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An axiomatics of non-Desarguean geometry based on the half-plane as the primitive notion

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Introduction

The present paper is divided into two parts. Part I contains a construction, on the ground of a Boolean algebra augmented with some additional axioms, of a non-Desarguean geometry, i.e., of a geometry whose primitive notions are: the point, the straight line to which I shall refer as the line for the sake of brevity, the incidence relation of points and lines, and the relation of betweenness which concerns triples of points.

It should be stressed that this idea is not new. Jaśkowski [3] and Tarski [5] studied the geometry of bodies, i.e., a geometry whose primitive notion is the body. In particular, Jaśkowski gave an axiomatics of the affine geometry which was based on the Boolean algebra of regions and on the notion of half-space. In fact, Jaśkowski added to the axiomatics of Boolean algebra some axioms with a geometrical meaning and obtained a theory which can be used to construct a model of affine geometry.

Jaśkowski supplemented the axioms of the Boolean algebra, which are explained in Section 1 of Part I, with the axioms A. I, A. II, A. III, and A. IV cited below in the original version (1).

Let B denote the collection of regions which are generated by half-spaces. Let P denote the collection of half-spaces.

A. I. $a \in P \to a^{-} \in P$.

A. II. $x,\ y,\ z,\ t \in B \wedge x,\ y,\ z,\ t \neq 0 \to \bigvee_{a \in P}\{(x\cdot a,\ x\cdot a^{-},\ y\cdot a,\ y\cdot a^{-},\ z\cdot a\ ,\ z\cdot a^{-},\ t\cdot a,\ t\cdot a^{-} \neq 0) \vee (x+y+z)\cdot a + t\cdot a^{-} = 0\}$.

A. III. $x \in B \wedge x \neq 0 \to \bigvee_{a,b,c,d \in P}\{a\cdot b\cdot c\cdot d \neq 0\ \wedge\ a\cdot b\cdot c\cdot d\cdot x^{-} = 0\}$.

A. IV. Distinct systems of n half-spaces differ by vanishing of different intersections. Every system of this sort can be presented by some equations and inequalities as a meaningful expression of Boolean algebra. Let us denote it by

$$P_i(a_1,\ a_2,\ \ldots,\ a_n).$$

The disjunction of all the expressions $P_1, P_2, \ldots, P_m$ is a propositional

(1) Oral communication of S. Jaśkowski.

function of n variables which can be written in the following from:

$$G_m(a_1, a_2, \ldots, a_n).$$

Then, the fourth axiom reads as follows:

$$a_1, a_2, \ldots, a_7 \in P \rightarrow G_m(a_1, a_2, \ldots, a_7).$$

Jaśkowski conjectured that the very general axiom A. IV can be substituted by a set of much simpler ones. The present paper realizes Jaśkowski's idea with respect to the non-Desarguean geometry.

In Part I, I choose a set of axioms for the Boolean algebra. Next, in the collection of all plane regions, I distinguish a subcollection of regions which are called half-planes. I consider the half-plane as a regular open set which, therefore, is equal to the interior of its closure.

Then, I complete the axioms of Boolean algebra with four axioms which describe the geometric properties of half-plane. On the ground of so augmented theory, I construct a model of non-Desarguean geometry. The line is defined as a pair of complementary half-planes. The point is defined as a collection of half-planes satisfying a definite algebraic condition. A point is said to lie on a line if the half-planes which describe the line satisfy the condition defining that point. Also, the betweenness relation is defined by some algebraic conditions.

In the next paragraphs of Part I, it is proved that the notions and relations I have introduced satisfy the incidence axioms, the Euclid's axiom included, and also satisfy the axioms of betweenness relation. Moreover, I have given the proof that the axiom of complete additivity of Boolean algebra implies the axiom of continuity.

In Part II, it is shown that, in the non-Desarguean geometry, one can prove the theorems which serve as the axioms in the Part I. In fact, Part II presents a derivation of non-Desarguean geometry. There are introduced some new geometrical notions, e.g., the concept of the convex of two or more sets, and the concept of the hodograph of two sets. There are also proofs of some properties of the introduced notions.

Professor Wanda Szmielew and Doc. Lech Dubikajtis have read the earlier draft of this paper and I have used their very helpful remarks in the preparation of the final text. I take this opportunity to express them my cordial gratitude.

