in T}$ *of* $\mathfrak{A}$-*structures be given. Suppose that there exists a direct product* π_t: $\boldsymbol{A} \to \boldsymbol{A}_t$ $(t \in T)$ *in* $\mathfrak{A}$ *such that for every family* $(a_t)_{t\in T}$ *of elements* $a_t \in A_t$ *there exists exactly one* $a \in A$ *satisfying* $\pi_t(a) = a_t$ *for all* $t \in T$. *Then* $(\boldsymbol{A}_t)_{t\in T}$ *has a Cartesian product in* $\mathfrak{A}$.

The proof is obvious.

6. Rich scategories

A scategory $\mathfrak{A}$ is called *rich* if for all $\boldsymbol{A} \in \mathfrak{A}^o$, every subset of A is underlying in $\boldsymbol{A}$.

A subscategory $\mathfrak{A}$ of a scategory $\mathfrak{B}$ is called an *S-skeleton of* $\mathfrak{B}$ if every $\mathfrak{B}$-structure is a $\mathfrak{B}$-substructure of an $\mathfrak{A}$-structure.

Supposing that a scategory $\mathfrak{A}$ is given, we shall construct in this section a rich scategory containing an S-skeleton equivalent to $\mathfrak{A}$. This will be done in two various ways; in connection therewith see also Theorem 16 below.

Although the results of this section are not applied below, I think they may be interesting.

An S-skeleton $\mathfrak{A}$ of a scategory $\mathfrak{B}$ is called *rough* if the following holds for all $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$: if $\boldsymbol{A}$ is an $\mathfrak{A}$-substructure of $\boldsymbol{B}$, then it is a $\mathfrak{B}$-substructure of $\boldsymbol{B}$ if and only if id_A can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{B}$ into $\boldsymbol{A}$.

THEOREM 9. *Every scategory $\mathfrak{A}$ is equivalent to a rough S-skeleton of a rich scategory $\mathfrak{B}$ satisfying*

(14) *if $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C} \in \mathfrak{B}^{\mathcal{O}}$, $\boldsymbol{A}$ being a $\mathfrak{B}$-substructure of $\boldsymbol{B}$ and $\varphi\colon \boldsymbol{A} \to \boldsymbol{C}$ in $\mathfrak{B}$, then φ can be extended to a $\mathfrak{B}$-morphism of $\boldsymbol{B}$ into a $\mathfrak{B}$-overstructure of $\boldsymbol{C}$.*

Proof. Let $\mathscr{K}$ be the class of all ordered pairs $\langle \boldsymbol{A}, M\rangle$ such that $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ and $M \subseteq A$. Let us define a pre-scategory $\bar{\mathfrak{A}}$ in this way: $\bar{\mathfrak{A}}^{\mathcal{O}} = \mathscr{K}$; $\mathscr{U}_{\bar{\mathfrak{A}}}(\langle \boldsymbol{A}, M\rangle) = M$; a mapping φ of M into N is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, M\rangle$ into $\langle \boldsymbol{B}, N\rangle$ if and only if φ can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. This $\bar{\mathfrak{A}}$ evidently satisfies (1) and (2). Let $\mathfrak{B}$ be a scategorization of $\bar{\mathfrak{A}}$ with respect to $\bar{\varepsilon}$. Define a mapping ε of $\mathfrak{A}^{\mathcal{O}}$ into $\mathfrak{B}^{\mathcal{O}}$ by $\varepsilon(\boldsymbol{A}) = \bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. A mapping φ of A into B is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, A\rangle$ into $\langle \boldsymbol{B}, B\rangle$ evidently if and only if it is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$; by Theorem 2, it is thus a $\mathfrak{B}$-morphism of $\varepsilon(\boldsymbol{A})$ into $\varepsilon(\boldsymbol{B})$ if and only if it is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. By Theorem 1, ε is an equivalence of $\mathfrak{A}$ onto a subscategory of $\mathfrak{B}$. It is easy to prove that if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ and $M \subseteq A$, then $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ is a $\mathfrak{B}$-substructure of $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. In view of this and Theorem 6, the $\mathfrak{B}$-substructures of $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ are exactly the $\mathfrak{B}$-structures $\bar{\varepsilon}(\langle \boldsymbol{A}, N\rangle)$ such that $N \subseteq M$. From this it is easy to prove that $\mathfrak{B}$ is rich and that ε is an equivalence of $\mathfrak{A}$ onto an S-skeleton of $\mathfrak{B}$. If $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$ is a $\mathfrak{B}$-substructure of $\bar{\varepsilon}(\langle \boldsymbol{B}, B\rangle)$, then we have $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle) = \bar{\varepsilon}(\langle \boldsymbol{B}, A\rangle)$, as $\bar{\varepsilon}(\langle \boldsymbol{B}, A\rangle)$ is also a $\mathfrak{B}$-substructure; from this it follows that id_A can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{B}$ into $\boldsymbol{A}$. The S-skeleton is thus rough. Let $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$, $\bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle)$ and $\bar{\varepsilon}(\langle \boldsymbol{C}, Q\rangle)$ be $\mathfrak{B}$-structures, the first being a $\mathfrak{B}$-substructure of the second; let φ be a $\mathfrak{B}$-morphism of $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ into $\bar{\varepsilon}(\langle \boldsymbol{C}, Q\rangle)$. As $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle) = \bar{\varepsilon}(\langle \boldsymbol{B}, M\rangle)$, φ can be extended to an $\mathfrak{A}$-morphism $\bar{\varphi}\colon \boldsymbol{B} \to \boldsymbol{C}$. The restriction $\bar{\varphi} \restriction N$ can be extended to $\bar{\varphi}$, so that

$$\bar{\varphi} \restriction N\colon\ \bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{C}, C\rangle) \quad \text{in } \mathfrak{B}.$$

Since $\bar{\varphi} \restriction N$ is an extension of φ and $\bar{\varepsilon}(\langle \boldsymbol{C}, C\rangle)$ is a $\mathfrak{B}$-overstructure of $\bar{\varepsilon}(\langle \boldsymbol{C}, Q\rangle)$, we have proved (14).

An S-skeleton $\mathfrak{A}$ of a scategory $\mathfrak{B}$ is called *faithful* if the following holds for all $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$: if $\boldsymbol{A}$ is an $\mathfrak{A}$-substructure of $\boldsymbol{B}$, then $\boldsymbol{A}$ is also a $\mathfrak{B}$-substructure of $\boldsymbol{B}$.

THEOREM 10. *Every scategory $\mathfrak{A}$ is equivalent to a faithful S-skeleton of a rich scategory.*

Proof. Let $\mathscr{K}$ be the class of all ordered pairs $\langle \boldsymbol{A}, M\rangle$ such that $\boldsymbol{A}$ is an $\mathfrak{A}$-structure and $M \subseteq A$. Let us define a pre-scategory $\bar{\mathfrak{A}}$ in this way: $\bar{\mathfrak{A}}^{\mathcal{O}} = \mathscr{K}$; $\mathscr{U}_{\bar{\mathfrak{A}}}(\langle \boldsymbol{A}, M\rangle) = M$; a mapping φ of M into N is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, M\rangle$ into $\langle \boldsymbol{B}, N\rangle$ if and only if it can be extended to an $\mathfrak{A}$-morphism of an $\mathfrak{A}$-substructure of $\boldsymbol{A}$ into $\boldsymbol{B}$. This pre-scategory evidently satisfies (1). Let $\mathfrak{B}$ be a scategorization of $\overline{\mathfrak{A}}$ with respect to $\bar{\varepsilon}$. Define a mapping ε of $\mathfrak{A}^{\mathcal{O}}$ into $\mathfrak{B}^{\mathcal{O}}$ by $\varepsilon(\boldsymbol{A}) = \bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. It is evidently a sfunctor; we shall prove that it is an equivalence of $\mathfrak{A}$ onto a subscategory of $\mathfrak{B}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$ and $\varphi\colon \varepsilon(\boldsymbol{A}) \to \varepsilon(\boldsymbol{B})$ in $\mathfrak{B}$; it is sufficient to prove $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$.

By Theorem 2 there exists a finite sequence

$$\varphi_1\colon \langle \boldsymbol{A}_0, M_0\rangle \to \langle \boldsymbol{A}_1, M_1\rangle, \ \ldots, \ \varphi_n\colon \langle \boldsymbol{A}_{n-1}, M_{n-1}\rangle \to \langle \boldsymbol{A}_n, M_n\rangle \qquad (n \geqslant 1)$$

of $\overline{\mathfrak{A}}$-structures and $\overline{\mathfrak{A}}$-morphisms such that

$$\langle \boldsymbol{A}_0, M_0\rangle = \langle \boldsymbol{A}, A\rangle, \quad \langle \boldsymbol{A}_n, M_n\rangle = \langle \boldsymbol{B}, B\rangle \quad \text{and} \quad \varphi = \varphi_n \circ \ldots \circ \varphi_1.$$

We show by induction on $i = 1, \ldots, n$ that $\varphi_i \circ \ldots \circ \varphi_1$ is an $\mathfrak{A}$-morphism of $\boldsymbol{A}_0$ into $\boldsymbol{A}_i$. Let $i = 1$. We have $\varphi_1\colon \langle \boldsymbol{A}, A\rangle \to \langle \boldsymbol{A}_1, M_1\rangle$ in $\overline{\mathfrak{A}}$; $\boldsymbol{A}$ is the only $\mathfrak{A}$-substructure of $\boldsymbol{A}$ whose underlying set contains A, so $\varphi_1\colon \boldsymbol{A} \to \boldsymbol{A}_1$ in $\mathfrak{A}$. Let the assertion hold for an i and let $i+1 \leqslant n$. We have

$\varphi_i \circ \ldots \circ \varphi_1\colon \boldsymbol{A}_0 \to \boldsymbol{A}_i$ in $\mathfrak{A}$ and $\varphi_{i+1}\colon \langle \boldsymbol{A}_i, M_i\rangle \to \langle \boldsymbol{A}_{i+1}, M_{i+1}\rangle$ in $\overline{\mathfrak{A}}$; there exists an $\mathfrak{A}$-substructure $\boldsymbol{A}_i'$ of $\boldsymbol{A}_i$ and an extension φ_{i+1}' of φ_{i+1} to an $\mathfrak{A}$-morphism of $\boldsymbol{A}_i'$ into $\boldsymbol{A}_{i+1}$. As $W(\varphi_i) \subseteq M_i \subseteq A_i'$, we have $W(\varphi_i \circ \ldots \circ \varphi_1) \subseteq A_i'$ and hence $\varphi_i \circ \ldots \circ \varphi_1\colon \boldsymbol{A}_0 \to \boldsymbol{A}_i'$ in $\mathfrak{A}$, so that $\varphi_{i+1}' \circ \varphi_i \circ \ldots \circ \varphi_1\colon \boldsymbol{A}_0 \to \boldsymbol{A}_{i+1}$ in $\mathfrak{A}$; evidently

$$\varphi_{i+1}' \circ \varphi_i \circ \ldots \circ \varphi_1 = \varphi_{i+1} \circ \varphi_i \circ \ldots \circ \varphi_1.$$

