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ON THE SELECTION OF PAIRS

1. The problem, considered in this paper may be presented in the following way.

There are given a set $S_n = \{a_1, a_2, \ldots, a_n\}$ of n objects and a function z defined on the cartesian product $S_n \times S_n$ and valued in the set $\{0, 1\}$ such that $z(a_i, a_j) = z(a_j, a_i)$ for $i \leqslant j \leqslant n$.

We want to split the set S_n or its subset into disjoint sets $B_1, B_2, \ldots$ satisfying the following conditions:

1° each of these sets consists of two elements exactly;

2° for each set $B_i = \{a_l, a_k\}$ we have $z(a_l, a_k) = 1$;

3° the number of sets $B_1, B_2, \ldots$ is as great as possible.

Suppose that we assemble the copies of a technical system from ready-made elements. In each produced copy of the system we must lodge two elements of the series S_n of n elements of the same kind, but of different quality. Suppose that the quality of the elements a_i, a_j of S_n, lodged in the produced copy, decides upon the quality of the produced copy in the same way for each copy and that the quality of the copy with lodged elements a_i, a_j does not change, when we put in this copy a_i on the place of a_j and a_j on the place of a_i. Then we can interpret function z as a criterion of capability of the pair $\{a_i, a_j\}$ to be lodged in certain produced copy ($z(a_i, a_j) = 1$ in the case of capability).

If we shall lodge in the i-th produced copy the elements of a set B_i, then we shall provide a greatest possible number of produced copies. The capability of the pairs $\{a_i, a_j\}$ may be represented by function z in many practical technological cases, e.g. in the batch production of some instruments for comparative measurements. If we have no technological way to make the dispersion in the quality of elements a_i sufficiently small, then the solution of our problem may be useful for economizing the batch production.

2. Remark, first of all, that we can suppose without loss of generality that S_n is the set $\{1, 2, \ldots, n\}$ of n smallest positive integers; it is clear that $S_p \subset S_q$ whenever $p \leqslant q$. We call S_n the *series of length n*. We shall also denote for functions $f, g, \ldots$ their domains by $A(f), A(g), \ldots$ respectively. The function f may be identified with the set of ordered pairs, which explains the notation $f \cup g$, where f, g are functions.

Definition 1. We say that f is a *selection function* in S_n or that f is a *selection from* S_n if $A(f) \subset S_n$ and the condition

$$\bigwedge_{i \in A(f)} \{(f(i) \in A(f) \setminus \{i\}) \wedge (ff(i) = i)\}$$

holds.

The selection functions have the following obvious properties.

PROPERTY 1. *If f is a selection from S_p, then f is a selection from S_q, $q \geqslant p$.*

PROPERTY 2. *If f is a selection from S_p, then f is a selection from S_r, $r = \max A(f)$.*

PROPERTY 3. *If f, g are selections from S_p, S_q respectively and $A(f) \cap A(g) = \emptyset$, then $f \cup g$ is a selection from S_r, $r = \max\{p, q\}$, and $A(f \cup g) = A(f) \cup A(g)$.*

PROPERTY 4. *If f is a selection from S_p, $A \subset A(f)$, $f(A) = A$, then $f \mid A$ is a selection from S_r, $r = \max A$.*

PROPERTY 5. *If f, g are selections from S_p, $A(f) \cap A(g) = \emptyset$, $A \subset A(f) \cup A(g)$, $f(A \cap A(f)) = A \cap A(f)$, $g(A \cap A(g)) = A \cap A(g)$, then $(f \cup g) \mid A = (f \mid (A \cap A(f))) \cup (g \mid (A \cap A(g)))$ is a selection from S_r, $r = \max A$.*

We denote here an empty set by $\emptyset$ and by $f \mid A$ a function g such that $A(g) = A \subset A(f)$ and $f(i) = g(i)$ for $i \in A(g)$. Denote for set A by $|A|$ number of its elements. By $P(f)$ we denote a number p such that $2p = |A(f)|$.

Definition 2. We say that the function z is a *capability function* on S_n if $z\colon S_n \times S_n \to \{0, 1\}$ and the condition $z(i, j) = z(j, i)$ is fulfilled for each i, j from S_n.

Definition 3. We say that the selection f from S_n is *z-permissible* if $z(i, f(i)) = 1$ for each $i \in A(f)$. Denote by $B_p(z)$ the class of all z-permissible selections from S_p, and $\max\{P(f)\colon f \in B_p(z)\}$ by $P_p(z)$. We say that the selection f from S_p is *z-maximal* on S_p if $f \in B_p(z)$ and $P(f) = P_p(z)$.

PROPERTY 6. *For $p < n$ the inequalities $P_p(z) \leqslant P_{p+1}(z) \leqslant 1 + P_p(z)$ hold.*

Proof. $P_p(z) \leqslant P_{p+1}(z)$ follows from $B_p(z) \subset B_{p+1}(z)$. Suppose that $f \in B_{p+1}(z) \setminus B_p(z)$ and $P(f) > 1 + P_p(z)$ and let $A = A(f) \setminus \{p+1, f(p+1)\}$, then $f \mid A \in B_p(z)$ and $P(f \mid A) > P_p(z)$. Thus $P_{p+1}(z) \leqslant 1 + P_p(z)$ holds.

PROPERTY 7. *Let f be a z-maximal selection on S_p and let $i, j \in S_p \setminus A(f)$. Then $z(i, j) = 0$.*

Proof. Supposing that $z(i,j) = 1$ and putting $g(i) = j$, $g(j) = i$, we get $f \cup g \in B_p(z)$ and $P(f \cup g) > P_p(z)$, which is impossible.

3. The purpose of this paper is to find the algorithm for construction of the z-maximal selection from S_n for each z and n. We shall show that we can make it recurrently, constructing for given z-maximal selection f, on S_p a z-maximal selection from S_{p+1} and making it step-by-step until $p+1 = n$.

To begin with, we prove in this section one fundamental theorem.

Definition 4. Let f be a z-permissible selection from S_p. Then we say that the sequence $\langle i_1, f(i_1)\rangle, \langle i_2, f(i_2)\rangle, \ldots, \langle i_s, f(i_s)\rangle$ of ordered pairs is an *(f, z)-path* from i to j if the following conditions are satisfied:

$$i_1 = i, \quad f(i_s) = j, \tag{1}$$

$$\bigwedge_{k \in \{1,2,\ldots,s\}} \; \bigwedge_{l \in \{1,2,\ldots,s\} \setminus \{k\}} \{i_k \neq i_l \neq f(i_k)\}, \tag{2}$$

$$\bigwedge_{k \in \{1,2,\ldots,s-1\}} \{z(f(i_k), i_{k+1}) = 1\}. \tag{3}$$

We write $i \overset{f,z}{\to} j$ iff there exists an (f, z)-path from i to j.

