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A REPRESENTATION THEOREM FOR PERFECT PARTITIONS OF Z^2-ACTIONS WITH FINITE ENTROPY

BY

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1. Introduction. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and $\mathscr{Z}$ be the set of all measurable partitions of X with finite entropy. We denote by ε the measurable partition of X into single points.

Let T be an automorphism of $(X, \mathscr{B}, \mu)$. For a measurable partition P of X we put

$$P_T^- = \bigvee_{n=1}^{\infty} T^{-n} P, \quad P_T = \bigvee_{n=-\infty}^{\infty} T^n P.$$

Recall that P is said to be a *generator* of the one-dimensional dynamical system (Z^1-action) (X, T) if $P_T = \varepsilon$.

A measurable partition ζ is called *T-perfect* (cf. [4]) if

$$T^{-1}\zeta \leqslant \zeta, \quad \zeta_T = \varepsilon, \quad \bigwedge_{n=0}^{\infty} T^{-n}\zeta = \pi(T), \quad h(T) = h(\zeta, T),$$

where $\pi(T)$ and $h(T)$ denote the Pinsker partition and the entropy of T, respectively.

Rohlin has shown in [4] that if $h(T) < \infty$ and ζ is T-perfect, then there exists a generator $P \in \mathscr{Z}$ of (X, T) such that $\zeta = P \vee P_T^-$, i.e., ζ is the past of the process (P, T).

Now we want to consider a two-dimensional analogue of this result.

Let (X, G) be a two-dimensional dynamical system (Z^2-action), i.e., G is an abelian free group of rank 2 of automorphisms of $(X, \mathscr{B}, \mu)$.

Let N denote the set of positive integers, Z^2 the set of two-dimensional integers, and $\prec$ the lexicographical order in Z^2.

Let (T, S) be an ordered pair of independent generators of G and $A \subset Z^2$. For a measurable partition P we put

$$P(A) = \bigvee_{(k,l) \in A} T^k S^l P.$$

In the sequel we shall use the following notation:

$$P^n(m) = P(\{(i,j) \in Z^2;\ -n \leqslant i \leqslant m-1,\ 0 \leqslant j \leqslant 2n\}),$$

$$P^n_{-n}(m) = P(\{(i,j) \in Z^2;\ -n \leqslant i \leqslant m-1,\ -n \leqslant j \leqslant n\}), \qquad n, m \in N,$$

$$P(n) = P(\{(i,j) \in Z^2;\ i \leqslant n\}),$$

$$P(n, m) = P(\{(i,j) \in Z^2;\ (i,j) \prec (n,m) \text{ or } (i,j) = (n,m)\}), \qquad n, m \in Z^1,$$

$$P_G^- = P(0, -1), \qquad P_G = P(Z^2).$$

If $P_G = \varepsilon$, we say that P is a *generator* of (X, G).

A measurable partition ζ is said to be (T, S)-*perfect* (cf. [1]) if

(i) $S^{-1}\zeta \leqslant \zeta$, $T^{-1}\zeta_S \leqslant \zeta$,

(ii) $\zeta_G = \varepsilon$,

(iii) $\bigwedge_{n=0}^{\infty} S^{-n}\zeta = T^{-1}\zeta_S$,

(iv) $\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S = \pi(G)$,

(v) $h(G) = H(\zeta|\zeta_G^-) = H(\zeta|S^{-1}\zeta)$,

where $\pi(G)$ and $h(G)$ denote the Pinsker partition and the entropy of G, respectively. Let us observe that condition (i) means that $T^k S^l \zeta \leqslant \zeta$ for $(k, l) \prec (0, 0)$.

Remark. If $h(G) < \infty$ and a measurable partition ζ satisfies conditions (i)–(iii) and (v), then it also satisfies (iv).

Proof. The idea of the proof is similar to that of the corresponding property of Z^1-actions (cf. [5]). Therefore we give only a sketch of the proof.

It follows from properties (i)–(iii) and Theorem 2 of [1] that

$$\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S \geqslant \pi(G).$$

Let $(P_n) \subset \mathscr{Z}$ and $Q \in \mathscr{Z}$ be such that $P_n \leqslant P_{n+1}$, $n \geqslant 1$,

$$\bigvee_{n=1}^{\infty} P_n = \zeta \quad \text{and} \quad Q \leqslant \bigwedge_{n=0}^{\infty} T^{-n}\zeta_S.$$

The Pinsker formula for Z^2-actions and the relations

$$(P_n)_G^- \leqslant S^{-1}\zeta, \quad Q_G \leqslant T^{-1}\zeta_S$$

give

$$h(G) \geqslant h(P_n \vee Q, G) \geqslant h(Q, G) + H(P_n|S^{-1}\zeta), \qquad n \geqslant 1.$$

Taking the limit as $n \to \infty$ in the last inequality and using (v) and the fact that $h(G) < \infty$, we obtain $h(Q, G) = 0$, i.e., $Q \leqslant \pi(G)$. Thus we have shown that

$$\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S \leqslant \pi(G),$$

and so equality (iv) is satisfied.