The assertion is thus proved; we get

$$\varphi_n \circ \ldots \circ \varphi_1\colon \boldsymbol{A}_0 \to \boldsymbol{A}_n \quad \text{in } \mathfrak{A}, \quad \text{i.e.} \quad \varphi\colon \boldsymbol{A} \to \boldsymbol{B} \quad \text{in } \mathfrak{A}.$$

Let $\boldsymbol{A}$ be an $\mathfrak{A}$-structure and $M \subseteq A$; we now prove that $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ is a $\mathfrak{B}$-substructure of $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. Since id_M can be extended to $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{A}$, we have $\mathrm{id}_M\colon \langle \boldsymbol{A}, M\rangle \to \langle \boldsymbol{A}, A\rangle$ in $\overline{\mathfrak{A}}$ and hence

$$\mathrm{id}_M\colon \bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle) \quad \text{in } \mathfrak{B}.$$

Let a $\mathfrak{B}$-structure $\bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle)$ be given and let φ be a $\mathfrak{B}$-morphism of $\bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle)$ into $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$ such that $W(\varphi) \subseteq M$. By Theorem 2 there exists a finite sequence

$$\varphi_1\colon \langle \boldsymbol{B}_0, N_0\rangle \to \langle \boldsymbol{B}_1, N_1\rangle, \ \ldots, \ \varphi_n\colon \langle \boldsymbol{B}_{n-1}, N_{n-1}\rangle \to \langle \boldsymbol{B}_n, N_n\rangle \qquad (n \geqslant 1)$$

of $\overline{\mathfrak{A}}$-structures and $\overline{\mathfrak{A}}$-morphisms such that

$$\langle \boldsymbol{B}_0, N_0\rangle = \langle \boldsymbol{B}, N\rangle, \quad \langle \boldsymbol{B}_n, N_n\rangle = \langle \boldsymbol{A}, A\rangle \quad \text{and} \quad \varphi = \varphi_n \circ \ldots \circ \varphi_1.$$

For every $i = 1, \ldots, n$ there exists an $\mathfrak{A}$-substructure C_{i-1} of $\boldsymbol{B}_{i-1}$ and an extension $\overline{\varphi}_i$ of φ_i to an $\mathfrak{A}$-morphism of C_{i-1} into $\boldsymbol{B}_i$. We define a set N'_{i-1} and a mapping φ'_i for every $i = 1, \ldots, n$ in this way:

$$N'_0 = N_0, \quad \varphi'_1 = \varphi_1; \quad N'_{i-1} = W(\varphi'_{i-1}), \quad \varphi'_i = \varphi_i \restriction N'_{i-1};$$

put further $N'_n = M$. We have

$$\varphi'_n \circ \ldots \circ \varphi'_1 = \varphi_n \circ \ldots \circ \varphi_1 = \varphi;$$

evidently

$$W(\varphi'_n) = W(\varphi'_n \circ \ldots \circ \varphi'_1) = W(\varphi) \subseteq N'_n.$$

Hence, φ'_i is for every $i = 1, \ldots, n$ a mapping of N'_{i-1} into N'_i and it can be extended to the $\mathfrak{A}$-morphism $\overline{\varphi}_i$; we get

$$\varphi'_i\colon \langle \boldsymbol{B}_{i-1}, N'_{i-1}\rangle \to \langle \boldsymbol{B}_i, N'_i\rangle \quad \text{in } \overline{\mathfrak{A}}$$

and consequently

$$\varphi'_i\colon \bar{\varepsilon}(\langle \boldsymbol{B}_{i-1}, N'_{i-1}\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{B}_i, N'_i\rangle) \quad \text{in } \mathfrak{B}.$$

This gives

$$\varphi'_n \circ \ldots \circ \varphi'_1\colon \bar{\varepsilon}(\langle \boldsymbol{B}_0, N'_0\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{B}_n, N'_n\rangle) \quad \text{in } \mathfrak{B},$$

i.e.

$$\varphi\colon \bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle) \quad \text{in } \mathfrak{B}.$$

From this we get easily that $\mathfrak{B}$ is rich and that ε is an equivalence of $\mathfrak{A}$ onto an S-skeleton of $\mathfrak{B}$; this S-skeleton is faithful. For if $\boldsymbol{A}$ is an $\mathfrak{A}$-substructure of $\boldsymbol{B}$, then it is easy to prove $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle) = \bar{\varepsilon}(\langle \boldsymbol{B}, A\rangle)$, so that $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$ is a $\mathfrak{B}$-substructure of $\bar{\varepsilon}(\langle \boldsymbol{B}, B\rangle)$.

7. Locally small, bounded and regular scategories

Let $\varkappa$ be a cardinal number. A scategory $\mathfrak{A}$ is called *$\varkappa$-locally small* if there exists a set $\mathscr{M} \subseteq \mathfrak{A}^{\mathcal{O}}$ such that every $\mathfrak{A}$-structure of order $< \varkappa$ is isomorphic to a structure from $\mathscr{M}$. It is called *locally small* if it is $\varkappa$-locally small for all cardinal numbers $\varkappa$.

A scategory $\mathfrak{A}$ is called *small* if $\mathfrak{A}^{\mathcal{O}}$ is a set. Every small scategory is locally small.

THEOREM 11. *An abstract scategory $\mathfrak{A}$ is locally small if and only if*

(15) *for any set A, the class of all $\mathfrak{A}$-structures with the underlying set A is a set.*

Proof. Let $\mathfrak{A}$ be abstract and locally small. If A is a given set, then there exists evidently a set $\mathscr{M}$ of $\mathfrak{A}$-structures with the underlying set A such that every $\mathfrak{A}$-structure with the underlying set A is isomorphic to a structure from $\mathscr{M}$. If $\boldsymbol{A} \in \mathscr{M}$, then the class of all $\mathfrak{A}$-structures $\boldsymbol{B}$ with the underlying set A which are isomorphic to $\boldsymbol{A}$ is a set, because any such $\boldsymbol{B}$ is uniquely determined by a permutation of A, and all permutations of A constitute a set.

Let $\mathfrak{A}$ be an abstract scategory satisfying (15) and let a cardinal number $\varkappa$ be given. In virtue of (15), the class of all $\mathfrak{A}$-structures $\boldsymbol{A}$ such that $A \subseteq \varkappa$ is a set. As $\mathfrak{A}$ is abstract, every $\mathfrak{A}$-structure of order $< \varkappa$ is isomorphic to some $\boldsymbol{A}$ with $A \subseteq \varkappa$.

If we defined locally small scategories by (15), then the following theorem would not hold:

Theorem 12. *The abstraction of a locally small scategory is an abstract, locally small scategory.*

The proof is obvious.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers. A scategory $\mathfrak{A}$ is called $(\mathfrak{m}, \mathfrak{n})$-*bounded* if for every $\mathfrak{A}$-structure $\boldsymbol{A}$ and every set $M \subseteq A$ of cardinality $< \mathfrak{m}$ there exists an underlying set B in $\boldsymbol{A}$ such that $M \subseteq B$ and $\operatorname{Card} B < \mathfrak{n}$.

A scategory is called *bounded* if for every $\mathfrak{m}$ there exists an $\mathfrak{n}$ such that it is $(\mathfrak{m}, \mathfrak{n})$-bounded.

Let $\mathfrak{A}$ be a scategory and $\boldsymbol{A} \in \mathfrak{A}^{\mathscr{O}}$. A set $M \subseteq A$ is called *dense in* $\boldsymbol{A}$ (or: $\mathfrak{A}$-*dense in* $\boldsymbol{A}$) if for every $\mathfrak{A}$-structure $\boldsymbol{B}$, the conditions $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$, $\psi\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\varphi \restriction M = \psi \restriction M$, imply $\varphi = \psi$.

A scategory $\mathfrak{A}$ is called *regular* if for every $\mathfrak{A}$-structure $\boldsymbol{A}$ and every subset $M \subseteq A$ there exists a substructure $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $M \subseteq B$ and M is dense in $\boldsymbol{B}$.

8. Free structures

Let a scategory $\mathfrak{A}$ and an $\mathfrak{A}$-structure $\boldsymbol{A}$ be given. A set $M \subseteq A$ is called an $\mathfrak{A}$-*basis of* $\boldsymbol{A}$ if for every $\mathfrak{A}$-structure $\boldsymbol{B}$, every mapping of M into B can be extended in exactly one way to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. Every $\mathfrak{A}$-basis of $\boldsymbol{A}$ is dense in $\boldsymbol{A}$. An $\mathfrak{A}$-structure is called $\mathfrak{A}$-*free* if it has an $\mathfrak{A}$-basis.

Theorem 13. *Let M_1 be an $\mathfrak{A}$-basis of $\boldsymbol{A}_1$ and M_2 an $\mathfrak{A}$-basis of $\boldsymbol{A}_2$. If there exists a one-to-one mapping φ of M_1 onto M_2, then φ can be extended in exactly one way to an isomorphism of $\boldsymbol{A}_1$ onto $\boldsymbol{A}_2$.*

The proof is easy.

THEOREM 14. *Let $\mathfrak{A}$ be a non-trivial, locally small, bounded, regular and Cartesian scategory. Then for every cardinal number* $\mathfrak{m}$ *there exists an $\mathfrak{A}$-free structure with $\mathfrak{A}$-basis of cardinality* $\mathfrak{m}$.

Proof. Let M be an arbitrary set of cardinality $\mathfrak{m}$. As $\mathfrak{A}$ is non-trivial and bounded, there exists a cardinal number $\mathfrak{n}$ such that the following two conditions are satisfied:

(16) there exists a non-trivial $\mathfrak{A}$-structure of order $\leqslant \mathfrak{n}$;

(17) if $\boldsymbol{D} \in \mathfrak{A}^{\mathcal{O}}$, $Y \subseteq D$ and $\operatorname{Card} Y \leqslant \mathfrak{m}$, then there exists an underlying set Z in $\boldsymbol{D}$ such that $Y \subseteq Z$ and $\operatorname{Card} Z \leqslant \mathfrak{n}$.