Property 8. *If $i \overset{f,z}{\to} j$, then $j \overset{f,z}{\to} i$.*

Proof. Let $\langle i_1, f(i_1)\rangle, \ldots, \langle i_s, f(i_s)\rangle$ be an (f, z)-path from i to j. Put $j_k = f(i_{s-k+1})$. Then $\langle j_1, f(j_1)\rangle, \ldots, \langle j_s, f(j_s)\rangle$ is an (f, z)-path from j to i.

Theorem 1. *Let $p < n$ and let f be a z-maximal selection on S_p. Then:*

(i) *if there exists a $q \in S_p \setminus A(f)$ such that $z(q, p+1) = 1$, then $h = f \cup g$, where $A(g) = \{q, p+1\}$, $g(q) = p+1$, $g(p+1) = q$, is the z-maximal selection on S_{p+1};*

(ii) *if there exist $i \in A(f)$, $j \in A(f)$, $q \in S_p \setminus A(f)$ and an (f, z)-path $\langle i_1, f(i_1)\rangle, \ldots, \langle i_s, f(i_s)\rangle$ from i to j such that $z(i, p+1) = z(j, q) = 1$, then for $A = \{i_1, i_2, \ldots, i_s\} \cup \{f(i_1), \ldots, f(i_s)\}$ and g defined by the formulae*

$$A(g) = A \cup \{q, p+1\}, \quad g(p+1) = i, \quad g(i) = p+1, \quad g(q) = j,$$

$$g(j) = q, \quad gf(i_k) = i_{k+1}, \quad g(i_{k+1}) = f(i_k), \quad k \in \{1, 2, \ldots, s-1\}$$

$h = (f \mid (A(f) \setminus A)) \cup g$ is the z-maximal selection on S_{p+1};

(iii) *if both the presupposition of* (i) *and of* (ii) *are not fulfilled, then f is a z-maximal selection on S_{p+1}.*

Proof. In both cases (i) and (ii) we have $P(h) = 1 + P(f) = 1 + P_p(z)$, thus the thesis follows from property 6.

In case (iii) the inequality $P(f) \geqslant P(g)$ is obvious for each $g \in B_p(z)$. Let $g \in B_{p+1}(z) \setminus B_p(z)$. We shall show that $P(f) \geqslant P(g)$, defining the one-to-one function from $A(g)$ into $A(f)$.

We have

$$A(f) = \big(A(g) \cap A(f)\big) \cup \big(A(f) \setminus A(g)\big),$$
$$A(g) = \big(A(g) \cap A(f)\big) \cup \big(A(g) \setminus A(f)\big).$$

Now, let $i \in A(g) \setminus A(f)$.

If $i = p+1$, then $j = g(i) \in A(f)$ and $g(i) \in A(g) \cap A(f)$ because for $g(i) \in A(g) \setminus A(f)$ from z-permissibility of g follows that there exists $j \in S_p \setminus \setminus A(f)$ such that $z(j, p+1) = 1$, which contradicts the presuppositions. If $i \neq p+1 \neq g(i)$, then $g(i) \in A(f) \cap A(g)$ follows from the fact that f is z-maximal on S_p. Thus $g(i) \in A(g) \cap A(f)$ for each $i \in A(g) \setminus A(f)$.

Define the sequence $q_1, q_2, \ldots$ by the formulae

$$q_1 = g(i),$$
$$q_2 = f(q_1),$$
$$\ldots\ldots\ldots$$
$$q_{2r+1} = g(q_{2r}),$$
$$q_{2r+2} = f(q_{2r+1}).$$

Thus, if for certain l, q_k is defined for $k \in \{1, 2, \ldots, l\}$ and if $q_l \in A(g) \cap \cap A(f)$, then q_{k+1} may be defined. We shall prove, at first, that there must exist k such that $q_k \notin A(g) \cap A(f)$ and $q_r \in A(g) \cap A(f)$ for $r < k$.

We shall show namely that if $\{q_1, q_2, \ldots, q_s\} \subset A(g) \cap A(f)$, then $q_k \neq q_l$, whenever $k < l \leqslant s$.

It is true for $s = 2$. Suppose that it is true for certain $s-1 \geqslant 2$ and let $\{q_1, \ldots, q_s\} \subset A(g) \cap A(f)$. We have to prove that $q_s \neq q_k$ for $k < s$.

Remark that $q_1, q_2, \ldots$ are of the form

$$q, f(q), (gf)(q), f(gf)(q), \ldots, (gf)^r(q), f(gf)^r(q), \ldots$$

where $q = q_1$.

Let $q_s = (gf)^r(q)$ and $(gf)^r(q) = (gf)^l(q)$ for $l < r$. Then, for $l > 0$, we have $g(gf)^r(q) = f(gf)^{r-1}(q) = f(gf)^{l-1}(q) = g(gf)^l(q)$, which is impossible by virtue of the inductive hypothesis; for $l = 0$ we have $g(gf)^r(q) = f(gf)^{r-1}(q) \in A(g) \cap A(f)$ and $g(gf)^l(q) = i \in A(g) \setminus A(f)$, thus $(gf)^r(q) = (gf)^l(q)$ is impossible also in this case.

Let $q_s = (gf)^r(q)$ and $(gf)^r(q) = f(gf)^l(q)$ for certain $l < r$. Then we have $g(gf)^r(q) = f(gf)^{r-1}(q) = (gf)^{l+1}(q) = g\big(f(gf)^l\big)(q)$. It is impossible for $l < r-1$ by the inductive hypothesis and for $l = r-1$ by the fact that $g(j) \neq j$ for each j.

Now, let $q_s = f(gf)^r(q)$ and $f(gf)^r(q) = f(gf)^l(q)$, where $l < r$; then $f\big(f(gf)^r\big)(q) = (gf)^r(q) = (gf)^l(q) = f\big(f(gf)^l\big)(q)$, which is impossible by the inductive hypothesis.

Suppose, at the end, that $q_s = f(gf)^r(q) = (gf)^l(q)$, $l \leqslant r$. For $l = r$ it is impossible, because $f(j) \neq j$ for each j; for $l < r$ we have $f(f(gf)^r)(q) = (gf)^r(q) = f(gf)^l(q) = f(gf)^l(q)$, which is impossible both for $l = r-1$ because $g(j) \neq j$ for each j as for $l < r-1$ by the inductive hypothesis.

