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On a subclass of univalent functions II

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Abstract. Let f be analytic in the unit disc E with $f(0) = 0 = f'(0)-1$ and $f(z)f'(z)/z \neq 0$ for z *in* E. In this paper the class $C(\alpha, \lambda)$ of function f satisfying in E the condition $|(H(\alpha, f)-1)/(H(\alpha, f)+1)| < \lambda$, $0 < \lambda \leqslant 1$, where $H(\alpha, f) = (1-\alpha)(zf'(z)/g(z)) + \alpha(zf'(z))'/g'(z)$, for some g starlike in E and α a non-negative real number, is introduced. It is proved that $C(\alpha, \lambda) \subset C(0, \lambda)$. An integral representation for $f \in C(\alpha, \lambda)$ and estimates for the first few coefficients of $f \in C(\alpha, \lambda)$ are obtained. Further, some subclasses of $C(\alpha, \lambda)$ have been studied.

1. Introduction. Let V denote the class of functions f analytic in the unit disc E with $f(0) = 0 = f'(0)-1$ and $f(z)f'(z)/z \neq 0$ in E.

In this paper, we introduce the class $C(\alpha, \lambda)$ of functions $f \in V$ satisfying in E the condition

$$|(H(\alpha, f)-1)/(H(\alpha, f)+1)| < \lambda, \quad 0 < \lambda \leqslant 1,$$

where

$$H(\alpha, f) = (1-\alpha)(zf'(z)/g(z)) + \alpha(zf'(z))'/g'(z)$$

for some starlike function g, α being any non-negative real number.

For $\lambda = 1$, the class $C(\alpha, \lambda)$ coincides with the class introduced by Chichra [1].

In this paper we investigate a few properties of the class $C(\alpha, \lambda)$ and study some of its subclasses.

2. To prove our main theorem of this section we require the following lemma.

Lemma 1. *Let $\alpha \geqslant 0$ and $D(z)$ be a starlike function in the unit disc E. Let $N(z)$ be analytic in E with $N(0) = D(0) = 0 = N'(0)-1 = D'(0)-1$. Then*

$$\left|\left\{\frac{N(z)}{D(z)}-1\right\}\Big/\left\{\frac{N(z)}{D(z)}+1\right\}\right| < \lambda \quad \textit{for } z \textit{ in } E$$

whenever

$$\left|\left\{(1-\alpha)\frac{N(z)}{D(z)}+\alpha\frac{N'(z)}{D'(z)}-1\right\}\bigg/\left\{(1-\alpha)\frac{N(z)}{D(z)}+\alpha\frac{N'(z)}{D'(z)}+1\right\}\right| < \lambda$$

for z *in* E.

Proof. Define an analytic function $w(z)$ in E by

$$\frac{N(z)}{D(z)} = \frac{1-\lambda w(z)}{1+\lambda w(z)}.$$

Clearly $w(0) = 0$ and $w(z) \neq -1/\lambda$ in E. We shall prove that $|w(z)| < 1$, $z \in E$. For, if not, there exists $z_0 \in E$, by Jack's Lemma [2], such that $|w(z_0)| = 1$ and

$$z_0 w'(z_0) = k w(z_0), \quad k \geqslant 1.$$

If we write

$$\psi(z) = (1-\alpha)\frac{N(z)}{D(z)}+\alpha\frac{N'(z)}{D'(z)},$$

then

$$\left|\frac{\psi(z_0)-1}{\psi(z_0)+1}\right| = \lambda\left|\frac{1+\Phi(z_0)}{1-\lambda w(z_0)\Phi(z_0)}\right|,$$

where $\Phi(z_0) = \dfrac{k\alpha}{1+\lambda w(z_0)}\cdot\dfrac{D(z_0)}{z_0 D'(z_0)}$.