$\mathfrak{A}$ is locally small, so there exists a set $\mathscr{M} \subseteq \mathfrak{A}^{\mathcal{O}}$ such that every $\mathfrak{A}$-structure of order $\leqslant \mathfrak{n}$ is isomorphic to a structure from $\mathscr{M}$. Let T be the set of all ordered pairs $\langle \boldsymbol{D}, \lambda\rangle$ such that $\boldsymbol{D} \in \mathscr{M}$, λ is a mapping of M into D and $W(\lambda)$ is a set dense in $\boldsymbol{D}$. For the $t \in T$, denote the first member of the corresponding ordered pair by $\boldsymbol{D}_t$ and the second by λ_t. Let $\bar{\boldsymbol{F}}$ be the Cartesian product of $(\boldsymbol{D}_t)_{t\in T}$. We define a mapping φ of M into $\bar{F}$ in this way: if $a \in M$, then $\varphi(a)$ is that element of $\mathop{\mathsf{X}}\limits_{t\in T} D_t$ which satisfies $\big(\varphi(a)\big)(t) = \lambda_t(a)$ for all $t \in T$. As $\mathfrak{A}$ is regular, there exists a substructure $\boldsymbol{F}$ of $\bar{\boldsymbol{F}}$ such that $W(\varphi) \subseteq F$ and $W(\varphi)$ is dense in $\boldsymbol{F}$. It is easy to prove that φ is injective (apply the non-triviality of $\mathscr{M}$ and the regularity of $\mathfrak{A}$). Hence, it is sufficient to prove that $W(\varphi)$ is an $\mathfrak{A}$-basis of $\boldsymbol{F}$. Let an $\mathfrak{A}$-structure $\boldsymbol{A}$ and a mapping ψ of $W(\varphi)$ into A be given. As $\operatorname{Card} W(\psi) \leqslant \mathfrak{m}$, there exists a substructure of $\boldsymbol{A}$ of order $\leqslant \mathfrak{n}$, whose underlying set contains $W(\psi)$; in this substructure there exists a substructure $\boldsymbol{B}$ such that $W(\psi) \subseteq B$ and $W(\psi)$ is dense in $\boldsymbol{B}$. There exists evidently a structure $\boldsymbol{D} \in \mathscr{M}$ and an isomorphism η: $\boldsymbol{B} \simeq \boldsymbol{D}$. Put $t = \langle \boldsymbol{D}, \eta \circ \psi \circ \varphi\rangle$, so that $t \in T$. For all $a \in M$ we have

$$(\eta \circ \psi \circ \varphi)(a) = \lambda_t(a) = \big(\varphi(a)\big)(t) = (\mathrm{pr}_t \circ \varphi)(a),$$

so that

$$\eta \circ \psi \circ \varphi = \mathrm{pr}_t \circ \varphi;$$

consequently,

$$\eta \circ \psi = \mathrm{pr}_t \restriction W(\varphi).$$

The mapping $\eta^{-1} \circ (\mathrm{pr}_t \restriction F)$ is evidently an extension of ψ to a morphism of $\boldsymbol{F}$ into $\boldsymbol{A}$; the uniqueness of such an extension follows from the density of $W(\varphi)$.

9. ϑ-ary scategories and scategories with $(\vartheta, \varkappa)$-ary morphisms

Let a cardinal number ϑ be given. A scategory $\mathfrak{A}$ is called *ϑ-ary* if the following holds for all $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$:

(18) whenever φ is a mapping of A into B such that $\varphi \restriction K$ can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ for all non-empty sets $K \subseteq A$ of cardinality $<\vartheta$, then $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$.

Let cardinal numbers ϑ and $\varkappa$ be given. A scategory $\mathfrak{A}$ is called *scategory with $(\vartheta, \varkappa)$-ary morphisms* if the following holds for all $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$:

(19) whenever φ is a mapping of A into B such that $\varphi \restriction \bar{A}\colon \bar{\boldsymbol{A}} \to \boldsymbol{B}$ for all substructures $\bar{\boldsymbol{A}}$ of $\boldsymbol{A}$ satisfying $\operatorname{Card} \bar{A} < \varkappa$ and having a non-empty dense subset of cardinality $< \vartheta$, then $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$.

THEOREM 15. *Let ϑ be a cardinal number. If $\mathfrak{A}$ is a scategory with (ϑ, ϑ)-ary morphisms, then it is a ϑ-ary scategory.*

The proof is evident.

THEOREM 16. *Let ϑ be a cardinal number. Every ϑ-ary scategory is equivalent to an S-skeleton of a rich scategory with (ϑ, ϑ)-ary morphisms.*

Proof. Let $\mathfrak{A}$ be a ϑ-ary scategory. Let $\mathscr{K}$ be the class of all ordered pairs $\langle \boldsymbol{A}, M\rangle$ such that $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ and $M \subseteq A$. Define a pre-scategory $\bar{\mathfrak{A}}$ in the following way: $\bar{\mathfrak{A}}^{\mathcal{O}} = \mathscr{K}$; $\mathscr{U}_{\bar{\mathfrak{A}}}(\langle \boldsymbol{A}, M\rangle) = M$; a mapping φ of M into N is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, M\rangle$ into $\langle \boldsymbol{B}, N\rangle$ if and only if $\varphi \restriction K$ can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ for all non-empty $K \subseteq M$ of cardinality $< \vartheta$. This pre-scategory evidently satisfies (1); we prove that it satisfies (2). Let $\varphi\colon \langle \boldsymbol{A}, M\rangle \to \langle \boldsymbol{B}, N\rangle$ in $\bar{\mathfrak{A}}$ and $\psi\colon \langle \boldsymbol{B}, N\rangle \to \langle \boldsymbol{C}, L\rangle$ in $\bar{\mathfrak{A}}$; let a non-empty set K of cardinality $< \vartheta$ be given. We have to prove that $(\psi \circ \varphi) \restriction K$ can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{C}$. The mapping $\varphi \restriction K$ can be extended to an $\mathfrak{A}$-morphism $\bar{\varphi}\colon \boldsymbol{A} \to \boldsymbol{B}$. The set $\varphi'' K$ is evidently non-empty and of cardinality $< \vartheta$; hence, $\psi \restriction (\varphi'' K)$ can be extended to an $\mathfrak{A}$-morphism $\bar{\psi}\colon \boldsymbol{B} \to \boldsymbol{C}$. We have $\bar{\psi} \circ \bar{\varphi}\colon \boldsymbol{A} \to \boldsymbol{C}$ in $\mathfrak{A}$, and $\bar{\psi} \circ \bar{\varphi}$ is an extension of $(\psi \circ \varphi) \restriction K$. Let $\mathfrak{B}$ be a scategorization of $\bar{\mathfrak{A}}$ with respect to $\bar{\varepsilon}$. Define a mapping ε of $\mathfrak{A}^{\mathcal{O}}$ into $\mathfrak{B}^{\mathcal{O}}$ by $\varepsilon(\boldsymbol{A}) = \bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. Since $\mathfrak{A}$ is ϑ-ary, the following holds for all $\mathfrak{A}$-structures $\boldsymbol{A}$ and $\boldsymbol{B}$: a mapping φ of A into B is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, A\rangle$ into $\langle \boldsymbol{B}, B\rangle$ if and only if it is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. On account of Theorem 2, ε is an equivalence of $\mathfrak{A}$ onto a subscategory of $\mathfrak{B}$.

It is evident that if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ and $M \subseteq A$, then $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ is a $\mathfrak{B}$-substructure of $\bar{\varepsilon}(\langle \boldsymbol{A}, A\rangle)$. From this it follows easily that ε is an equivalence of $\mathfrak{A}$ onto an S-skeleton of $\mathfrak{B}$ and that $\mathfrak{B}$ is rich. We prove that $\mathfrak{B}$ is a scategory with (ϑ, ϑ)-ary morphisms. Let $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ and $\bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle)$ be two $\mathfrak{B}$-structures and φ a mapping of M into N such that its restriction to any substructure containing a non-empty dense subset of cardinality $< \vartheta$ and being of order $< \vartheta$ is a $\mathfrak{B}$-morphism. If $K \subseteq M$ is non-empty and of cardinality $< \vartheta$, then K is dense in the substructure $\bar{\varepsilon}(\langle \boldsymbol{A}, K\rangle)$ of $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$, so that

$$\varphi \restriction K\colon\ \bar{\varepsilon}(\langle \boldsymbol{A}, K\rangle) \to \bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle) \quad \text{in } \mathfrak{B}$$

and consequently

$$\varphi \restriction K\colon\ \langle \boldsymbol{A}, K\rangle \to \langle \boldsymbol{B}, N\rangle \quad \text{in } \bar{\mathfrak{A}};$$

by the construction of $\bar{\mathfrak{A}}$, $\varphi \restriction K$ can be extended to an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. This holds for every K, so φ is an $\bar{\mathfrak{A}}$-morphism of $\langle \boldsymbol{A}, M\rangle$ into $\langle \boldsymbol{B}, N\rangle$ and consequently a $\mathfrak{B}$-morphism of $\bar{\varepsilon}(\langle \boldsymbol{A}, M\rangle)$ into $\bar{\varepsilon}(\langle \boldsymbol{B}, N\rangle)$.

10. Scategories closed in themselves

Ascategory $\mathfrak{A}$ is called *closed in itself* if it satisfies the following condition: If $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$ and $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$, then $W(\varphi)$ is a set underlying in $\boldsymbol{B}$.