Since $A(g) \cap A(f)$ is a finite set, there must exist an s such that $q_s \notin A(g) \cap A(f)$, $q_k \in A(g) \cap A(f)$ for $k \in \{1, 2, \ldots, s-1\}$. We shall show that there must be $q_s \in A(f) \setminus A(g)$. Suppose that $q_s \in A(g) \setminus A(f)$. Then $q_s = (gf)^r(q)$ for an r; we have proved before that $f(gf)^{r-1}(q) = q_{s-1}$, hence $g(q) = i$, $g(f(gf)^{r-1})(q) = q_s$. The sequence

$$\langle q, f(q)\rangle, \ldots, \langle (gf)^{r-1}(q), f(gf)^{r-1}(q)\rangle$$

is an (f, z)-path from q to q_{s-1}. If $i \neq p+1 \neq q_s$, then, putting $A = A(f) \setminus \{q_1, q_2, \ldots, q_s\}$ and $B = \{i, q_1, \ldots, q_s\}$, we obtain $(f \mid A) \cup (g \mid B) \in B_p(z)$ and $P((f \mid A) \cap (g \mid B)) = 1 + P(f)$, which is impossible, because f is z-maximal on S_p. If $i = p+1$, then

$$\langle q, f(q)\rangle, \ldots, \langle (gf)^{r-1}(q), f(gf)^{r-1}(q)\rangle$$

is the (f, z)-path and $z(p+1, q) = z(f(gf)^{r-1}(q), (gf)^r(q)) = 1$, which contradicts the presupposition of part (iii) of the theorem.

Put, for each $i \in A(g) \setminus A(f)$, $q_1(i) = g(i)$, $q_{2r+1}(i) = g(q_{2r}(i))$, $q_{2r+2}(i) = f(q_{2r+1}(i))$; we can define all $q_k(i)$ for $k \leqslant s(i)$, where

$$s(i) = \max\{k : (l < k) \Rightarrow q_l(i) \in A(g) \cap A(f)\};$$

we have proved that $q_{s(i)}(i) \in A(f) \setminus A(g)$. Thus the function $h_1(i) = q_{s(i)}(i)$ is defined on whole set $A(g) \setminus A(f)$ and transforms it into $A(f) \setminus A(g)$. We shall show that h_1 is one-to-one.

Let $i \in A(g) \setminus A(f)$ $j \in A(g) \setminus A(f)$, $i \neq j$. If $s(i) = s(j)$, then there exists an r such that $q_{s(i)}(i) = f(gf)^r(q_1(i))$, $q_{s(j)}(j) = f(gf)^r(q_1(j))$ and, since $q_1(i) \neq q_1(j)$, $h_1(i) \neq h_1(j)$; it results from the fact that f, g are one-to-one. If $s(i) < s(j)$, then $h_1(i) = f(gf)^{r(i)}(q_1(i))$, $h_1(j) = f(gf)^{r(j)}(q_1(j))$,

$$r(i) = \frac{s(i)-1}{2} < \frac{s(j)-1}{2} = r(j)$$

and

$$(fg)^{r(i)+1}(h_1(i)) = i \in A(g) \setminus A(f),$$

$$(fg)^{r(i)+1}(h_1(j)) = (f(gf)^{r(j)-r(i)-1})(q_1(j)) \in A(g) \cap A(f)$$

and, since f, g are one-to-one, we have $h_1(i) \neq h_1(j)$. Thus h_1 is one-to-one. Put $h_2(i) = i$ for each $i \in A(g) \cap A(f)$. Then $h = h_1 \cup h_2$ is one-

-to-one and transforms $A(g)$ into $A(f)$. Hence $P(f) \geqslant P(g)$, which completes the proof.

4. Let for a given z-maximal selection f on S_p, $p < n$,

$$Q = \{q\colon (q \in A(f)) \wedge (z(q, p+1) = 1)\},$$
$$R = \{r \in A(f)\colon \bigvee_{l \in S_p \setminus A(f)} (z(r, l) = 1)\}.$$

The two theorems of this section shall enable us to construct the (f, z)-path from certain $q \in Q$ to certain $r \in R$ if such (f, z)-path exists. The algorithm of this construction is, of course, an essential part of the algorithm of construction of the z-maximal selection on S_n.

We shall denote by $nr(i, f(i))$ a function, defined on the set of all ordered pairs $\langle i, f(i)\rangle$, where $i \in A(f)$, such that

$$nr\colon f \to \{1, 2, \ldots, 2P(f)\}$$

and $nr(i, f(i)) \leqslant P(f)$ for $i < f(i)$ and $nr(i, f(i)) = P(f) + nr(f(i), i)$ for $f(i) < i$. For $nr(i, f(i)) = k$ we denote $nr(f(i), i)$ by k'. Put $u_{ik} = (1-\delta_{ik}) \times \times (1-\delta_{ik'})$, where δ_{ik} is the Kronecker delta symbol, put also $z_{ik} = u_{ik} z(f(j), l)$, where $nr(j, f(j)) = i$, $nr(l, f(l)) = k$. Denote at the end by $c_{it}^{(s)}$ the function, defined on $\{1, 2, \ldots, 2P(f)\} \times \{1, 2, \ldots, s\}$, valued in $\{0, 1\}$ and such that $c_{it}^{(s)} = 1$ iff there exists an (f, z)-path

$$\langle q_1, f(q_1)\rangle, \langle q_2, f(q_2)\rangle, \ldots, \langle q_s, f(q_s)\rangle$$

with $q_1 \in Q$ and $nr(q_t, f(q_t)) = i$. There obviously holds

PROPERTY 9. *The sequence*

$$\langle k_1, f(k_1)\rangle, \langle k_2, f(k_2)\rangle, \ldots, \langle k_s, f(k_s)\rangle, \quad k_1 \in Q,$$

of ordered pairs from f is the (f, z)-path iff

$$\Big(\prod_{p=1}^{s} c_{i_p p}^{(s)}\Big)\Big(\prod_{1 \leqslant q < r \leqslant s} u_{i_q i_r}\Big)\Big(\prod_{p=1}^{s-1} z_{i_p i_{p+1}}\Big) = 1,$$

where $i_p = nr(k_p, f(k_p))$.

For real numbers a, b we shall denote $\max\{a, b\}$ by $a +' b$ and for the set $\{a_1, a_2, \ldots, a_n\}$ of real numbers we shall denote

$$\max\{a_1, a_2, \ldots, a_n\} \text{ by } \sum_{i=1}^{n}{}' a_i.$$

Let $A(i_1, i_2, \ldots, i_{s+1})$ be a real function defined on the set $\{1, 2, \ldots, 2P(f)\}^{s+1}$. Then we denote the maximal value of A on the set

$$\{1, 2, \ldots, 2P(f)\}^{t-1} \times \{i\} \times \{1, 2, \ldots, 2P(f)\}^{s-t+1}$$

by

$$\sum_{i_t=i}' A(i_1, i_2, \ldots, i_{s+1}).$$

PROPERTY 10. *We have*

$$c_{i1}^{(1)} = \sum_{\substack{j=nr(q,f(q))\\ q\in Q}}' \delta_{ij}$$

and

$$(1) \qquad c_{it}^{(s+1)} = \sum_{i_t=i}' \Big(\Big(\prod_{1\leqslant q<r\leqslant s+1} u_{i_q i_r}\Big)\Big(\prod_{p=1}^{s} c_{i_p p}^{(s)} z_{i_p i_{p+1}}\Big)\Big).$$

It easily follows from property 9. The intermediate application of property 10 is not possible in interesting practical cases; for $s+1 = 50$, $2P(f) = 100$ the expression for $c_{it}^{(s+1)}$ in property 10 is the maximum of 10^{100} products of the form, like that in (1). However, we can essentially simplify the procedure described in property 10, in the way, which we shall now describe.

