ANNALES
POLONICI MATHEMATICI
XII (1962)

Functions with all partial derivatives arbitrarily prescribed at a point

by A. PLIŚ and T. WAŻEWSKI (Kraków)

In this note we give a construction of functions of n real variables, in class C^∞ on the whole real space R^n, possessing arbitrarily prescribed partial derivatives of all orders at a given point. The existence of such functions (even analytic except at the given point) results from a more general theorem of H. Whitney ([1], p. 65, theorem I) but we present our construction because of its simplicity.

THEOREM. *For an arbitrary system of numbers $a_0, a_{j_1\dots j_k}$, $1 \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n$, $n \geqslant 1$, $k = 1, 2, \dots$ there exists a function $f(x)$, $x = (x_1, \dots, x_n)$, of class C^∞ on R^n, satisfying at a given point the conditions*

$$(1) \qquad f(x) = a_0, \quad \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_k}} f(x) = a_{j_1\dots j_k}, \quad 1 \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n, \quad k = 1, 2, \dots$$

Proof. Without loss of generality we can assume that the derivatives are prescribed at $x = 0$. Let $a_0, a_{j_1\dots j_k}$, $1 \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n$, $k = 1, 2, \dots$ be arbitrarily fixed. It is known that for each $m \geqslant 1$ there exists a homogeneous polynomial $p_m(x)$ of m-th degree satisfying the conditions

$$(2) \qquad \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_m}} p_m(x) = a_{j_1\dots j_m}, \quad \text{for} \quad 1 \leqslant j_1 \leqslant \dots \leqslant j_m \leqslant n \text{ [1]}.$$

NOTATION. We shall denote by $D_k f$ any unspecified derivative of f of the order k.

Obviously we have the following relation:

$$(3) \qquad D_j p_m(x) = 0 \quad \text{at} \quad x = 0 \quad \text{for any } D_j,\ j \neq m.$$

Moreover, for every m there exists a constant C_m such that

$$|D_i p_m(x)| \leqslant C_m |x|^{m-i}, \quad i = 0, 1, \dots, m.$$

[1] Such a polynomial is unique, but this is irrelevant to our purpose, and can be given explicitly.

Particularly, for every m and every positive b

(4) $$|D_i p_m(x)| \leqslant C_m b^{m-i} \quad \text{for} \quad |x| \leqslant b\,, \quad i = 0, 1, \ldots, m\,.$$

Write $P_m(x) = \sum_{k=0}^{m} p_k(x)$, where $p_0(x) = a_0$. We have

(5) $$\frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_k}} P_m(x)|_{x=0} = \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_k}} p_k(x)|_{x=0} = a_{j_1 \ldots j_k} \quad \text{for} \quad k \leqslant m\,,$$

in virtue of (3), (2).

In the construction we shall use an arbitrarily given function $g(x)$ of class C^∞ on R^n with the following properties:

(6) $$g(x) = 1 \quad \text{in a neighbourhood of } x = 0\,,$$

(7) $$g(x) = 0 \quad \text{for} \quad |x| > 1\,,$$

where $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ [2].

In virtue of (7) there exists such a positive non-decreasing sequence E_k that

$$|D_k g(x)| \leqslant E_k\,, \quad \text{for all } x, \quad k = 0, 1, \ldots$$

Hence we have

(8) $$\begin{aligned} &|D_k g(x/b)| \leqslant E_k b^{-k}, \quad \text{for all } x \text{ and any positive } b, \quad k = 0, 1, \ldots \\ &E_k \leqslant E_{k+1}\,, \quad k = 0, 1, \ldots \end{aligned}$$

It is well known that if the series of polynomials $p_m(x)$ is (uniformly) convergent in a certain neighbourhood of $x = 0$, then the limit function is analytic and satisfies (1) at $x = 0$, but it may be divergent at every point except $x = 0$. Therefore we modify each polynomial $p_m(x)$ outside a certain neighbourhood N_m of $x = 0$ putting

(9) $$q_m(x, b) = p_m(x) g(x/b)\,, \quad m = 0, 1, \ldots,$$

where b is a positive constant and $g(x)$ is a function of class C^∞ on R^n satisfying (6), (7).

We have, in virtue of (6), (7),

(10) $$q_m(x, b) = p_m(x) \quad \text{in a neighbourhood of } x = 0\,,$$

(2) An example of such a function can be constructed in the following manner: The function

$$h(t) = \int_t^1 \exp\left((s - 1/2)(s-1)\right)^{-1} ds \quad \text{for} \quad 1/2 \leqslant t \leqslant 1\,,$$

$h(t) = 0$ for $t > 1$, $h(t) = h(1/2)$ for $t < 1/2$, is of class C^∞ for $-\infty < t < \infty$. Hence, the function $g(x) = h(|x|)/h(1/2)$ is of class C^∞ for $|x| > 0$, and (6), (7) are satisfied. By (6) $g(x)$ is of class C^∞ for all x.

and

$$q_m(x, b) = 0 \quad \text{for} \quad |x| \geqslant b\,. \tag{11}$$

Consider the functions

$$Q_m(x) = \sum_{k=0}^{m} q_k(x, b_k)\,, \tag{12}$$

where b_k are positive numbers.

In virtue of (10), for each m, $Q_m(x) = P_m(x)$ in a neighbourhood of $x = 0$ and hence, in virtue of (5), $Q_m(x)$ satisfy (1) for $k \leqslant m$. Therefore, if sequence (12) together with each of its derivatives is uniformly convergent on R^n for a certain positive sequence b_k, the theorem holds true.

Any derivative $D_j q_m$, $j \leqslant m$, is in virtue of (9) a sum of 2^j products of the type $D_i p_m(x) D_{j-i} g(x/b)$, $0 \leqslant i \leqslant j$. In virtue of (8) $|D_{j-i} g(x/b)| \leqslant E_{j-i} b^{i-j} \leqslant E_j b^{i-j}$. Hence, in virtue of (4) $|D_i p_m(x) D_{j-i} g(x/b)| \leqslant E_j C_m b^{m-j}$ for $|x| < b$. Therefore, $|D_j q_m(x, b)| \leqslant 2^j E_j C_m b^{m-j}$ for $|x| < b$ and because of (11) for all x. For $b < 1$, $m > j$ we have $|D_j q_m(x, b)| \leqslant 2^j E_j C_m b$. We take a number $b = b_m$ so small that $|D_j q_m(x, b)| \leqslant 2^{-m}$ for $j < m$ and all x. It follows by (12) that $Q_m(x)$ together with each of its derivatives is uniformly convergent on R^n and thus the proof is complete.

Reference

[1] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), p. 63-89.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 10. 11. 1961