THEOREM 17. *Let $\mathfrak{A}$ be a scategory closed in itself and such that for every cardinal number $\mathfrak{m}$ there exists an $\mathfrak{A}$-free structure with $\mathfrak{A}$-basis of cardinality $\mathfrak{m}$. Then the following holds:*

(i) *$\mathfrak{A}$ is bounded;*

(ii) *$\mathfrak{A}$ is regular;*

(iii) *if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$, then the intersection of any non-empty system of sets underlying in $\boldsymbol{A}$ is a set underlying in $\boldsymbol{A}$;*

(iv) *if $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ and if M is an $\mathfrak{A}$-generating set in $\boldsymbol{A}$, then M is dense in $\boldsymbol{A}$;*

(v) *if $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ and M is a set underlying in $\boldsymbol{A}$, then $\varphi'' M$ is underlying in $\boldsymbol{B}$;*

(vi) *if $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ and M is a set underlying in $\boldsymbol{B}$, then $\varphi^{-1''} M$ is underlying in $\boldsymbol{A}$.*

Proof. (i) Let $\mathfrak{m}$ be a cardinal number. There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with $\mathfrak{A}$-basis M of cardinality $\mathfrak{m}$; put $\mathfrak{n} = \operatorname{Card} F$. Let $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$, $N \subseteq A$ and $\operatorname{Card} N = \mathfrak{m}$. There exists a one-to-one mapping of M onto N, and it can be extended to an $\mathfrak{A}$-morphism $\varphi\colon \boldsymbol{F} \to \boldsymbol{A}$. The set $W(\varphi)$ is underlying in $\boldsymbol{A}$ and $\operatorname{Card} W(\varphi) \leqslant \mathfrak{n}$. From this the boundedness follows easily.

(iii) Let $(A_\iota)_{\iota\in T}$ be a non-empty family of sets underlying in $\boldsymbol{A}$; for every $\iota \in T$ let $\boldsymbol{A}_\iota$ be the corresponding substructure. Put $M = \bigcap_{\iota\in T} A_\iota$. There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with $\mathfrak{A}$-basis N of cardinality Card M. A one-to-one mapping η of N onto M can be extended to a morphism $\varphi\colon \boldsymbol{F} \to \boldsymbol{A}$ in exactly one way. For every $\iota \in T$ we have $\varphi\colon \boldsymbol{F} \to \boldsymbol{A}_\iota$, as η can be extended to a morphism of $\boldsymbol{F}$ into $\boldsymbol{A}_\iota$ and this morphism is a morphism of $\boldsymbol{F}$ into $\boldsymbol{A}$. We get $W(\varphi) \subseteq M$ and consequently $W(\varphi) = M$.

(iv) There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with $\mathfrak{A}$-basis N of cardinality Card M; a one-to-one mapping η of N onto M can be extended to a morphism $\varphi\colon \boldsymbol{F} \to \boldsymbol{A}$ in exactly one way. Evidently $W(\varphi) = A$. Let $\boldsymbol{B} \in \mathfrak{A}^{\mathscr{O}}$, $\psi_1\colon \boldsymbol{A} \to \boldsymbol{B}$, $\psi_2\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\psi_1 \upharpoonright M = \psi_2 \upharpoonright M$. We have $\psi_1 \circ \varphi\colon \boldsymbol{F} \to \boldsymbol{B}$, $\psi_2 \circ \varphi\colon \boldsymbol{F} \to \boldsymbol{B}$ and $(\psi_1 \circ \varphi) \upharpoonright N = (\psi_2 \circ \varphi) \upharpoonright N$, so that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. As $W(\varphi) = A$, we get $\psi_1 = \psi_2$.

(ii) If $\boldsymbol{A} \in \mathfrak{A}^{\mathscr{O}}$ and $M \subseteq A$, then M is $\mathfrak{A}$-generating in a substructure of $\boldsymbol{A}$ by (iii), and it is dense in this substructure by (iv).

(v) is easy.

(vi) There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with $\mathfrak{A}$-basis N of cardinality $\mathrm{Card}(\varphi^{-1''}M)$. A one-to-one mapping η of N onto $\varphi^{-1''}M$ can be extended to a morphism $\psi\colon \boldsymbol{F} \to \boldsymbol{A}$. We have $\varphi \circ \psi\colon \boldsymbol{F} \to \boldsymbol{B}$; as $(\varphi \circ \psi)(x) \in M$ for all $x \in N$, we have evidently $W(\varphi \circ \psi) \subseteq M$. Hence, $W(\psi) \subseteq \varphi^{-1''}M$, so $W(\psi) = \varphi^{-1''}M$.

11. Additive scategories

Let ϑ be a cardinal number. A scategory $\mathfrak{A}$ is called *ϑ-additive* if the following holds: whenever $\boldsymbol{A}$ is an $\mathfrak{A}$-structure and $M \subseteq A$ is a set such that for every $K \subseteq M$ of cardinality $< \vartheta$ there exists a set containing K, contained in M and underlying in $\boldsymbol{A}$, then M is underlying in $\boldsymbol{A}$.

A system $\mathscr{M}$ of sets is called *ϑ-local* if for every set $K \subseteq \bigcup \mathscr{M}$ of cardinality $< \vartheta$ there exists an $M \in \mathscr{M}$ such that $K \subseteq M$. Every ϑ-local system (ϑ being non-zero) is evidently non-empty.

THEOREM 18. *Let ϑ be a cardinal number, $\vartheta \geqslant 2$. A scategory $\mathfrak{A}$ is ϑ-additive if and only if for every $\mathfrak{A}$-structure $\boldsymbol{A}$, the union of any ϑ-local system of sets underlying in $\boldsymbol{A}$ is a set underlying in $\boldsymbol{A}$.*

The proof is easy.

Let $\mathfrak{A}$ be a scategory, ϑ a cardinal number and $\boldsymbol{A} \in \mathfrak{A}^{\mathscr{O}}$. A set $M \subseteq A$ is called *ϑ-dense in $\boldsymbol{A}$* if it is dense in $\boldsymbol{A}$ and if for every $a \in A$ there exist a set $K \subseteq M$ of cardinality $< \vartheta$ and a substructure $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $K \subseteq B$, K is dense in $\boldsymbol{B}$ and $a \in B$.

THEOREM 19. *Let ϑ be a cardinal number, $\vartheta \geqslant 2$. Let $\mathfrak{A}$ be a closed in itself scategory such that for every cardinal number $\mathfrak{m}$ there exists an $\mathfrak{A}$-free structure with $\mathfrak{A}$-basis of cardinality $\mathfrak{m}$, this basis being ϑ-dense. Then $\mathfrak{A}$ is ϑ-additive.*

Proof. Let an $\mathfrak{A}$-structure $\boldsymbol{A}$ and a ϑ-local system $\mathscr{M}$ of sets underlying in $\boldsymbol{A}$ be given. There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with $\mathfrak{A}$-basis N of cardinality $\mathrm{Card}(\bigcup \mathscr{M})$ such that N is ϑ-dense in $\boldsymbol{F}$. A one-to-one mapping η of N onto $\bigcup \mathscr{M}$ can be extended to a morphism φ: $\boldsymbol{F} \to \boldsymbol{A}$ in exactly one way. It is sufficient to prove $W(\varphi) = \bigcup \mathscr{M}$, or only $W(\varphi) \subseteq \bigcup \mathscr{M}$. Let $a \in F$. As N is ϑ-dense in $\boldsymbol{F}$, there exists a set $K \subseteq N$ of cardinality $< \vartheta$ and a substructure $\boldsymbol{B}$ of $\boldsymbol{F}$ such that K is dense in $\boldsymbol{B}$ and $a \in B$. Since $\eta''K \subseteq \bigcup \mathscr{M}$ and $\mathrm{Card}(\eta''K) < \vartheta$, so there exists an $M \in \mathscr{M}$ such that $\eta''K \subseteq M$; denote the corresponding substructure by $\boldsymbol{M}$. (If $\bigcup \mathscr{M}$ is empty, then evidently F is empty and everything is evident; if it is non-empty, then we can choose M to be non-empty.) From this we get: There exists a morphism ψ: $\boldsymbol{F} \to \boldsymbol{M}$ such that

$$\psi \restriction (\eta^{-1''}M) = \eta \restriction (\eta^{-1''}M) = \varphi \restriction (\eta^{-1''}M).$$

We have

$$\psi \restriction B\colon \boldsymbol{B} \to \boldsymbol{A}, \qquad \varphi \restriction B\colon \boldsymbol{B} \to \boldsymbol{A}$$

and

$$(\psi \restriction B) \restriction (\eta^{-1''}M) = (\varphi \restriction B) \restriction (\eta^{-1''}M),$$

so that

$$(\psi \restriction B) \restriction K = (\varphi \restriction B) \restriction K, \qquad \text{so that} \qquad \psi \restriction B = \varphi \restriction B.$$

We get $W(\varphi \restriction B) \subseteq M$ and consequently $\varphi(a) \in M \subseteq \bigcup \mathscr{M}$.

THEOREM 20. *Let ϑ be a regular number. Let $\mathfrak{A}$ be a regular ϑ-additive scategory. If a set M is an $\mathfrak{A}$-basis in an $\mathfrak{A}$-structure $\boldsymbol{A}$, then it is ϑ-dense in $\boldsymbol{A}$.*

Proof. For every $K \subseteq M$ of cardinality $< \vartheta$ let us choose a substructure $\boldsymbol{A}_K$ of $\boldsymbol{A}$ such that $K \subseteq A_K$ and K is dense in $\boldsymbol{A}_K$. (This can be done by the regularity of $\mathfrak{A}$.) We shall prove that the system of all those A_K's is ϑ-local. Let $L \subseteq \bigcup_K A_K$ be of cardinality $< \vartheta$. Every $l \in L$ belongs to an A_{K_l}. Since ϑ is regular, the cardinality of $K^0 = \bigcup_{l \in L} K_l$ is $< \vartheta$. For every $l \in L$ we have $A_{K_l} \subseteq A_{K^0}$, as id_{K_l} can be extended to a morphism of $\boldsymbol{A}$ into $\boldsymbol{A}_{K^0}$ (if $A_{K^0} \neq 0$) and hence to a morphism of $\boldsymbol{A}_{K_l}$ into $\boldsymbol{A}_{K^0}$, and this latter morphism, being a morphism of $\boldsymbol{A}_{K_l}$ into $\boldsymbol{A}$, is equal to id_{K_l}. (If $A_{K^0} = 0$, then this is evident.) The system of all those A_K is thus ϑ-local, so that its union B is a set underlying in $\boldsymbol{A}$; denote the corresponding substructure by $\boldsymbol{B}$. It is evidently sufficient to prove

$B = A$. The mapping id_M can be extended to a morphism φ of A into $\boldsymbol{B}$; as φ is a morphism of $\boldsymbol{A}$ into $\boldsymbol{A}$, we have $\varphi = \mathrm{id}_A$. Hence, $A \subseteq B$.