Let $C^{(s)} = \|c_{it}^{(s)}\|$, $Z = \|z_{ik}\|$, $U = \|u_{ik}\|$, $V = \|v_{ik}\|$, where $v_{ik} = 1-u_{ik}$, $i, k\in\{1, 2, \ldots, 2P(f)\}$, $t\in\{1, 2, \ldots, s\}$. For $(m\times n)$-matrix $A = \|a_{ik}\|$ write

$$A_t = \begin{Vmatrix} a_{1t} \\ \vdots \\ a_{mt} \end{Vmatrix}, \qquad A^t = \begin{Vmatrix} a_{t1} \\ \vdots \\ a_{tn} \end{Vmatrix}$$

and

$$A_{tr} = \begin{Vmatrix} a_{tr} \\ \vdots \\ a_{tr} \end{Vmatrix}, \qquad \text{where } A_{tr} \text{ is } (m\times 1)\text{-matrix}.$$

If $A = \|a_{ik}\|$ and $B = \|b_{ik}\|$ are $(m\times n)$-matrices, then we define the $(m\times n)$-matrices $A \wedge B$ and $A+' B$ by the formulae

$$A \wedge B \underset{\mathrm{df}}{=} \|a_{ik}\cdot b_{ik}\|, \qquad A+' B \underset{\mathrm{df}}{=} \|a_{ik}+' b_{ik}\|.$$

For the matrices denoted by capitals, we shall denote their elements by corresponding lower case letters, e.g. a_{ik} is an element of the matrix A. The elements of a sequence of matrices will be denoted by capitals with upper indices in parenthesis, e.g. $C^{(s)}$; the elements of such matrices will be denoted by corresponding lower case letters with upper indices in parenthesis, e.g. $c_{ik}^{(s)}$ is an element of the matrix $C^{(s)}$.

We can, under these conventions, formulate theorem 2 in the following way:

THEOREM 2. *Let for given $C^{(s)}$ be*

$$B_{s+1}^{(0)} = \sum_{l=1}^{2P(f)}{}' (C_{ls}^{(s)} \wedge Z^l),$$

$$B_{q-1}^{(0)} = \sum_{l=1}^{2P(f)}{}' (B_{lq}^{(0)} \wedge Z^l) \quad \textit{for } 1 < q \leqslant s+1,$$

$$E_1^{(0)} = B_1^{(0)},$$

$$E_{q+1}^{(0)} = \sum_{l=1}^{2P(f)}{}' (E_{lq}^{(0)} \wedge Z^l) \quad \textit{for } 1 \leqslant q \leqslant s,$$

$$A^{(0)} = B^{(0)} \wedge E^{(0)};$$

now, let

$$W = \{w \leqslant P(f)\colon |\{t\colon a_{wt}^{(0)} +' a_{w't}^{(0)} = 1\}| > 1\}$$

and, for non-empty $W = \{w_1, w_2, \ldots, w_M\}$ let

$$T_j = \{t\colon a_{w_j t}^{(0)} +' a_{w'_j t}^{(0)} = 1\} = \{t_{j1}, t_{j2}, \ldots, t_{jr_j}\}, \quad \textit{where } j \in \{1, 2, \ldots, M\};$$

let $\tau_1, \tau_2, \ldots, \tau_K$, where $K = r_1 \cdot r_2 \ldots r_M$, be all the sequences of the form $\tau_j = \langle t_{1k_1}, t_{2k_2}, \ldots, t_{Mk_M}\rangle$.

Finally, let for $\tau_j = \langle t_1, t_2, \ldots, t_M\rangle$ be

$$U_q^{(j)} = (U_{w_1} \wedge U_{w_2} \wedge \ldots \wedge U_{w_M}) + \sum_{l=1}^{M}{}' (\Delta_{qt_l} \wedge V_{w_l}),$$

$$B_q^{(j)} = A_q^{(0)} \wedge U_q^{(j)} \text{ for } 1 \leqslant q \leqslant s+1,$$

$$D_{s+1}^{(j)} = B_{s+1}^{(j)},$$

$$D_{q-1}^{(j)} = \sum_{l=1}^{2P(f)}{}' (D_{lq}^{(j)} \wedge Z^l) \quad \textit{for } 1 < q \leqslant s+1$$

$$E_1^{(j)} = D_1^{(j)}$$

$$E_{q+1}^{(j)} = \sum_{l=1}^{2P(f)} (E_q^{(j)} \wedge Z^l) \quad \text{for } 1 \leqslant q \leqslant s$$

and

$$A^{(j)} = E^{(j)} \wedge D^{(j)}.$$

Then $C^{(s+1)} = \sum_{j=1}^{K}{}' A^{(j)}$ for $W \neq \emptyset$ and $C^{(s+1)} = A^{(0)}$ for $W = \emptyset$.

Proof. It is easy to see that

$$a_{ik}^{(0)} = \sum_{i=i_k}' \Big(\prod_{p=1}^{s} c_{i_p p}^{(s)} z_{i_p i_{p+1}}\Big) \quad \text{and} \quad a_{ik}^{(j)} = \sum_{i=i_k}' \Big(\prod_{p=1}^{s} c_{i_p p}^{(s)} z_{i_p i_{p+1}}\Big)\Big(\prod_{p=1}^{s+1} u_{i_p p}^{(j)}\Big).$$

Suppose that W is non-empty. We shall show that

$$\sum_{j=1}^{K}{}' \prod_{p=1}^{s+1} u_{i_p p}^{(j)} = \prod_{1 \leqslant q < r \leqslant s+1} u_{i_q i_r}$$

for given sequence $\langle i_1, i_2, \ldots, i_{s+1}\rangle$.

The equality $u_{i_q i_r} = 0$, $q < r$, holds iff $\{i_q, i_r\} \subset \{w_m, w_m'\}$ for certain m. If $\tau_j = \langle t_1, t_2, \ldots, t_M\rangle$, then from $q = t_m$ it follows that $r \neq t_m$ and

$$u_{i_q q}^{(j)} u_{i_r r}^{(j)} = 0 = \prod_{p=1}^{s+1} u_{i_p p}^{(j)};$$

obviously,

$$u_{i_q q}^{(j)} u_{i_r r}^{(j)} = 0 = \prod_{p=1}^{s+1} u_{i_p p}^{(j)}$$

also in case where $q \neq t_m \neq r$.