PART I

1. Axioms of Boolean algebra. We consider a set B whose elements admit the following three operations: addition, multiplication and complementation. We use the following symbols to denote these operations: $+, \cdot, ^-$. The symbol $^-$ is written like an exponent.

I assume that the elements of the set B satisfy the axioms (I follow Mostowski [4]):

(1) $$0 \in B,$$

(2) $$0 = 1^-,$$

(3) $$1 \in B.$$

If $x, y, z \in B$, then

(4) $$x+y \in B,$$

(5) $$x^- \in B,$$

(6) $$x \cdot y \in B,$$

(7) $$(x = y) \rightarrow (y = x),$$

(8) $$x = x,$$

(9) $$(x = y) \wedge (y = z) \rightarrow (x = z),$$

(10) $$(x = y) \rightarrow (x+z = y+z),$$

(11) $$(x = y) \rightarrow (x \cdot z = y \cdot z),$$

(12) $$x+0 = x,$$

(13) $$x \cdot 1 = x,$$

(14) $$x+x^- = 1,$$

(15) $$x \cdot x^- = 0,$$

(16) $$x+y = y+x,$$

(17) $$x \cdot y = y \cdot x,$$

(18) $$(x+y)+z = x+(y+z),$$

(19) $$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

(20) $$x+y \cdot z = (x+y) \cdot (x+z),$$

(21) $$x \cdot (y+z) = x \cdot y + x \cdot z.$$

Remark. The elements of the set B will be called *regions*.

2. Half-planes and their axioms. In the set B we distinguish a subset P which is called the *set of half-planes*. The elements of P are called *half-planes* and are denoted with the lower case letters.

If a is an element of the set P, then the complement of a will be denoted by a^{-1} or, for short, by a^-. The element a also will often be written a^{+1} or simply a^+. Clearly,

$$(a^{-1})^{-1} = a.$$

We shall use common logical symbols like $\wedge$, $\vee$, $\bigwedge$, $\bigvee$, $\rightarrow$, and we assume that the elements of the set P satisfy the following axioms:

2.1. Axiom. $a^+ \in P \rightarrow a^- \in P$.

2.2. Axiom. $x_1, x_2, x_3 \in B \wedge x_1, x_2, x_3 \neq 0 \rightarrow$ either

(1) $$\bigvee_{a\in P}\{x_1\cdot a, x_1\cdot a^-, x_2\cdot a, x_2\cdot a^-, x_3\cdot a, x_3\cdot a^- \neq 0\}$$

or

(2) $$\bigvee_{a_1,a_2,a_3,\in P}\{x_1\cdot a_1^- + x_2\cdot a_2^- + x_3\cdot a_3^- + (x_2+x_3)\cdot a_1^+ + (x_3+x_1)\cdot a_2^+ + (x_1+x_2)\cdot a_3^+ = 0\}.$$

2.3. Axiom. $a_1, a_2, a_3 \in P \wedge a_1^+\cdot(a_2^+ + a_3^+) = 0 \rightarrow a_2^+\cdot a_3^- = 0 \vee a_2^-\cdot a_3^+ = 0$.

2.4. Axiom. $a_1, a_2, a_3, a_4 \in P \wedge a_1^+\cdot a_2^+\cdot(a_3^+\cdot a_4^- + a_3^-\cdot a_4^+) = 0 \rightarrow (a_3^+ = a_4^+) \vee (a_1^+\cdot a_2^+\cdot a_3^+ = 0 \vee a_1^+\cdot a_2^+\cdot a_3^- = 0)$.

We refer to Axiom 2.2 as the *separation axiom.* It should be stressed that we assume that conditions (1) and (2) cannot be simultaneously satisfied.

3. The line.

3.1. Definition. An unordered pair of half-planes a^+, a^- is said to be a *line* which is denoted by A.

The half-planes a^+ and a^- are said to be the *sides* of the line A.

3.2. Definition. A line A is said *to cut the region* x, written A/x, if $x\cdot a^+ \neq 0$ and $x\cdot a^- \neq 0$.

The regions x_1 and x_2 are said *to lie on the same side of the line* A if its sides can be denoted so that

$$x_1\cdot a^+ + x_2\cdot a^+ = 0.$$

The line A is said *to separate the regions* x_1 *and* x_2 if the sides of A can be denoted so that

$$x_1\cdot a^+ + x_2\cdot a^- = 0,$$

and we shall refer to the same situation saying that x_1 *and* x_2 *lie on different sides of the line* A.