Let cardinal numbers ϑ and $\varkappa$ be given. A scategory $\mathfrak{A}$ is called $(\vartheta, \varkappa)$-*additive* if the following holds: if $\boldsymbol{A} \in \mathfrak{A}^c$, $M \subseteq A$ and

(20) whenever $\boldsymbol{B}$ is a substructure of $\boldsymbol{A}$ such that $\operatorname{Card} B < \varkappa$ and such that $\boldsymbol{B}$ contains a dense subset D satisfying $D \subseteq M$ and $\operatorname{Card} D < \vartheta$, then $B \subseteq M$,

then M is a set underlying in $\boldsymbol{A}$.

If $\mathfrak{A}$ is $(\vartheta, \varkappa)$-bounded, regular and ϑ-additive, then it is evidently $(\vartheta, \varkappa)$-additive.

12. Scategories with divisible morphisms

A scategory $\mathfrak{A}$ is called a *scategory with divisible morphisms* if it satisfies the following condition: if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathfrak{A}^c$, $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$, $W(\varphi) = B$, ψ a mapping of B into C and $\psi \circ \varphi\colon \boldsymbol{A} \to \boldsymbol{C}$, then $\psi\colon \boldsymbol{B} \to \boldsymbol{C}$.

THEOREM 21. *Let $\mathfrak{A}$ be a closed in itself scategory with divisible morphisms; let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^c$. Then $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$ if and only if $A \subseteq B$ and* $\mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{B}$.

Proof. Let this condition be satisfied. $\mathfrak{A}$ is closed in itself, so A is a set underlying in $\boldsymbol{B}$; denote the corresponding substructure by $\bar{\boldsymbol{A}}$. We have $\mathrm{id}_A\colon \boldsymbol{A} \to \bar{\boldsymbol{A}}$, $W(\mathrm{id}_A) = \bar{A}$, id_A is a mapping of $\bar{A}$ into A and $\mathrm{id}_A \circ \mathrm{id}_A\colon \boldsymbol{A} \to \boldsymbol{A}$; since $\mathfrak{A}$ is a scategory with divisible morphisms, we get $\mathrm{id}_A\colon \bar{\boldsymbol{A}} \to \boldsymbol{A}$. This implies $\mathrm{id}_A\colon \boldsymbol{A} \simeq \bar{\boldsymbol{A}}$, so that $\boldsymbol{A} = \bar{\boldsymbol{A}}$.

13. Quasi-algebras and algebras

By a *type* we mean an arbitrary family of sets (the domain being a set). Let us make the following convention: if a type is denoted by Δ (or $\Delta^*, \ldots$ resp.), then its domain is denoted by I (or $I^*, \ldots$ resp.) and its value at an i by Δ_i (or $\Delta_i^*, \ldots$ resp.).

A type Δ is called a type *without constants* if Δ_i is non-empty for all $i \in I$; otherwise it is called a type *with constants*.

The dimension of Δ is the least regular number exceeding all $\operatorname{Card} \Delta_i$ $(i \in I)$. Δ is called *finitary* if its dimension is $\aleph_0$, i.e. if all the sets Δ_i $(i \in I)$ are finite.

Let A and L be two sets. A mapping f is called a *partial operation of arity L in A* if $\mathscr{D}(f) \subseteq A^L$ and $W(f) \subseteq A$. A mapping f is called an *operation of arity L in A* if $\mathscr{D}(f) = A^L$ and $W(f) \subseteq A$. Operations of arity 0 in A are in a natural one-to-one correspondence with the elements of A.

By a *quasi-algebra of type Δ* we mean an ordered pair $\boldsymbol{A} = \langle A, (f_i)_{i\in I}\rangle$ such that A is an arbitrary set and f_i is a partial operation of arity Δ_i in A for all $i \in I$. The set A is called the *underlying set of $\boldsymbol{A}$*; f_i is called the *i-th fundamental partial operation of $\boldsymbol{A}$*. Let us make the following convention: if a quasi-algebra is denoted by $\boldsymbol{A}$ (or $\boldsymbol{B}$, or $\boldsymbol{A}_\iota, \ldots$ resp.), then its underlying set is denoted by A (or B, or $A_\iota, \ldots$ resp.) and its i-th fundamental partial operation by $\boldsymbol{A}_{[i]}$ (or $\boldsymbol{B}_{[i]}$, or $\boldsymbol{A}_{\iota,[i]}, \ldots$ resp.).

A quasi-algebra $\boldsymbol{A}$ is called *algebra* if every fundamental partial operation of $\boldsymbol{A}$ is an operation; we then speak of *fundamental operations*.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two quasi-algebras of type Δ. A mapping φ of A into B is called a *homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$* if: whenever $i \in I$ and $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}_{[i]})$, then

$$\varphi \circ \boldsymbol{a} \in \mathscr{D}(\boldsymbol{B}_{[i]}) \quad \text{and} \quad \varphi\big(\boldsymbol{A}_{[i]}(\boldsymbol{a})\big) = \boldsymbol{B}_{[i]}(\varphi \circ \boldsymbol{a}).$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras, then this condition can be re-formulated: whenever $i \in I$ and $\boldsymbol{a} \in A^{\Delta_i}$, then

$$\varphi\big(\boldsymbol{A}_{[i]}(\boldsymbol{a})\big) = \boldsymbol{B}_{[i]}(\varphi \circ \boldsymbol{a}).$$

Let a type Δ be given. We define a scategory $\mathscr{P}_\Delta$ as follows: its structures are exactly quasi-algebras of type Δ; the underlying set of a structure is the underlying set of the quasi-algebra; morphisms are just the homomorphisms. (It is easy to show that $\mathscr{P}_\Delta$ is really a scategory.) The subscategory of $\mathscr{P}_\Delta$, determined by the class of all algebras of type Δ, is denoted by $\mathscr{A}_\Delta$.

By a *sfunctor of a scategory $\mathfrak{A}$ into a class $\mathscr{K}$ of quasi-algebras of type Δ* we mean a sfunctor of $\mathfrak{A}$ into the scategory determined by $\mathscr{K}$. Similarly: sfunctor of $\mathscr{K}$ into $\mathfrak{A}$, of $\mathscr{K}_1$ into $\mathscr{K}_2$; equivalence of $\mathfrak{A}$ onto $\mathscr{K}$, etc.

Let a quasi-algebra $\boldsymbol{A}$ of type Δ and a set $B \subseteq A$ be given. Let us define a quasi-algebra $\boldsymbol{B}$ of type Δ with the underlying set B in this way: if $i \in I$ and $\boldsymbol{a} \in B^{\Delta_i}$, then $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{B}_{[i]})$ if and only if $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}_{[i]})$ and $\boldsymbol{A}_{[i]}(\boldsymbol{a}) \in B$; in the positive case we put $\boldsymbol{B}_{[i]}(\boldsymbol{a}) = \boldsymbol{A}_{[i]}(\boldsymbol{a})$. This $\boldsymbol{B}$ is denoted by $\boldsymbol{A}\|B$. A quasi-algebra is called a *relative quasi-algebra of $\boldsymbol{A}$* if it is of the form $\boldsymbol{A}\|B$ for some $B \subseteq A$. It is easy to prove that relative quasi-algebras of $\boldsymbol{A}$ are exactly the substructures of $\boldsymbol{A}$ in the sense of $\mathscr{P}_\Delta$.

Let $\boldsymbol{A}$ be a quasi-algebra of type Δ. A set $B \subseteq A$ is called *closed in $\boldsymbol{A}$* if: whenever $i \in I$, $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}_{[i]})$ and $W(\boldsymbol{a}) \subseteq B$, then $\boldsymbol{A}_{[i]}(\boldsymbol{a}) \in B$. A relative quasi-algebra of $\boldsymbol{A}$ is called *subquasi-algebra of $\boldsymbol{A}$* if its underlying set is closed in $\boldsymbol{A}$.

It is easy to prove that the intersection of any non-empty system of sets closed in $\boldsymbol{A}$ is a set closed in $\boldsymbol{A}$. From this it follows that for every set $M \subseteq A$ there exists the least set closed in A and containing M; this set is denoted by $C_{\boldsymbol{A}}(M)$ and the corresponding subquasi-algebra by $\mathscr{C}_{\boldsymbol{A}}(M)$. A set M is called a *set of generators of* $\boldsymbol{A}$ if $C_{\boldsymbol{A}}(M) = A$ (not to be confused with the notion of $\mathscr{P}_\Delta$-generating set in $\boldsymbol{A}$; evidently, A is the only $\mathscr{P}_\Delta$-generating set of $\boldsymbol{A}$.)

If $\boldsymbol{A}$ is an algebra, then every subquasi-algebra of $\boldsymbol{A}$ is an algebra; we then speak of subalgebras. Subalgebras of $\boldsymbol{A}$ are exactly the substructures of $\boldsymbol{A}$ in the sense of $\mathscr{A}_\Delta$. $\mathscr{A}_\Delta$-underlying sets of $\boldsymbol{A}$ are exactly the sets closed in $\boldsymbol{A}$; $\mathscr{A}_\Delta$-generating sets are exactly the sets of generators.

Let $(\boldsymbol{A}_t)_{t\in T}$ be a family of quasi-algebras of type Δ. Put $A = \bigtimes_{t\in T} A_t$ and define a quasi-algebra $\boldsymbol{A}$ of type Δ with the underlying set A as follows: if $i \in I$ and $\boldsymbol{a} \in A^{\Delta_i}$, then $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}_{[i]})$ if and only if $\mathrm{pr}_t \circ \boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}_{t,[i]})$ for all $t \in T$; in the positive case $\boldsymbol{A}_{[i]}(\boldsymbol{a})$ is that element $f \in \bigtimes_{t\in T} A_t$ which satisfies $f(t) = \boldsymbol{A}_{t,[i]}(\mathrm{pr}_t \circ \boldsymbol{a})$ for all $t \in T$. This $\boldsymbol{A}$ is called the *Cartesian product of the family* $(\boldsymbol{A}_t)_{t\in T}$ *of quasi-algebras*. It is evidently the Cartesian product of this family with respect to $\mathscr{P}_\Delta$. If $(\boldsymbol{A}_t)_{t\in T}$ is a family of algebras, then its Cartesian product is an algebra and it is the Cartesian product of that family with respect to $\mathscr{A}_\Delta$.

THEOREM 22. *Let $(\boldsymbol{A}_t)_{t\in T}$ be a family of algebras of type Δ. An algebra $\boldsymbol{A}$ of type Δ with the underlying set $\bigtimes_{t\in T} A_t$ is the Cartesian product of that family if and only if pr_t is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{A}_t$ for all $t \in T$.*

The proof is easy.