From the equality

$$\prod_{1 \leqslant q < r \leqslant s+1} u_{i_q i_r} = 1,$$

on the other hand, it follows that if $i_q \in \{w_m, w_m'\}$, then $i_r \notin \{w_m, w_m'\}$ for $q \neq r$; hence, for each m, $\{w_m, w_m'\} \cap \{i_1, i_2, \ldots, i_{s+1}\} = \{i_{p_m}\}$ for certain $p_m \in T_m$ and, putting $\tau_j = \langle p_1, p_2, \ldots, p_M\rangle$, we have

$$\prod_{p=1}^{s+1} u_{i_p p}^{(j)} = 1.$$

Thus we have

$$\prod_{1 \leqslant q < r \leqslant s+1} u_{i_q i_r} = \sum_{j=1}^{K}{}' \prod_{p=1}^{s+1} u_{i_p p}^{(j)} \quad \text{and} \quad c_{ik}^{(s+1)} = \sum_{j=1}^{K}{}' a_{ik}^{(j)},$$

which completes the proof.

The (f, z)-path $\langle i_1, f(i_1)\rangle, \ldots, \langle i_{s+1}, f(i_{s+1})\rangle$ with $i_1 \in Q$ and given $f(i_{s+1}) \in R$ we can construct according to the following

THEOREM 3. *Let, for given* $C^{(s+1)}$, $c_{k_{s+1}s+1}^{(s+1)} = 1$, *where* $k_{s+1} = nr(i_{s+1}, f(i_{s+1}))$ *for* $f(i_{s+1}) \in R$[1]; *let* $c_{k_{s+1}s+1}^{(s+1)} = a_{k_{s+1}s+1}^{(j)}$. *Let, finally, for* $1 < q \leqslant s+1$ *be* $k_{q-1} = \min\{k\colon a_{k,q-1}^{(j)} z_{k,k_q} = 1\}$ *(such a k must exist).*

---

[1] We use here notation of theorem 2 according to which $C_{k_{s+1}s+1}^{(s+1)} = a_{k_{s+1}s+1}^{(j)}$ must hold for certain $j \geqslant 0$.

Then

$$\prod_{p=1}^{s} a^{(j)}_{k_p p} z_{k_p k_{p+1}} = 1$$

and the sequence

$$\langle i_1, f(i_1)\rangle, \langle i_2, f(i_2)\rangle, \ldots, \langle i_{s+1}, f(i_{s+1})\rangle,$$

where $k_p = nr(i_p, f(i_p))$, *is an* (f, z)*-path with* $i_1 \in Q$.

Proof. If W in theorem 2 is empty, then $C^{(s)} = A^{(0)}$. If $W \neq \emptyset$, then there must be $c^{(s+1)}_{k_{s+1}s+1} = a^{(j)}_{k_{s+1}s+1}$ for a $j > 0$. In both these cases we have

$$\prod_{p=1}^{s} a^{(j)}_{k_p p} \geqslant \prod_{1 \leqslant p < q \leqslant s+1} u_{k_p k_q}.$$

By the definition, there must also be

$$\prod_{p=1}^{s} a^{(j)}_{k_p p} = 1.$$

Then, by virtue of the inequality proved before, there holds

$$c^{(s+1)}_{k_{s+1}s+1} \Big(\prod_{p=1}^{s} c^{(s+1)}_{k_p p} z_{k_p k_{p+1}}\Big)\Big(\prod_{1 \leqslant q < r \leqslant s+1} u_{k_q k_r}\Big) = 1,$$

which completes the proof.

5. Let $r = \min\{t\colon \bigvee_{q<t} z(q, t) = 1\}$; then, for $A(f) = \{q, r\}$, $f(q) = r$, $f(r) = q, f$ is a z-maximal selection on S_r.

For $p+1 \leqslant n$ and for given z-maximal selection f, on S_p we can using theorems 1-3 construct the z-maximal selection on S_{p+1}, as follows.

A. Let f be a z-maximal selection on S_p. If the presuppositions of part (i) of theorem 1 are fulfilled, then we define the z-maximal selection g, on S_{p+1}, like in part (i) of theorem 1. If these presuppositions are not fulfilled, then we go to the point B.

B. We verify determining the matrices $C^{(1)}, C^{(2)}, \ldots,$ according to theorem 2 the presuppositions of part (ii) of theorem 1. If they are fulfilled, then we construct according to theorem 3 an (f, z)-path from certain $q \in Q$ to certain $r \in R$ and define the z-maximal selection g, on S_{p+1} as in part (ii) of theorem 2. If the presuppositions of part (ii) of theorem 1 are not fulfilled, then f is a z-maximal selection on S_{p+1}.

6. To illustrate the presented method, consider the following example. Let $n = 12$. Define the function $\varphi: S_n \to E^2$, where E^2 is the cartesian plane, by the table (see also fig. 1).

i	1	2	3	4	5	6	7	8	9	10	11	12
$x_{\varphi(i)}$	5	8	11	6	2	4	9	8	10	8	9	0
$y_{\varphi(i)}$	2	3	7	4	0	1	7	7	5	5	1	0

where $x_{\varphi(i)}, y_{\varphi(i)}$, are the abscissa and the ordinate, respectively, of the point $\varphi(i)$.

Let $z(i,j) = 1$ iff $i \neq j$ and

$$\max\{|x_{\varphi(i)} - x_{\varphi(j)}|, |y_{\varphi(i)} - y_{\varphi(j)}|\} \leqslant 2.$$