Now $|(\psi(z_0)-1)/(\psi(z_0)+1)| > \lambda$, provided

$$|1+\Phi(z_0)|^2 > |1-\lambda w(z_0)\Phi(z_0)|^2.$$

This condition reduces to the following:

$$1+2\,\mathrm{Re}\,\Phi(z_0)+|\Phi(z_0)|^2 > 1-2\,\mathrm{Re}\,\lambda w(z_0)\Phi(z_0)+\lambda^2|\Phi(z_0)|^2$$

or equivalently

$$(1-\lambda^2)|\Phi(z_0)|^2 > -2\mathrm{Re}\big(1+\lambda w(z_0)\big)\Phi(z_0).$$

This is true if $\mathrm{Re}\big(1+\lambda w(z_0)\big)\Phi(z_0) > 0$ or if $\mathrm{Re}\,\{k\alpha D(z_0)/z_0 D'(z_0)\} > 0$ which holds since D is starlike an α. $k > 0$. But this contradicts the hypothesis that $|(\psi(z)-1)/(\psi(z)+1)| < \lambda$, z in E. This proves the lemma.

THEOREM 1. *Let* $f \in C(\alpha, \lambda)$, $\alpha > 0$, $0 < \lambda \leqslant 1$. *Then* $f \in C(0, \lambda)$.

Proof. Choose $D(z) = g(z)$ and $N(z) = zf'(z)$ in Lemma 1.

THEOREM 2. *For* $0 \leqslant \beta < \alpha$, $C(\alpha, \lambda) \subset C(\beta, \lambda)$.

Proof. If $\beta = 0$, the result is obvious from Theorem 1. Assume therefore

$\beta \neq 0$ and $f \in C(\alpha, \lambda)$. Then there exists a starlike function g in E such that

$$H(\alpha, f) = (1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \frac{1-\lambda w_1(z)}{1+\lambda w_1(z)}$$

$$\text{and} \quad \frac{zf'(z)}{g(z)} = \frac{1-\lambda w_2(z)}{1+\lambda w_2(z)},$$

where w_i are analytic in E, $w_i(0) = 0$ and $|w_i(z)| < 1$ in E, $i = 1, 2$. The theorem then follows from the identity

$$H(\beta, f) = \frac{\beta}{\alpha} H(\alpha, f) + (1-\beta/\alpha)\frac{zf'(z)}{g(z)}, \quad \beta < \alpha.$$

THEOREM 3. *A function f is in $C(\alpha, \lambda)$ if and only if there exists a starlike function g in E, with $g(0) = 0 = g'(0)-1$, and a regular function H, $H(0) = 1$ satisfying in E the condition $|(H(z)-1)/(H(z)+1)| < \lambda$, $0 < \lambda \leqslant 1$ such that*

$$f'(z) = \frac{1}{\alpha z[g(z)]^{\frac{1}{\alpha}-1}} \int_0^z [g(t)]^{\frac{1}{\alpha}-1} g'(t) H(t)\, dt, \quad \alpha > 0.$$

If $\alpha = 0$, then

$$f'(z) = \frac{g(z)}{z} H(z).$$

(Powers in the above formula are chosen to ensure continuity near the origin.)

Proof. Let $f \in C(\alpha, \lambda)$, so that for some starlike function g in E we have

$$(1-\alpha)\frac{zf'(z)}{g(z)} + \frac{\alpha(zf'(z))'}{g'(z)} = H(z), \tag{2.1}$$

where H is regular in E, $H(0) = 1$ and $|(H(z)-1)/(H(z)+1)| < \lambda$, $0 < \lambda \leqslant 1$, $z \in E$. Multiplying both sides of (2.1) by $\frac{1}{\alpha}[g(z)]^{\frac{1}{\alpha}-1} g'(z)$, we obtain

$$\left(\frac{1}{\alpha}-1\right) zf'(z)[g(z)]^{\frac{1}{\alpha}-2} g'(z) + (zf'(z))'[g(z)]^{\frac{1}{\alpha}-1} = \frac{1}{\alpha} H(z)[g(z)]^{\frac{1}{\alpha}-1} g'(z). \tag{2.2}$$

The left-hand member of (2.2) is the exact differential of $zf'(z)[g(z)]^{\frac{1}{\alpha}-1}$. Hence on integrating with respect to z we arrive at the required representation formula. The proof of the converse part follows easily.