A class $\mathscr{K}$ of quasi-algebras of type Δ is called *I-closed* if, whenever $\boldsymbol{A}$ belongs to $\mathscr{K}$, then every quasi-algebra isomorphic to $\boldsymbol{A}$ belongs to $\mathscr{K}$; it is called *S-closed* if, whenever $\boldsymbol{A}$ belongs to $\mathscr{K}$, then every subquasi-algebra of $\boldsymbol{A}$ belongs to $\mathscr{K}$; it is called *P-closed* if, whenever a family of quasi-algebras of $\mathscr{K}$ is given, then its Cartesian product belongs to $\mathscr{K}$; it is called *S, P-closed* if it is S-closed and P-closed; etc.

14. A characterization of some classes of quasi-algebras

THEOREM 23. *Let a scategory $\mathfrak{A}$ and a regular number ϑ be given. The following two conditions are equivalent:*

(i) *$\mathfrak{A}$ is equivalent to an S-closed class of quasi-algebras of a type of dimension $\leqslant \vartheta$;*

(ii) *there exists a cardinal number $\varkappa$ such that $\mathfrak{A}$ is a $\varkappa$-locally small and $(\vartheta, \varkappa)$-additive scategory with $(\vartheta, \varkappa)$-ary morphisms.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{A}$ be equivalent to an S-closed class $\mathscr{K}$ of quasi-algebras of a type of dimension $\leqslant \vartheta$. There exists a cardinal number $\varkappa$ such that whenever A is a quasi-algebra of type Δ, $K \subseteq A$ and $\operatorname{Card} K < \vartheta$ then $\operatorname{Card}(C_{A}(K)) < \varkappa$. $\mathfrak{A}$ is evidently locally small and $(\vartheta, \varkappa)$-bounded; we prove that it is $(\vartheta, \varkappa)$-additive. Let $\boldsymbol{A}$ be an $\mathfrak{A}$-structure and let $M \subseteq A$ be a set satisfying (20). Let us denote by $\boldsymbol{A}^*$ the quasi-algebra corresponding to $\boldsymbol{A}$. It is evidently sufficient to prove that M is closed in $\boldsymbol{A}^*$. Let $i \in I$, $\boldsymbol{a} \in M^{\Delta i}$ and $\boldsymbol{a} \in \mathscr{D}(A^*_{[i]})$. Put $K = W(\boldsymbol{a})$, so that $K \subseteq M$ and $\operatorname{Card} K < \vartheta$. Put $\boldsymbol{B}^* = \mathscr{C}_{A^*}(K)$, so that $\boldsymbol{B}^* \in \mathscr{K}$; let $\boldsymbol{B}$ be the $\mathfrak{A}$-structure corresponding to $\boldsymbol{B}^*$. $\boldsymbol{B}$ is a substructure of $\boldsymbol{A}$, the set K is $\mathfrak{A}$-dense in $\boldsymbol{B}$ and $\operatorname{Card} B < \varkappa$. By (20) we have $B \subseteq M$ and consequently $A^*_{[i]}(\boldsymbol{a}) \in B \subseteq M$. It may be proved quite similarly that $\mathfrak{A}$ is a scategory with $(\vartheta, \varkappa)$-ary morphisms.

(ii) $\Rightarrow$ (i): Since $\mathfrak{A}$ is $\varkappa$-locally small, there exists a set $\mathscr{M} \subseteq \mathfrak{A}^{\mathcal{O}}$ such that every $\mathfrak{A}$-structure of order $< \varkappa$ is isomorphic to a structure from $\mathscr{M}$. Let I be the set of all ordered triples $\langle \boldsymbol{Z}, M, z\rangle$ such that $\boldsymbol{Z} \in \mathscr{M}$, M is a set dense in $\boldsymbol{Z}$, $\operatorname{Card} M < \vartheta$ and $z \in Z$. Define a type Δ (with the just defined domain I) in this way: if $i = \langle \boldsymbol{Z}, M, z\rangle \in I$, then $\Delta_i = M$. It is evidently a type of dimension $\leqslant \vartheta$. To every $\mathfrak{A}$-structure $\boldsymbol{A}$ we shall assign a quasi-algebra $\boldsymbol{A}^*$ of type Δ: its underlying set is the set A; if $i = \langle \boldsymbol{Z}, M, z\rangle \in I$ and $\boldsymbol{a} \in A^M$, then $\boldsymbol{a} \in \mathscr{D}(A^*_{[i]})$ if and only if $\boldsymbol{a}$ can be extended to an $\mathfrak{A}$-morphism $\alpha\colon \boldsymbol{Z} \to \boldsymbol{A}$ (this α is then uniquely determined); in the positive case we put $A^*_{[i]}(\boldsymbol{a}) = \alpha(z)$. Let ε be the mapping assigning to every $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ the corresponding $\boldsymbol{A}^*$; put $\mathscr{K} = W(\varepsilon)$. We shall prove that ε is an equivalence of $\mathfrak{A}$ onto $\mathscr{K}$ and that $\mathscr{K}$ is S-closed.

Let φ be an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$; we shall prove that it is a homomorphism of $\boldsymbol{A}^*$ into $\boldsymbol{B}^*$. Let $i = \langle \boldsymbol{Z}, M, z\rangle \in I$ and $\boldsymbol{a} \in \mathscr{D}(A^*_{[i]})$. The mapping $\boldsymbol{a}$ can be extended to an $\mathfrak{A}$-morphism $\alpha\colon \boldsymbol{Z} \to \boldsymbol{A}$ and we have $A^*_{[i]}(\boldsymbol{a}) = \alpha(z)$. The mapping $\varphi \circ \alpha$ is a morphism of $\boldsymbol{Z}$ into $\boldsymbol{B}$ and it is an extension of $\varphi \circ \boldsymbol{a}$; hence

$$\varphi \circ \boldsymbol{a} \in \mathscr{D}(\boldsymbol{B}^*_{[i]}) \quad \text{and} \quad \boldsymbol{B}^*_{[i]}(\varphi \circ \boldsymbol{a}) = \varphi(\alpha(z)) = \varphi(A^*_{[i]}(\boldsymbol{a})).$$

Conversely, let φ be a homomorphism of $\boldsymbol{A}^*$ into $\boldsymbol{B}^*$; we shall prove that it is an $\mathfrak{A}$-morphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. Let $\bar{\boldsymbol{A}}$ be an arbitrary $\mathfrak{A}$-substructure of $\boldsymbol{A}$ having a non-empty dense subset M of cardinality $< \vartheta$ and being of order $< \varkappa$. As $\mathfrak{A}$ is a scategory with $(\vartheta, \varkappa)$-ary morphisms, it is sufficient to prove $\varphi \restriction \bar{A}\colon \bar{\boldsymbol{A}} \to \boldsymbol{B}$ in $\mathfrak{A}$. There exists a structure $\boldsymbol{Z} \in \mathscr{M}$ and an isomorphism $\eta\colon \boldsymbol{Z} \simeq \bar{\boldsymbol{A}}$. Put $\bar{M} = \eta^{-1''}M$. For every $c \in \bar{A}$ put $i_c = \langle \boldsymbol{Z}, \bar{M}, \eta^{-1}(c)\rangle$, so that $i_c \in I$. M is non-empty, so $\bar{A}$ is also non-empty; fix an element $c_0 \in \bar{A}$. As $\eta\colon \boldsymbol{Z} \to \boldsymbol{A}$, we have

$$A^*_{[i_{c_0}]}(\eta \restriction \bar{M}) = \eta(\eta^{-1}(c_0)) = c_0;$$

since φ is a homomorphism of quasi-algebras, we get

$$\boldsymbol{B}^*_{[i_{c_0}]}(\varphi \circ \eta \restriction \overline{M}) = \varphi(c_0);$$

by the definition of $\boldsymbol{B}^*$, the mapping $\varphi \circ \eta \restriction \overline{M}$ can be extended to an $\mathfrak{A}$-morphism ψ: $\boldsymbol{Z} \to \boldsymbol{B}$. We have $\psi \circ \eta^{-1}$: $\overline{\boldsymbol{A}} \to \boldsymbol{B}$ in $\mathfrak{A}$; it is sufficient to prove $\psi \circ \eta^{-1} = \varphi \restriction \overline{A}$. Let $c \in \overline{A}$. In the same way as for c_0, we can prove

$$\boldsymbol{A}^*_{[i_c]}(\eta \restriction \overline{M}) = c,$$

so that

$$\boldsymbol{B}^*_{[i_c]}(\varphi \circ \eta \restriction \overline{M}) = \varphi(c);$$

by the definition of $\boldsymbol{B}^*$, the mapping $\varphi \circ \eta \restriction \overline{M}$ can be extended to an $\mathfrak{A}$-morphism $\overline{\psi}$: $\boldsymbol{Z} \to \boldsymbol{B}$ and we have $\overline{\psi}(\eta^{-1}(c)) = \varphi(c)$; as $\overline{M}$ is dense in $\boldsymbol{Z}$, we have $\psi = \overline{\psi}$, so that $\psi(\eta^{-1}(c)) = \varphi(c)$. This holds for all $c \in \overline{A}$, thus $\psi \circ \eta^{-1} = \varphi \restriction \overline{A}$.