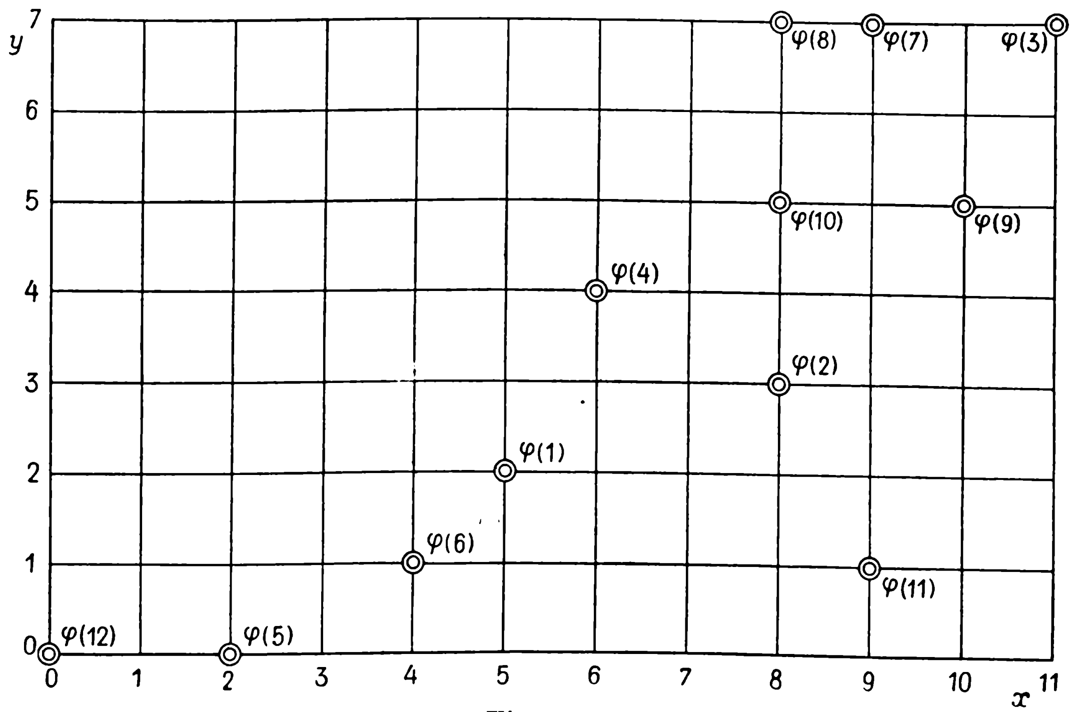

Fig. 1.

Thus we have

$$z(1,4) = z(4,1) = z(1,6) = z(6,1) = z(2,4) = z(4,2) = z(2,9) = z(9,2) = z(2,10) = z(10,2) = z(2,11) = z(11,2) = z(3,7) = z(7,3) = z(3,9) = z(9,3) = z(4,10) = z(10,4) = z(5,6) = z(6,5) = z(5,12) = z(12,5) = z(7,8) = z(8,7) = z(7,9) = z(9,7) = z(7,10) = z(10,7) = z(8,9) = z(9,8) = z(8,10) = z(10,8) = z(9,10) = z(10,9) = 1,$$

and $z(i,j) = 0$ for any other pair $\langle i,j\rangle \in S_n \times S_n$.

Now, we shall define step-by-step for $p \in \{1, 2, \ldots, 12\}$ the z-maximal selections f_p from S_p.

By virtue of the definition of z we have $B_1(z) = B_2(z) = B_3(z) = \emptyset$ and $f_1 = f_2 = f_3 = \emptyset$ (an empty subset of $S_n \times S_n$).

For $p = 3$ we have $z(4, 1) = 1$, hence $A(f_4) = \{1, 4\}$, $f_4(1) = 4$, $f_4(4) = 1$ by virtue of theorem 1.

There is $z(i, 5) = 0$ for each $i < 5$; thus we have $f_4 = f_5$, because both the presupposition of part (i) of theorem 1 and that of part (ii) are not fulfilled (theorem 1, (iii)).

We have $S_5 \setminus A(f_5) = \{2, 3, 5\}$ and there is $z(2, 6) = z(3, 6) = 0$, but $z(5, 6) = 1$. Hence, for $A(f_6) = \{1, 4, 5, 6\}$, $f_6(1) = 4$, $f_6(4) = 1$, $f_6(5) = 6$, $f_6(6) = 5$, f_6 is the z-maximal selection from S_6 (theorem 1, (i)).

By a similar procedure we obtain the z-maximal selection f_7 from S_7, defined by the formulae $A(f_7) = \{1, 3, 4, 5, 6, 7\}$, $f_7(1) = 4$, $f_7(3) = 7$, $f_7(4) = 1$, $f_7(5) = 6$, $f_7(6) = 5$, $f_7(7) = 3$.

Now we shall try to construct the z-maximal selection f_8 from S_8. We have $S_7 \setminus A(f_7) = \{2\}$ and $z(2, 8) = 0$. The presupposition of part (i) of theorem 1 is not fulfilled. On the other hand, we have $z(8, 7) = 1$, $7 \in A(f_7)$ and so we must verify the presupposition of theorem 1, part (ii). For this reason define the function nr for f_7 (denote this function by nr_7). We obtain

$$nr_7(1, 4) = 1, \quad nr_7(5, 6) = 2, \quad nr_7(3, 7) = 3,$$
$$nr_7(4, 1) = 4, \quad nr_7(6, 5) = 5, \quad nr_7(7, 3) = 6.$$

The matrix Z is of the form

$$Z = \left\| \begin{array}{ccc:ccc} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ \hdashline 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{array} \right\|.$$

Following the formulae in theorem 2 we obtain

$$C^{(1)} = \left\| \begin{array}{c} 0\\0\\0\\ \hdashline 0\\0\\1 \end{array} \right\|, \quad B^{(0)} = \left\| \begin{array}{cc} 0&0\\0&0\\0&0\\ \hdashline 0&0\\0&0\\1&0 \end{array} \right\|, \quad A^{(0)} = C^{(2)} = 0.$$

We see that there is no (f_7, z)-path from 7 (we have $Q = \{7\}$ for S_7) to the points distinct from 3 and that $z(3, 2) = 0$. Thus the presuppositions of part (i) and (ii) of theorem 1 are not fulfilled and $f_8 = f_7$ is the z-maximal selection from S_8.

By similar arguments as for $S_p, p \leqslant 8$, we obtain $A(f_9) = \{1, 2, 3, 4, 5, 6, 7, 9\}$, $f_9(1) = 4, f_9(2) = 9, f_9(3) = 7, f_9(4) = 1, f_9(5) = 6, f_9(6) = 5, f_9(7) = 3, f_9(9) = 2$; $A(f_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, f_{10} \mid A(f_9) = f_9$, $f_{10}(8) = 10$, $f_{10}(10) = 8$; $f_{11} = f_{10}$.

For f_{11} the presupposition of theorem 1, part (i), is not fulfilled and we must check once more the presuppositions of theorem 1, part (ii).

We have $z(12, 5) = 1$, $5 \in A(f_{11})$, $nr_{11}(1, 4) = 1$, $nr_{11}(2, 9) = 2$, $nr_{11}(3, 7) = 3$, $nr_{11}(5, 6) = 4$, $nr_{11}(8, 10) = 5$, $nr_{11}(4, 1) = 6$, $nr_{11}(9, 2) = 7$, $nr_{11}(7, 3) = 8$, $nr_{11}(6, 5) = 9$, $nr_{11}(10, 8) = 10$.