It is known (cf. [1]) that for every pair (T, S) of generators of G there exists a (T, S)-perfect partition.

Now, let G have a finite entropy and be totally ergodic, i.e., every automorphism belonging to G and different from the identity transformation on X is ergodic.

A generator $P \in \mathscr{Z}$ of (X, G) is said to be *(T, S)-regular* (cf. [3]) if the smallest (T, S)-invariant partition refining P, i.e., the partition $\zeta = P \vee P_G^-$ $= P(0, 0)$, is (T, S)-perfect. This concept has a well-defined analogue for Z^1-actions with finite entropy, but it is easy to see that in this case every generator is regular. However, as we have seen in [3], for Z^2-actions there are generators which are not regular.

Using results of [2] and [3] it is easy to check that the following three conditions are equivalent:

(a) P is a (T, S)-regular generator,

(b) $\bigwedge_{n=0}^{\infty} P(0, -n) = P(-1)$,

(c) $\pi(S|P(-1)) = P(-1)$, where $\pi(S|\cdot)$ denotes the relative Pinsker partition of S.

It is proved in [2] that for every pair (T, S) there exists a (T, S)-regular generator. Moreover, the set of all (T, S)-regular generators is dense in the set of all generators.

The purpose of this paper is to characterize these (T, S)-perfect partitions ζ which are representable, i.e., ζ which are of the form $\zeta = P \vee P_G^-$, where P is a (T, S)-regular generator of (X, G). To do this we introduce the following concept.

A (T, S)-invariant partition ζ (cf. [1]) is said to be *(T, S)-saturated* if for every partition $P \in \mathscr{Z}$, $P \leqslant \zeta$, there exists a (T, S)-regular generator $Q \in \mathscr{Z}$, $Q \leqslant \zeta$, such that $P \leqslant Q(0, 0)$.

It follows from our earlier considerations that for Z^2-actions with finite entropy every perfect partition is saturated.

It is an easy consequence of Proposition 2 of [2] that in the case $h(G) = 0$ the partition $\zeta = \varepsilon$ is saturated.

Our aim is to prove that a (T, S)-perfect partition ζ is representable if and only if it is (T, S)-saturated.

It would be interesting to know whether there exist perfect partitions which are not saturated.

2. Auxiliary lemma. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space, $\mathscr{A}$ be a sub-σ-algebra of $\mathscr{B}$ and $E^{\mathscr{A}}$ be the conditional expectation operator in $L^1(X, \mathscr{B}, \mu)$.

Let $\mathscr{A}_i$ and $\mathscr{C}$ be sub-σ-algebras of $\mathscr{B}$, $i = 1, 2$.

Let us recall that $\mathscr{A}_1$ and $\mathscr{A}_2$ are *independent relative to* $\mathscr{C}$ and we write $\mathscr{A}_1 \perp^{\mathscr{C}} \mathscr{A}_2$ if for every $f \in L^1(X, \mathscr{A}_1, \mu)$ and $g \in L^1(X, \mathscr{A}_2, \mu)$ such that $f \cdot g \in L^1(X, \mathscr{B}, \mu)$. we have

$$\mathrm{E}^{\mathscr{C}}(f \cdot g) = \mathrm{E}^{\mathscr{C}} f \cdot \mathrm{E}^{\mathscr{C}} g \text{ a.e.}$$

Now, let us suppose that $\mathscr{A}_i \supset \mathscr{C}$, $i = 1, 2$.

LEMMA. *$\mathscr{A}_1 \perp^{\mathscr{C}} \mathscr{A}_2$ if and only if $\mathrm{E}^{\mathscr{A}_1} f = \mathrm{E}^{\mathscr{C}} f$ a.e. for every $f \in L^1(X, \mathscr{A}_2, \mu)$.*

Proof. Let us suppose that $\mathscr{A}_1 \perp^{\mathscr{C}} \mathscr{A}_2$, $A \in \mathscr{A}_1$ and $f \in L^1(X, \mathscr{A}_2, \mu)$. Using the basic properties of the conditional expectation and our assumptions we have

$$\int_A f d\mu = \int_X \mathrm{E}^{\mathscr{C}}(\chi_A \cdot f) d\mu = \int_X \mathrm{E}^{\mathscr{C}} \chi_A \cdot \mathrm{E}^{\mathscr{C}} f d\mu$$
$$= \int_X \mathrm{E}^{\mathscr{C}}(\chi_A \cdot \mathrm{E}^{\mathscr{C}} f) d\mu = \int_A \mathrm{E}^{\mathscr{C}} f d\mu,$$

i.e., $\mathrm{E}^{\mathscr{C}} f = \mathrm{E}^{\mathscr{A}_1} f$ a.e.