THEOREM 4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\alpha, \lambda)$. *Then we have the following inequalities*:

(i) $$|a_2| \leqslant \frac{\lambda + 1 + \alpha}{1 + \alpha},$$

(ii) $$|a_3| \leqslant \frac{\lambda(6 + 14\alpha) + 3 + 9\alpha + 6\alpha^2}{3(1+\alpha)(1+2\alpha)},$$

(iii) $$|a_4| \leqslant \frac{\lambda(3 + 16\alpha + 23\alpha^2) + 1 + 6\alpha + 11\alpha^2 + 6\alpha^3}{(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

We require the following elementary

LEMMA 2. *Let* $H(z) = 1 + \sum_{n=1}^{\infty} H_n z^n$ *be analytic in* E *and satisfy the condition* $|(H(z)-1)/(H(z)+1)| < \lambda$, $0 < \lambda \leqslant 1$, $z \in E$. *Then for* $n \geqslant 1$, $|H_n| \leqslant 2\lambda$. *Equality is attained by the function*

$$H_0(z) = (1 - \lambda z^n)/(1 + \lambda z^n).$$

Proof of Theorem 4. Since $f \in C(\alpha, \lambda)$, we have

$$(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = H(z) = 1 + \sum_{n=1}^{\infty} H_n z^n,$$

where $H(z)$ is as in Lemma 2. Setting $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ and comparing coefficients, we obtain

(2.3) $$2(1+\alpha)a_2 = H_1 + (1+\alpha)g_2,$$

(2.4) $$3(1+2\alpha)a_3 = H_2 + 3H_1 g_2 + 2g_2^2 + (1+2\alpha)g_3 - 4a_2 g_2,$$

(2.5) $$4(1+3\alpha)a_4 = H_3 + 3H_2 g_2 + H_1(4g_3 + 2g_2^2) + (1+3\alpha)g_4 + 5g_2 g_3 + \\ + (2\alpha - 6)a_2 g_3 - (6 + 3\alpha)a_3 g_2.$$

Using the estimates $|H_n| \leqslant 2\lambda$ and $|g_n| \leqslant n$ in (2.3), we at once get (i). Eliminating a_2 from (2.4) we obtain,

(2.6) $$3(1+2\alpha)(1+\alpha)a_3 = (1+\alpha)H_2 + (1+3\alpha)H_1 g_2 + (1+\alpha)(1+2\alpha)g_3.$$

Now (2.6) yields inequality (ii). Substituting for a_2 and a_3 from (2.3) and (2.6) in (2.5),

$$4(1+3\alpha)(1+2\alpha)(1+\alpha)a_4 = (1+\alpha)(1+2\alpha)H_3 + (1+\alpha)(1+5\alpha)H_2 g_2 + \\ + H_1(1+2\alpha)(1+5\alpha)\left\{g_3 - \frac{\alpha(1-\alpha)}{(1+2\alpha)(1+5\alpha)} g_2^2\right\} + (1+\alpha)(1+2\alpha)(1+3\alpha)g_4.$$

The result follows on using the following inequality of Keogh and Merkes [3]:

$$|g_3 - \mu g_2^2| \leqslant 3 - 4\mu, \quad \text{if } \mu \leqslant \tfrac{1}{2}.$$

3. In this section we study some subclasses of $C(\alpha, \lambda)$.

Let $F(\alpha, \lambda)$ denote the class of functions $f \in V$, satisfying in E the condition

$$|\{f'(z) + \alpha z f''(z) - 1\}/\{f'(z) + \alpha z f''(z) + 1\}| < \lambda, \quad 0 < \lambda \leqslant 1$$

for a fixed $\alpha \geqslant 0$. Obviously $F(\alpha, \lambda)$ is a subclass of $C(\alpha, \lambda)$ for the choice of $g(z) \equiv z$ and $F(\alpha\lambda) \subset F(0, \lambda)$. The class $F(0, \lambda)$ is investigated in [4]. It is easy to see that $F(\alpha, \lambda) \subset F(\beta, \lambda)$ for $0 \leqslant \beta < \alpha$. Also, if $0 \leqslant t \leqslant 1$ and f, g be in $F(\alpha, \lambda)$, then $F(z) = tf(z) + (1-t)g(z)$ is also in $F(\alpha, \lambda)$.