The matrix Z is of the form

$$Z = \left\| \begin{array}{c:c} 01000 & 00001 \\ 00101 & 00101 \\ 00001 & 01001 \\ 10000 & 00000 \\ 01000 & 11100 \\ \hdashline 00000 & 00010 \\ 00000 & 10001 \\ 00000 & 01000 \\ 00000 & 00000 \\ 00000 & 01100 \end{array} \right\|;$$

we have, too,

$$C^{(1)} = \left\| \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\|, \quad C^{(2)} = \left\| \begin{array}{c:c} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right\|,$$

and, for $s+1 = 3$,

$$C^{(3)} = A^{(0)} = B^{(0)} = \left\| \begin{array}{c} 010 \\ 001 \\ 000 \\ 100 \\ 000 \\ \hdashline 000 \\ 000 \\ 000 \\ 000 \\ 001 \end{array} \right\|.$$

For $s+1 = 4$ there is

$$A^{(0)} = B^{(0)} = \left\|\begin{matrix} 0100 \\ 0010 \\ 0001 \\ 1000 \\ 0001 \\ \cdots\cdots \\ 0000 \\ 0001 \\ 0001 \\ 0000 \\ 0011 \end{matrix}\right\|;$$

the set W is non-empty in this case, namely $W = \{2, 5\}$, $w_1 = 2$, $w_2 = 5$, $T_1 = T_2 = \{3, 4\}$.

All τ-sequences are

$$\tau_1 = \langle 3, 3\rangle, \ \ \tau_2 = \langle 3, 4\rangle, \ \ \tau_3 = \langle 4, 3\rangle, \ \ \tau_4 = \langle 4, 4\rangle.$$

Hence $A^{(1)} = 0$,

$$A^{(2)} = \left\|\begin{matrix} 0100 \\ 0000 \\ 0000 \\ 1000 \\ 0000 \\ 0000 \\ 0001 \\ 0001 \\ 0000 \\ 0010 \end{matrix}\right\|, \quad A^{(3)} = \left\|\begin{matrix} 0100 \\ 0010 \\ 0001 \\ 1000 \\ 0001 \\ \cdots\cdots \\ 0000 \\ 0000 \\ 0001 \\ 0000 \\ 0001 \end{matrix}\right\|, \quad A^{(4)} = \left\|\begin{matrix} 0100 \\ 0010 \\ 0001 \\ 1000 \\ 0000 \\ 0000 \\ 0000 \\ 0001 \\ 0000 \\ 0010 \end{matrix}\right\|, \quad C^{(4)} = \left\|\begin{matrix} 0100 \\ 0010 \\ 0001 \\ 1000 \\ 0001 \\ \cdots\cdots \\ 0000 \\ 0001 \\ 0001 \\ 0000 \\ 0011 \end{matrix}\right\|,$$

and so we have $C^{(4)} = A^{(0)}$, but it was necessary to verify this fact.

There is, for $s+1 = 5$,

$$A^{(0)} = B^{(0)} = \left\|\begin{matrix} 01000 \\ 00101 \\ 00010 \\ 10000 \\ 00011 \\ \cdots\cdots \\ 00001 \\ 00011 \\ 00011 \\ 00000 \\ 00111 \end{matrix}\right\|,$$

and it is easy to see, that $W = \{1, 2, 3, 5\}$, $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 5$ and $T_1 = \{2, 5\}$, $T_2 = \{3, 4, 5\}$, $T_3 = \{4, 5\}$, $T_4 = \{3, 4, 5\}$.

Hence, we have 36 τ-sequences, namely

$$\begin{array}{lll}
\tau_1 = \langle 2,3,4,3\rangle, & \tau_2 = \langle 2,3,4,4\rangle, & \tau_3 = \langle 2,3,4,5\rangle, \\
\tau_4 = \langle 2,3,5,3\rangle, & \tau_5 = \langle 2,3,5,4\rangle, & \tau_6 = \langle 2,3,5,5\rangle, \\
\tau_7 = \langle 2,4,4,3\rangle, & \tau_8 = \langle 2,4,4,4\rangle, & \tau_9 = \langle 2,4,4,5\rangle, \\
\tau_{10} = \langle 2,4,5,3\rangle, & \tau_{11} = \langle 2,4,5,4\rangle, & \tau_{12} = \langle 2,4,5,5\rangle, \\
\tau_{13} = \langle 2,5,4,3\rangle, & \tau_{14} = \langle 2,5,4,4\rangle, & \tau_{15} = \langle 2,5,4,5\rangle, \\
\tau_{16} = \langle 2,5,5,3\rangle, & \tau_{17} = \langle 2,5,5,4\rangle, & \tau_{18} = \langle 2,5,5,5\rangle, \\
\tau_{19} = \langle 5,3,4,3\rangle, & \tau_{20} = \langle 5,3,4,4\rangle, & \tau_{21} = \langle 5,3,4,5\rangle, \\
\tau_{22} = \langle 5,3,5,3\rangle, & \tau_{23} = \langle 5,3,5,4\rangle, & \tau_{24} = \langle 5,3,5,5\rangle, \\
\tau_{25} = \langle 5,4,4,3\rangle, & \tau_{26} = \langle 5,4,4,4\rangle, & \tau_{27} = \langle 5,4,4,5\rangle, \\
\tau_{28} = \langle 5,4,5,3\rangle, & \tau_{29} = \langle 5,4,5,4\rangle, & \tau_{30} = \langle 5,4,5,5\rangle, \\
\tau_{31} = \langle 5,5,4,3\rangle, & \tau_{32} = \langle 5,5,4,4\rangle, & \tau_{33} = \langle 5,5,4,5\rangle, \\
\tau_{34} = \langle 5,5,5,3\rangle, & \tau_{35} = \langle 5,5,5,4\rangle, & \tau_{36} = \langle 5,5,5,5\rangle.
\end{array}$$

From the form of τ_j we can conclude, that there are $a^{(j)}_{7,5} = 0$ for $j < 13$ and $A^{(j)} = 0$ for $j \geqslant 19$. For $s > 5$ there is no (f_{11}, z)-path of length s. Hence, $f_{12} = f_{11}$ iff $a^{(j)}_{7,5} = 0$ for $13 \leqslant j \leqslant 18$. For $j = 13$ we obtain

$$A^{(13)} = \left\| \begin{array}{c} 01000 \\ 00000 \\ 00000 \\ 10000 \\ 00000 \\ \hdashline 00000 \\ 00001 \\ 00010 \\ 00000 \\ 00100 \end{array} \right\| .$$

There are $a^{(13)}_{7,5} = 1$, $a^{(13)}_{8,4} = z_{8,7} = a^{(13)}_{10,3} = z_{10,8} = a^{(13)}_{1,2} = z_{1,10} = a^{(13)}_{4,1} = z_{4,1} = 1$. Since $4 = nr_{11}(5, 6)$, $1 = nr_{11}(1, 4)$, $10 = nr_{11}(10, 8)$, $8 = nr_{11}(7, 3)$, $7 = nr_{11}(9, 2)$, the sequence $\langle 5, 6\rangle$, $\langle 1, 4\rangle$, $\langle 10, 8\rangle$, $\langle 7, 3\rangle$, $\langle 9, 2\rangle$ is an (f_{11}, z)-path. Moreover, $z(12, 5) = z(2, 11) = 1$ and the selection f_{12} z-maximal on S_{12} is defined by the formulae $A(f_{12}) = S_{12}$, $f_{12}(1) = 6$, $f_{12}(2) = 11$, $f_{12}(3) = 9$, $f_{12}(4) = 10$, $f_{12}(5) = 12$, $f_{12}(6) = 1$, $f_{12}(7) = 8$, $f_{12}(8) = 7$, $f_{12}(9) = 3$, $f_{12}(10) = 4$, $f_{12}(11) = 2$, $f_{12}(12) = 5$.