The proof of the sufficiency runs in a similar manner.

Let $\mathscr{A}_i$, $\mathscr{C}$, $\mathscr{D}$ $(1 \leqslant i \leqslant 4)$ be sub-σ-algebras of $\mathscr{B}$ such that $\mathscr{D} \subset \mathscr{C} \subset \mathscr{A}_i$ $(1 \leqslant i \leqslant 3)$ and $\mathscr{A}_4 \subset \mathscr{A}_2$.

From the Lemma we obtain easily

COROLLARY 1. *If $\mathscr{A}_1 \perp^{\mathscr{C}} \mathscr{A}_2$ and $\mathscr{A}_3 \perp^{\mathscr{D}} \mathscr{A}_4$, then $\mathscr{A}_1 \perp^{\mathscr{D}} \mathscr{A}_4$.*

Now, let ξ be a measurable partition of X. We denote by $\sigma(\xi)$ the sub-σ-algebra of ξ-sets (cf. [5]). Let ξ, η, ζ be measurable partitions of X. The fact that ξ and η are independent relative to ζ will be denoted by $\xi \perp^{\zeta} \eta$.

It is easy to check that

$$\xi \perp^{\zeta} \eta \quad \text{if and only if} \quad \sigma(\xi) \perp^{\sigma(\zeta)} \sigma(\eta).$$

Let ξ_i, η, ζ $(1 \leqslant i \leqslant 4)$ be measurable partitions such that $\zeta \leqslant \eta \leqslant \xi_i$ $(1 \leqslant i \leqslant 3)$ and $\xi_4 \leqslant \xi_2$.

COROLLARY 2. *If $\xi_1 \perp^{\eta} \xi_2$ and $\xi_3 \perp^{\zeta} \xi_4$, then $\xi_1 \perp^{\zeta} \xi_4$.*

3. The representation theorem. Let (X, G) be a two-dimensional totally ergodic dynamical system with $h(G) < \infty$.

Let (T, S) be an ordered pair of independent generators of G. We say in the sequel that a measurable partition of X is *invariant* (*perfect*, *regular*, *saturated*) if it is (T, S)-invariant (perfect, regular, saturated, respectively).

THEOREM. *A perfect partition ζ is representable if and only if it is saturated.*

Proof. The necessity is obvious. To prove the sufficiency let us suppose that ζ is perfect and saturated. Since $H(\zeta | S^{-1}\zeta) = h(G) < \infty$, there exists a partition $R \in \mathscr{Z}$ with $S^{-1}\zeta \vee R = \zeta$ (cf. [4]). Since ζ is saturated, there exists a regular generator $\tilde{R} \leqslant \zeta$ with $R \leqslant \tilde{R}(0, 0)$. Let us observe that the generator $P = R \vee \tilde{R}$ is regular. Indeed, since $R \leqslant \tilde{R}(0, 0)$, we have $R(-1) \leqslant \tilde{R}(-1)$, and so $P(-1) = \tilde{R}(-1)$. Therefore, using Corollary 1 to Lemma 2 of [3] and the fact that $\tilde{R}$ is regular, we have

$$\pi(S|P(-1)) = \pi(S|\tilde{R}(-1)) = \tilde{R}(-1) = P(-1),$$

i.e., P is regular. We also have

$$S^{-1}\zeta \vee P = \zeta. \tag{1}$$

Our aim is to show that $\zeta = P(0, 0)$.

In order to obtain equalities (2)–(7) below we proceed similarly as Rohlin in [4].

Let $r \in N$ be arbitrary. First we check that

$$\zeta \vee P(0, r) = S^r\zeta. \tag{2}$$

Indeed, equality (1) implies

$$S^{-r}\zeta \vee \bigvee_{j=0}^{r-1} S^{-j}P = \zeta.$$

Hence, using the fact that $P \leqslant \zeta$ and ζ is invariant, we get (2). From (2) we obtain easily