Moreover, functions in $F(\alpha, \lambda)$ are obtained on taking the Hadamard product of the function

$$k(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \frac{t^{\frac{1}{\alpha}-1}}{1-t}\, dt \tag{3.1}$$

with $\int_0^z H(t)\,dt$, where H is an arbitrary function regular in E with $H(0) = 1$, satisfying $|(H(z)-1)/(H(z)+1)| < \lambda$ for z in E.

Let $G(\alpha, \lambda)$ denote the class of functions $f \in V$ satisfying in E the condition

$$\left|\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right\}\Big/\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + 1\right\}\right| < \lambda,$$

$0 < \lambda \leqslant 1$, $\alpha \geqslant 0$. The class $G(\alpha, \lambda)$ is closely related to $F(\alpha, \lambda)$ in the sense that $f(z) \in F(\alpha, \lambda)$ if and only if $zf'(z) \in G(\alpha, \lambda)$. It follows from Lemma 1 that if $f \in G(\alpha, \lambda)$ for $\alpha > 0$, then $f \in G(0, \lambda)$. Also for $0 \leqslant \beta < \alpha$, $G(\alpha, \lambda) \subset G(\beta, \lambda)$. Then it is evident that if $f \in G(\alpha, \lambda)$ and $\alpha \geqslant 1$, $f \in G(1, \lambda)$, that is,

$$|(f'(z)-1)/(f'(z)+1)| < \lambda, \quad z \in E. \tag{3.2}$$

We shall now obtain the estimate of r_0, so that in $|z| < r_0$, (3.2) holds with $f \in G(\alpha, \lambda)$ and $0 < \alpha < 1$.

Definition [5]. A function G analytic in the unit disc E normalized by $G(0) = 0$, $G'(0) \neq 0$ is called *prestarlike of order* α, $\alpha \leqslant 1$ if and only if

$$\operatorname{Re} \frac{G(z)}{zG'(0)} > \tfrac{1}{2}, \quad z \in E \quad \text{for } \alpha = 1,$$

$$\frac{z}{(1-z)^{2(1-\alpha)}} * G(z) \in S_\alpha \quad \text{for } \alpha < 1,$$

where $*$ denotes the Hadamard product and S_α the class of starlike functions of order α, $\alpha \leqslant 1$. Denote by R_α the class of prestarlike functions of order α.

LEMMA 3. *For $\alpha \leqslant 1$, let $G \in R_\alpha$, $p \in S_\alpha$. Let $F(z)$ be analytic in E. Then*

$$\frac{G * pF}{G * p}$$

takes values in the closed convex hull of $F(E)$.

This lemma is an easy consequence of the Main Theorem of Ruscheweyh [5].

THEOREM 6. *Let $f \in G(\alpha, \lambda)$, where $0 < \alpha < 1$. Then $|(f'(z)-1)/(f'(z)+1)| < \lambda$, $0 < \lambda \leqslant 1$, for $|z| < r_0$, where r_0 is the radius of the largest disc centered at the origin in which $\operatorname{Re} k'(z) > \frac{1}{2}$, k being given by* (3.1).

Proof. Since $f \in G(\alpha, \lambda)$, we can write

$$f(z) = zF(z) * k(z),$$

where F is regular in E with $F(0) = 1$, satisfying $|(F(z)-1)/(F(z)+1)| < \lambda$ in E and k is the function defined by (3.1),

$$f'(z) = \frac{zF(z) * zk'(z)}{z} = \frac{zF(z) * zk'(z)}{z * zk'(z)}.$$

Let r_0 be the radius of the largest disc in which $\operatorname{Re} k'(z) > \frac{1}{2}$. Then if we write $zk'(z) = G(z)$, $G(z)$ is prestarlike of order 1 in $|z| < r_0$. The function $p(z) = z$ is starlike of order 1. Therefore by Lemma 3, $f(z)$ satisfies (3.2).

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