This shows that ε is an equivalence of $\mathfrak{A}$ onto $\mathscr{K}$. We shall prove that $\mathscr{K}$ is S-closed. Let a quasi-algebra $\boldsymbol{P} \in \mathscr{K}$ and its arbitrary subquasi-algebra $\boldsymbol{Q}$ be given; we are to prove $\boldsymbol{Q} \in \mathscr{K}$. There exists an $\mathfrak{A}$-structure $\boldsymbol{A}$ such that $\boldsymbol{P} = \boldsymbol{A}^*$. We first prove that Q is a set underlying in $\boldsymbol{A}$. Let an $\mathfrak{A}$-substructure $\boldsymbol{D}$ of $\boldsymbol{A}$ be given, $\boldsymbol{D}$ being of order $< \varkappa$ and containing a dense subset K which satisfies $K \subseteq Q$ and $\operatorname{Card} K < \vartheta$. As $\mathfrak{A}$ is $(\vartheta, \varkappa)$-additive, it is sufficient to prove $D \subseteq Q$. There exists a structure $\boldsymbol{Z} \in \mathscr{M}$ and an isomorphism η: $\boldsymbol{Z} \simeq \boldsymbol{D}$. Put $M = \eta^{-1''}K$; for every $z \in Z$ put $i_z = \langle \boldsymbol{Z}, M, z\rangle$, so that $i_z \in I$. As η: $\boldsymbol{Z} \to \boldsymbol{A}$ in $\mathfrak{A}$, we have

$$\eta(z) = \boldsymbol{A}^*_{[i_z]}(\eta \restriction M) = \boldsymbol{Q}_{[i_z]}(\eta \restriction M) \in Q \qquad \text{for all } z \in Z.$$

We get $W(\eta) \subseteq Q$ and hence $D \subseteq Q$. Q is thus in fact a set underlying in $\boldsymbol{A}$; let us denote the corresponding substructure by $\boldsymbol{B}$. We shall prove $\boldsymbol{B}^* = \boldsymbol{Q}$. As id_Q: $\boldsymbol{B} \to \boldsymbol{A}$ in $\mathfrak{A}$, id_Q is a homomorphism of $\boldsymbol{B}^*$ into $\boldsymbol{A}^*$ and consequently a homomorphism of $\boldsymbol{B}^*$ into $\boldsymbol{Q}$. It is sufficient to prove that id_Q is a homomorphism of $\boldsymbol{Q}$ into $\boldsymbol{B}^*$. Let $i = \langle \boldsymbol{Z}, M, z\rangle \in I$ and $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{Q}_{[i]})$. The mapping $\boldsymbol{a}$ can be extended to an $\mathfrak{A}$-morphism a: $\boldsymbol{Z} \to \boldsymbol{A}$. For every $\bar{z} \in Z$ put $i_{\bar{z}} = \langle \boldsymbol{Z}, M, \bar{z}\rangle$, so that $i_{\bar{z}} \in I$; we have

$$a(\bar{z}) = \boldsymbol{A}^*_{[i_{\bar{z}}]}(\boldsymbol{a}) = \boldsymbol{Q}_{[i_{\bar{z}}]}(\boldsymbol{a}) \in Q$$

and thus $W(a) \subseteq Q$. As $\boldsymbol{B}$ is a substructure of $\boldsymbol{A}$, we get a: $\boldsymbol{Z} \to \boldsymbol{B}$ in $\mathfrak{A}$, whence

$$\boldsymbol{B}^*_{[i]}(\boldsymbol{a}) = a(z) = \boldsymbol{Q}_{[i]}(\boldsymbol{a}).$$

THEOREM 24. *Let a scategory $\mathfrak{A}$ and a regular number ϑ be given. The following two conditions are equivalent:*

(i) $\mathfrak{A}$ *is equivalent to an* S, P*-closed class of quasi-algebras of a type of dimension* $\leqslant \vartheta$;

(ii) *there exists a cardinal number* $\varkappa$ *such that* $\mathfrak{A}$ *is a* $\varkappa$*-locally small,* $(\vartheta, \varkappa)$*-additive Cartesian scategory with* $(\vartheta, \varkappa)$*-ary morphisms.*

Proof. (i) $\Rightarrow$ (ii) is evident. (ii) $\Rightarrow$ (i): Let us hold the notation introduced in the proof of Theorem 23. It is sufficient to prove that the class $\mathscr{K}$ constructed there is P-closed. Let a family $(\boldsymbol{A}_t)_{t\in T}$ of $\mathfrak{A}$-structures be given. Let us denote by $\boldsymbol{A}$ the Cartesian product (in $\mathfrak{A}$) of this family and by $\boldsymbol{C}$ the Cartesian product of the family $(\boldsymbol{A}_t^*)_{t\in T}$ of quasi-algebras; we shall prove $\boldsymbol{A}^* = \boldsymbol{C}$. For every $t\in T$ we have $\mathrm{pr}_t\colon \boldsymbol{A} \to \boldsymbol{A}_t$ in $\mathfrak{A}$, so that (by Theorem 23) pr_t is a homomorphism of $\boldsymbol{A}^*$ into $\boldsymbol{A}_t^*$; by the definition of $\boldsymbol{C}$ there exists exactly one homomorphism η of $\boldsymbol{A}^*$ into $\boldsymbol{C}$ such that $\mathrm{pr}_t = \mathrm{pr}_t \circ \eta$ for all $t\in T$; evidently $\eta = \mathrm{id}_C$; it remains to prove that id_C is a homomorphism of $\boldsymbol{C}$ into $\boldsymbol{A}^*$. Let $i = \langle \boldsymbol{Z}, M, z\rangle \in I$, $\boldsymbol{a} \in C^{\Delta_i} = C^M$ and $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{C}_{[i]})$. For all $t\in T$ we have $\mathrm{pr}_t \circ \boldsymbol{a} \in A_t^M$ and $\mathrm{pr}_t \circ \boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}^*_{t,[i]})$; the mapping $\mathrm{pr}_t \circ \boldsymbol{a}$ can be extended to an $\mathfrak{A}$-morphism $\varphi_t\colon \boldsymbol{Z} \to \boldsymbol{A}_t$ and we have $\boldsymbol{A}^*_{t,[i]}(\mathrm{pr}_t \circ \boldsymbol{a}) = \varphi_t(z)$. By the definition of the Cartesian product $\boldsymbol{A}$ there exists an $\mathfrak{A}$-morphism $\varphi\colon \boldsymbol{Z} \to \boldsymbol{A}$ such that $\varphi_t = \mathrm{pr}_t \circ \varphi$ for all $t\in T$.

If $x\in M$, then for all $t\in T$ we have

$$\mathrm{pr}_t\big(\varphi(x)\big) = \varphi_t(x) = \mathrm{pr}_t\big(\boldsymbol{a}(x)\big),$$

so that $\varphi(x) = \boldsymbol{a}(x)$. φ is thus an extension of $\boldsymbol{a}$; by the definition of $\boldsymbol{A}^*$ we have $\boldsymbol{a} \in \mathscr{D}(\boldsymbol{A}^*_{[i]})$; as id_C is a homomorphism of $\boldsymbol{A}^*$ into $\boldsymbol{C}$, we get $\boldsymbol{A}^*_{[i]}(\boldsymbol{a}) = \boldsymbol{C}_{[i]}(\boldsymbol{a})$.

Evidently, we could get a characterization of I, S-closed and I, S, P-closed classes of quasi-algebras if we added the assumption "$\mathfrak{A}$ is abstract" in (ii), Theorems 23 and 24.

THEOREM 25. *Let a small scategory* $\mathfrak{A}$ *and a regular number* ϑ *be given. The following two conditions are equivalent:*

(i) $\mathfrak{A}$ *is equivalent to a set of quasi-algebras of a type of dimension* $\leqslant \vartheta$;

(ii) $\mathfrak{A}$ *is a* ϑ*-ary scategory.*

Proof. (i) $\Rightarrow$ (ii) is easy. (ii) $\Rightarrow$ (i): By Theorem 16, $\mathfrak{A}$ is equivalent to an S-skeleton of a rich scategory $\mathfrak{B}$ with (ϑ, ϑ)-ary morphisms. $\mathfrak{B}$ is evidently small. $\mathfrak{B}$ is (ϑ, ϑ)-additive, because it is rich. By Theorem 23 $\mathfrak{B}$, and thus also $\mathfrak{A}$, is equivalent to a class of quasi-algebras of a type of dimension $\leqslant \vartheta$.

THEOREM 26. *A small scategory* $\mathfrak{A}$ *is equivalent to a set of quasi-algebras of a type if and only if for all* $\boldsymbol{A} \in \mathfrak{A}^{\mathscr{O}}$, *if* $A = 0$, *then* $0\colon \boldsymbol{A} \to \boldsymbol{B}$ *for every* $\boldsymbol{B} \in \mathfrak{A}^{\mathscr{O}}$.

The proof follows from Theorem 25.

15. A characterization of S, P-closed classes of algebras

THEOREM 27. *Let ϑ be a regular number. Let $\mathfrak{A}$ be a scategory with divisible morphisms such that for every cardinal number $\mathfrak{m}$ there exists an $\mathfrak{A}$-free structure with a ϑ-dense $\mathfrak{A}$-basis of cardinality $\mathfrak{m}$. Then $\mathfrak{A}$ is equivalent to a class of algebras of a type of dimension ϑ.*

Proof. For every cardinal number $\varkappa < \vartheta$ choose an $\mathfrak{A}$-free structure $\boldsymbol{F}_\varkappa$ with an $\mathfrak{A}$-basis $X_\varkappa$ of cardinality $\varkappa$. Let us define a type Δ in the following way: its domain I is the set of all ordered pairs $\langle \varkappa, c\rangle$ such that $\varkappa$ is a cardinal number, $\varkappa < \vartheta$ and $c \in F_\varkappa$; if $i = \langle \varkappa, c\rangle \in I$, then $\Delta_i = X_\varkappa$. Evidently, Δ is a type of dimension ϑ. We assign to every $\mathfrak{A}$-structure $\boldsymbol{A}$ an algebra $\boldsymbol{A}^*$ of type Δ: its underlying set is the set A; if $i = \langle \varkappa, c\rangle \in I$ and $\boldsymbol{a} \in A^{X_\varkappa}$, then $\boldsymbol{A}^*_{[i]}(\boldsymbol{a}) = \varphi(c)$ where φ is the uniquely determined extension of $\boldsymbol{a}$ to an $\mathfrak{A}$-morphism of $\boldsymbol{F}_\varkappa$ into $\boldsymbol{A}$. Let ε be the mapping which assigns to every $\boldsymbol{A} \in \mathfrak{A}^{\mathcal{O}}$ this $\boldsymbol{A}^*$. Put $\mathscr{K} = W(\varepsilon)$. We shall prove that ε is an equivalence of $\mathfrak{A}$ onto $\mathscr{K}$.

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}^{\mathcal{O}}$ and $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$. Let $i = \langle \varkappa, c\rangle \in I$ and $\boldsymbol{a} \in A^{X_\varkappa}$. The mapping $\boldsymbol{a}$ can be extended to an $\mathfrak{A}$-morphism $\psi\colon \boldsymbol{F}_\varkappa \to \boldsymbol{A}$ in exactly one way; we have $\boldsymbol{A}^*_{[i]}(\boldsymbol{a}) = \psi(c)$. As $\varphi \circ \psi\colon \boldsymbol{F}_\varkappa \to \boldsymbol{B}$ in $\mathfrak{A}$, we have

$$\boldsymbol{B}^*_{[i]}(\varphi \circ \boldsymbol{a}) = \varphi\big(\psi(c)\big) = \varphi\big(\boldsymbol{A}^*_{[i]}(\boldsymbol{a})\big).$$

φ is thus a homomorphism.