$$H(S^r\zeta|\zeta) = H(P(0, r)|\zeta). \tag{3}$$

The equality $h(G) = H(\zeta|S^{-1}\zeta)$ gives readily

$$H(S^r\zeta|\zeta) = r \cdot H(S\zeta|\zeta) = r \cdot h(G). \tag{4}$$

Similarly, since P is a generator of (X, G), we have

$$H(P(0, r)|P(0, 0)) = r \cdot h(G). \tag{5}$$

Thus (3)–(5) imply

$$H(P(0, r)|P(0, 0)) = H(P(0, r)|\zeta). \tag{6}$$

Now we want to show that

$$H(Q|P(0, 0)) = H(Q|\zeta) \tag{7}$$

for every $Q \in \mathscr{Z}$ and $Q \leqslant P(0, r)$. In fact, if $Q \in \mathscr{Z}$ and $Q \leqslant P(0, r)$, then applying the inequality $P \leqslant \zeta$ and (6) we have

$$\begin{aligned} H(Q|P(0, 0)) &= H(P(0, r)|P(0, 0)) - H(P(0, r)|P(0, 0) \vee Q) \\ &\leqslant H(P(0, r)|\zeta) - H(P(0, r)|\zeta \vee Q) = H(Q|\zeta). \end{aligned}$$

Since the converse inequality is trivial, we get (7). But the set consisting of

$Q \in \mathscr{Z}$ with $Q \leqslant P(0, r)$ for some $r \in N$ is dense in the set of $Q \in \mathscr{Z}$ with $Q \leqslant P(0)$. Therefore, (7) implies at once

$$H(Q|P(0, 0)) = H(Q|\zeta) \tag{8}$$

for every $Q \in \mathscr{Z}$ and $Q \leqslant P(0)$.

Now we shall prove, by induction, that (8) holds for $Q \in \mathscr{Z}$ and $Q \leqslant P(m)$, where $m \in N$ is arbitrary.

Let us suppose that

$$H(Q|P(0, 0)) = H(Q|\zeta) \tag{9}$$

for $Q \in \mathscr{Z}$ and $Q \leqslant P(m-1)$, where $m \in N$. Let us observe that (9) implies

$$H(Q|P(0, 0)) = H(Q|P \vee P_S^- \vee T^{-1}\zeta_S). \tag{10}$$

In fact,

$$H(Q|P \vee P_S^- \vee T^{-1}\zeta_S) \leqslant H(Q|P(0, 0)) = H(Q|\zeta) \leqslant H(Q|P \vee P_S^- \vee T^{-1}\zeta_S).$$

Since $P^n(m) \leqslant P(m-1)$, by (9) we have

$$H(P^n(m)|P(0, 0)) = H(P^n(m)|P \vee P_S^- \vee T^{-1}\zeta_S).$$

Therefore

$$H(P^n_{-n}(m)|P(0, -n)) = H(P^n_{-n}(m)|S^{-n+1}P_S^- \vee T^{-1}\zeta_S),$$

and so

$$H(TP^n_{-n}(m)|P(1, -n)) = H(TP^n_{-n}(m)|TS^{-n+1}P_S^- \vee \zeta_S). \tag{11}$$

Using Theorem 5.10 of [5] and (11) we have

$$TP^n_{-n}(m) \perp^{P(1,-n)} \zeta_S. \tag{12}$$

Now, let $k < n$ be fixed. From (12) we obtain

$$TP^k_{-k}(m) \perp^{P(1,-n)} \zeta_S. \tag{13}$$

Taking in (13) the limit as $n \to \infty$ and using the fact that P is regular we get

$$TP^k_{-k}(m) \perp^{P(0)} \zeta_S, \quad k \geqslant 1.$$

Therefore

$$P(m) \perp^{P(0)} \zeta_S. \tag{14}$$

Let us observe that (9) means

$$P(m-1) \perp^{P(0,-1)} \zeta. \tag{15}$$

It follows from (14), (15) and Corollary 2 that

$$P(m) \perp^{P(0,-1)} \zeta.$$

Thus $P(m) \perp^{P(0,-1)} \zeta$ for any $m \in N$. Since P is a generator, we get $\zeta = P \vee P_G^-$ and the theorem is proved.

REFERENCES

[1] B. Kamiński, *The theory of invariant partitions for Z^d-actions*, Bull. Acad. Pol. Sci., Math., 29 (1981), pp. 349–362.

[2] – *On regular generators of Z^2-actions in exhaustive partitions*, Studia Math. 85 (1987), pp. 17–26.

[3] – and M. Kobus, *Regular generators for multidimensional dynamical systems*, Colloq. Math. 50 (1986), pp. 263–270.

[4] V. A. Rohlin, *Generators in ergodic theory. II* (in Russian), Vestnik Leningrad. Univ., Mat. Meh. Astronom., 13, No. 3 (1965), pp. 68–72.

[5] – *Lectures on the entropy theory of transformations with invariant measure* (in Russian), Uspehi Mat. Nauk 22, No. 5 (137) (1967), pp. 3–56.

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