7. Ermolev and Melnik present in [2] a solution of the following problem:

There is given a graph G with the vertices $a_1, a_2, \ldots, a_n$ and a set of links. For each link there is given its length, say $d(a_i, a_j)$, for the link $\{a_i, a_j\}$. We say that the sequence $\varkappa = \langle k_1, k_2, \ldots, k_s\rangle$ is a path in a graph G iff for each $i \in \{1, 2, \ldots, s-1\}$ there is $k_i = \{a_l, a_m\}$, $k_{i+1} = \{a_m, a_t\}$. For the path $a = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_s}, a_{i_{s+1}}\rangle$ we define its length as

$$d(a) = \sum_{j=1}^{s} d(a_{i_j}, a_{i_{j+1}}).$$

For a given class C of paths in G the problem consists in finding a path $\varkappa$ such that $d(\varkappa) = \min\{d(a): a \in C\}$.

Let $d(a_i, a_j) = 1$ for each link and define the one-to-one function φ transforming the set W of nodes of G onto itself such that $\varphi^{-1} = \varphi$. Let Q, R be subsets of W and let $a \in C$ iff $a = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_s}, a_{i_{s+1}}\rangle$, $a_{i_1} \in Q$, $a_{i_{s+1}} \in R$ and $i_l \neq i_m \neq \varphi(i_l)$, whenever $l \neq m$.

Then theorems 2 and 3 of this paper provide another method, which seems to be more efficient too, for solving the problem of Ermolev and Melnik.

8. Note added in proof. After the paper has been written, the author noticed that part (iii) of Theorem 1 of the paper may be obtained as a simple corollary from the theorem of Berge, Norman and Rabin ([1], p. 175). For this purpose, it is sufficient to build the non-oriented graph $G = (X, U)$, where the set X of vertices of G is the series in question (say S_{p+1}) and the set U of edges of G is the family of all $\{i, j\} \subseteq X$ satisfying the equality $z(i, j) = 1$, and to put $f(i) = \max\{z(i, j): j \in X \setminus \{i\}\}$ for each $i \in X$ (f is to be understood in the sense of [1]). It is easily seen that the notion of a (f, z)-path from this paper may be reduced to that of an alternating chain from the theory of graphs, and that for a given z-permissible selection g from X the set $\{\{i, g(i)\}: i \in A(g)\}$ is a compatible set in the sense of [1] if f is defined as above.

References

[1] C. Berge, *The theory of graphs and its applications,* New York and London 1964.

[2] Ю. М. Ермолев и И. М. Мельник, *Экстремальные задачи на графах,* Киев 1968.

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O DOBORACH PAR

STRESZCZENIE

W pracy podane jest rozwiązanie następującego problemu: dla danego n-elementowego zbioru S_n i funkcji $z: S_n \times S_n \to \{0, 1\}$, spełniającej warunek $z(i,j) = z(j,i)$ dla dowolnych i, j z S_n znaleźć rodzinę $\{B_k\}, k \in \{1, \dots, s\}$ dwu-elementowych zbiorów rozłącznych taką, że

1. $\bigcup_{k \in \{1,\dots,s\}} B_k \subseteq S_n$,
2. dla każdego k, $B_k = \{t, u\}$ pociąga $z(t, u) = 1$ oraz
3. nie istnieje dla $r > s$ taka rodzina $\{B'_k\}, k \in \{1, \dots, r\}$, by własności 1 i 2 pozostawały prawdziwe po podstawieniu r za s i $\{B'_k\}$ za $\{B_k\}$.

Przedstawione zostało twierdzenie pozwalające sprawdzić, czy rodzina $\{B_k\}$, $k \in \{1, \dots, s\}$, spełnia warunki 1, 2 i 3 bez porównywania jej z innymi rodzinami oraz dwa twierdzenia, w oparciu o które skonstruować można rodzinę $\{B_k\}, k \in \{1, \dots, s\}$, o żądanych własnościach. Sugerowany przez te twierdzenia algorytm nadaje się do zaprogramowania na maszynę matematyczną.

Rezultaty pracy mogą mieć zastosowanie w produkcji seryjnej niektórych porównawczych układów pomiarowych itd.

Е. БЛАХУТ (Гливицэ)

ОБ ОТБОРАХ ПАР

РЕЗЮМЕ

В работе дано решение следующей проблемы: для произвольного n-элементного множества S_n и функции $z: S_n \times S_n \to \{0, 1\}$ такой, что $z(i,j) = z(j,i)$ для $i, j \in S_n$ найти семейство $\{B_k\}, k \in \{1, \dots, s\}$ непересекающихся двухэлементных множеств, такое, что:

1. $\bigcup_{k \in \{1,\dots,s\}} B_k \subseteq S_n$;
2. для произвольного k из $B_k = \{t, u\}$ вытекает $z(t, u) = 1$;
3. если $r > s$ и $\{B'_k\}, k \in \{1, \dots, r\}$ — семейство непересекающихся двухэлементных множеств, такое, что $\bigcup_{k \in \{1,\dots,r\}} B'_k \subset S_n$ то $\{B'_k\}$ не удовлетворяет по меньшей мере одному из условий 1, 2.

Предлагается теорема, позволяющая для $\{B_k\}, k \in \{1, \dots, s\}$ проверить, обладает-ли семейство $\{B_k\}, k \in \{1, \dots, s\}$ свойствами 1, 2, 3, без сравнивания $\{B_k\}$ с другими семействами подмножеств S_n. Даются две теоремы, с помощью которых можно конструировать семейство $\{B_k\}, k \in \{1, \dots, s\}$, обладающее требуемыми свойствами; подсказываемый этими теоремами алгоритм пригоден для вычислений на ЭЦВМ.

Результаты полученные в работе могут иметь применение в массовом производстве некоторых устройств для сравнительных измерений и т.п.