Conversely, let φ be a homomorphism of $\boldsymbol{A}^*$ into $\boldsymbol{B}^*$. If $A = 0$, then (as there exists a morphism of $\boldsymbol{F}_0$ into $\boldsymbol{A}$) the structure $\boldsymbol{F}_0$ is of order 0; since $0\colon \boldsymbol{F}_0 \to \boldsymbol{A}$, $W(0) = A$, 0 is a mapping of A into B and $0 \circ 0\colon \boldsymbol{F}_0 \to \boldsymbol{B}$, we get $0\colon \boldsymbol{A} \to \boldsymbol{B}$, so that $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$. We shall suppose $A \neq 0$. There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with an $\mathfrak{A}$-basis $\bar{A}$ of cardinality $\operatorname{Card} A$ such that $\bar{A}$ is ϑ-dense in $\boldsymbol{F}$. A one-to-one mapping η of $\bar{A}$ onto A can be extended to a morphism $\psi\colon \boldsymbol{F} \to \boldsymbol{A}$ and the mapping $\varphi \circ \eta$ to a morphism $\chi\colon \boldsymbol{F} \to \boldsymbol{B}$. Since $\mathfrak{A}$ is a scategory with divisible morphisms, thus, in order to prove that $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathfrak{A}$ it is sufficient to prove $\varphi \circ \psi = \chi$. For every $h \in F$ there exists a set $K_h \subseteq \bar{A}$ of cardinality $< \vartheta$ and a substructure $\boldsymbol{D}_h$ of $\boldsymbol{F}$ such that $h \in D_h$, $K_h \subseteq D_h$ and K_h is dense in $\boldsymbol{D}_h$. Let an arbitrary element $d \in F$ be given. If $K_d \neq 0$, put $K = K_d$ and $\boldsymbol{D} = \boldsymbol{D}_d$. If $K_d = 0$, choose an arbitrary $a \in \bar{A}$ and put $K = K_a$ and $\boldsymbol{D} = \boldsymbol{D}_a$, so that evidently $K \neq 0$; in this latter case $d \in D$, too, as there exists exactly one morphism of $\boldsymbol{D}_d$ into $\boldsymbol{D}$ (namely the restriction of any morphism of $\boldsymbol{F}$ into $\boldsymbol{D}$) and this is equal to id_{D_d}, so that $D_d \subseteq D$. In both cases K is a non-empty subset of $\bar{A}$, $\operatorname{Card} K < \vartheta$ and $\boldsymbol{D}$ is a substructure of $\boldsymbol{F}$, K being dense in $\boldsymbol{D}$. If $C \in \mathfrak{A}^{\mathcal{O}}$ and λ is a mapping of K into C, then λ can be extended to a morphism of $\boldsymbol{F}$ into C and the restriction of this morphism to D is an extension of λ to a morphism of $\boldsymbol{D}$ into $\boldsymbol{C}$; K is thus an $\mathfrak{A}$-basis in $\boldsymbol{D}$.

If we put $\varkappa = \operatorname{Card} K$, there exists an isomorphism $\mu\colon \boldsymbol{F}_\varkappa \simeq \boldsymbol{D}$ such that $\mu'' X_\varkappa = K$. Put $v = \mu^{-1}(d)$ and $i = \langle \varkappa, v\rangle$. We have

$$\begin{aligned}\chi(d) = \chi(\mu(v)) = \boldsymbol{B}^*_{[i]}(\chi\circ\mu\restriction X_\varkappa) &= \boldsymbol{B}^*_{[i]}(\varphi\circ\eta\circ\mu\restriction X_\varkappa)\\ &= \varphi(\boldsymbol{A}^*_{[i]}(\eta\circ\mu\restriction X_\varkappa)) = \varphi(\boldsymbol{A}^*_{[i]}(\psi\circ\mu\restriction X_\varkappa))\\ &= \varphi((\psi\circ\mu)(v)) = \varphi(\psi(d)).\end{aligned}$$

As $d\in F$ was arbitrary, we get really $\chi = \varphi\circ\psi$.

THEOREM 28. *Let ϑ be a regular number. Let $\mathfrak{A}$ be a closed in itself scategory with divisible morphisms such that for every cardinal number* $\mathfrak{m}$ *there exists an $\mathfrak{A}$-free structure with a ϑ-dense $\mathfrak{A}$-basis of cardinality* $\mathfrak{m}$. *Then $\mathfrak{A}$ is equivalent to an S-closed class of algebras of a type of dimension ϑ.*

Proof. Let us hold the notation introduced in the proof of Theorem 27. It is sufficient to prove that the class $\mathscr{K}$ constructed there is S-closed. Let $\boldsymbol{A}\in\mathfrak{A}^{\mathcal{O}}$ and let $\boldsymbol{D}$ be an arbitrary subalgebra of $\boldsymbol{A}^*$; we have to prove $\boldsymbol{D}\in\mathscr{K}$. If $D = 0$, then from the definition of $\boldsymbol{A}^*$ we get $F_0 = 0$; as there exists at most one algebra of type Δ with the empty underlying set, we have $\boldsymbol{F}_0^* = \boldsymbol{D}$, so that $\boldsymbol{D}\in\mathscr{K}$. Let $D\neq 0$. There exists an $\mathfrak{A}$-free structure $\boldsymbol{F}$ with an $\mathfrak{A}$-basis $\bar{D}$ of cardinality $\operatorname{Card} D$ such that $\bar{D}$ is ϑ-dense in $\boldsymbol{F}$; a one-to-one mapping η of $\bar{D}$ onto D can be extended to a morphism $\varphi\colon \boldsymbol{F}\to\boldsymbol{A}$. Let $d\in\bar{D}$. Similarly as in the proof of Theorem 27, there exists a non-empty $K\subseteq\bar{D}$ of cardinality $<\vartheta$ and a substructure $\boldsymbol{B}$ of $\boldsymbol{F}$ such that $d\in B$, $K\subseteq B$ and K is dense in $\boldsymbol{B}$; if we put $\varkappa = \operatorname{Card} K$, then there exists an isomorphism $\mu\colon \boldsymbol{F}_\varkappa\simeq\boldsymbol{B}$ such that $\mu'' X_\varkappa = K$. Put $v = \mu^{-1}(d)$ and $i = \langle\varkappa, v\rangle$. We have

$$\varphi(d) = (\varphi\circ\mu)(v) = \boldsymbol{A}^*_{[i]}(\varphi\circ\mu\restriction X_\varkappa) = \boldsymbol{A}^*_{[i]}(\eta\circ\mu\restriction X_\varkappa)\in D.$$

We get $W(\varphi)\subseteq D$ and consequently $W(\varphi) = D$, so that D is a set underlying in $\boldsymbol{A}$; let $\boldsymbol{B}$ be the corresponding substructure. Since $B^* \doteq D$ and id_D is a homomorphism of $\boldsymbol{B}^*$ into $\boldsymbol{A}^*$, we have evidently $\boldsymbol{B}^* = \boldsymbol{D}$, so that $\boldsymbol{D}\in\mathscr{K}$.

THEOREM 29. *Let a regular number ϑ and a scategory $\mathfrak{A}$ be given. The following four conditions are equivalent:*

(i) *$\mathfrak{A}$ is equivalent to an S, P-closed class of algebras of a type of dimension $\leqslant\vartheta$;*

(ii) *$\mathfrak{A}$ is a locally small, bounded, regular, Cartesian, ϑ-additive and closed in itself scategory with divisible morphisms;*

(iii) *there exists a cardinal number $\varkappa$ such that $\mathfrak{A}$ is a $\varkappa$-locally small, Cartesian, $(\vartheta,\varkappa)$-additive and closed in itself scategory with $(\vartheta,\varkappa)$-ary and divisible morphisms;*

(iv) *$\mathfrak{A}$ is a Cartesian and closed in itself scategory with divisible mor-*

phisms such that, if it is non-trivial, then for every cardinal number $\mathfrak{m}$ *there exists an* $\mathfrak{A}$*-free structure with a* ϑ*-dense* $\mathfrak{A}$*-basis of cardinality* $\mathfrak{m}$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are easy. (ii) $\Rightarrow$ (iv) follows from Theorems 14 and 20.

(iii) $\Rightarrow$ (iv): By Theorem 24 $\mathfrak{A}$ is equivalent to an S, P-closed class of quasi-algebras of a type of dimension $\leqslant \vartheta$, so that $\mathfrak{A}$ (if non-trivial) evidently fulfils the assumptions of Theorem 14 and every $\mathfrak{A}$-basis is ϑ-dense.

It remains to prove (iv) $\Rightarrow$ (i). If $\mathfrak{A}$ is non-trivial, then this is easy if we use Theorems 28 and 22. Let $\mathfrak{A}$ be trivial. Let Δ be an arbitrary type of dimension ϑ satisfying only the following condition: Δ is a type with constants if and only if there exists an $\mathfrak{A}$-structure of order 0. It is easy to prove that $\mathfrak{A}$ is equivalent to an S, P-closed class of algebras of type Δ if we realize the following facts:

(21) There exists at most one $\mathfrak{A}$-structure of order 0.

Indeed, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $\mathfrak{A}$-structures of order 0, then the Cartesian product $\boldsymbol{C}$ of the family constituted by $\boldsymbol{A}$ and $\boldsymbol{B}$ is again of order 0 and we have $0: \boldsymbol{C} \to \boldsymbol{A}$ and $0: \boldsymbol{C} \to \boldsymbol{B}$ in $\mathfrak{A}$; by the divisibility of morphisms we get $0: \boldsymbol{A} \to \boldsymbol{C}$ and $0: \boldsymbol{B} \to \boldsymbol{C}$, so that $\boldsymbol{A} = \boldsymbol{C} = \boldsymbol{B}$.

(22) If $\boldsymbol{0}$ is an $\mathfrak{A}$-structure of order 0 and $\boldsymbol{A} \in \mathfrak{A}^{o}$, then $0: \boldsymbol{0} \to \boldsymbol{A}$.

Indeed, the Cartesian product $\boldsymbol{C}$ of the family constituted by $\boldsymbol{0}$ and $\boldsymbol{A}$ is of order 0, so that $\boldsymbol{0} = \boldsymbol{C}$; we have $0: \boldsymbol{C} \to \boldsymbol{A}$.

(23) Every two $\mathfrak{A}$-structures of order 1 are isomorphic.

This follows from the existence of Cartesian products and divisibility of morphisms.

Evidently, we could get a characterization of I, S, P-closed classes of algebras if we added the assumption: $\mathfrak{A}$ is abstract.

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