3.3. Definition. We shall say that *the lines* A *and* B *intersect,* written A/B, if

$$a^+\cdot b^+ \neq 0, \quad a^+\cdot b^- \neq 0, \quad a^-\cdot b^+ \neq 0, \quad a^-\cdot b^- \neq 0.$$

The lines A and B are said to be *parallel,* written $A\|B$, if the following disjunction holds:

$$a^+\cdot b^+ = 0 \vee a^+\cdot b^- = 0 \vee a^-\cdot b^+ = 0 \vee a^-\cdot b^- = 0.$$

In particular, every line is parallel to itself.

Definition 3.3 implies that, for any pair of lines A, B, it holds exactly one of the conditions A/B and $A\|B$.

3.4. Definition. The *net* determined by lines $A_1, A_2, \ldots, A_n$, which is denoted by $(A_1, A_2, \ldots, A_n)$ or by S_n, is the collection of regions of the form

$$a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_n^{k_n},$$

where $k_1, k_2, \ldots, k_n = \pm 1$.

The net S_n consists of 2^n regions, some of which can be zero. To every pair of regions

$$x_1 = a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_n^{k_n}, \qquad x_2 = a_1^{m_1} \cdot a_2^{m_2} \cdot \ldots \cdot a_n^{m_n}$$

of the net S_n we assign the number $L_n(x_1, x_2)$ such that

$$L_n(x_1, x_2) = \left(\frac{k_1 - m_1}{2}\right)^2 + \left(\frac{k_2 - m_2}{2}\right)^2 + \ldots + \left(\frac{k_n - m_n}{2}\right)^2.$$

The regions x_1, x_2 will be called *adjacent* if $L_n(x_1, x_2) = 1$, *vertical* if $L_n(x_1, x_2) = 2$, and *opposite* if $L_n(x_1, x_2) = 3$.

In particular, if we tabulate the regions of the net (ABC) in the following way:

$$\begin{array}{cccc} a^+ \cdot b^+ \cdot c^+ & a^+ \cdot b^- \cdot c^- & a^- \cdot b^+ \cdot c^- & a^- \cdot b^- \cdot c^+ \\ a^- \cdot b^- \cdot c^- & a^- \cdot b^+ \cdot c^+ & a^+ \cdot b^- \cdot c^+ & a^+ \cdot b^+ \cdot c^- \end{array}$$

then the regions in the same row are vertical, the regions in the same column are opposite, and any two regions belonging to two different rows and two different columns are adjacent.

3.5. Definition. The *subnet* determined by the half-planes $a_1^{k_1}, \ldots$ $\ldots, a_n^{k_n}$, written

$$a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_n^{k_n} (A_{n+1} A_{n+2} \ldots A_{n+m}),$$

is the collection of regions,

$$a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_n^{k_n} \cdot a_{n+1}^{k_{n+1}} \cdot a_{n+2}^{k_{n+2}} \cdot \ldots \cdot a_{n+m}^{k_{n+m}},$$

where the superscripts $k_i = \pm 1$ are fixed for $i = 1, 2, \ldots, n$, and take all possible values for $i = n+1, n+2, \ldots, n+m$.

Using Definition 3.5, we can give the Axiom 2.4 the form of

3.6. Corollary. *If the net $(ABCD)$ satisfies the condition $a^+ \cdot b^+ \cdot$ $\cdot (c^+ \cdot d^- + c^- \cdot d^+) = 0$, then the lines C and D coincide or the subnet $a^+ \cdot b^+ (C)$ contains a zero region.*

3.7. Definition. The lines A, B, C are said to be *tied* if the net (ABC) contains two opposite zero regions.

The relation of being tied, defined for three lines, has the following simple geometric interpretation.

If the lines A and B intersect, the line C is tied with them, provided C does not cut two vertical regions of the net (AB), i.e., the lines A, B, C are copunctal.

On the other hand, if the lines A and B are distinct and parallel, the line C is tied with them if it is parallel to them.

4. Properties of the net S_3. The following assertions are immediate corollaries of Axiom 2.2:

4.1. THEOREM. *If there exist no line which separates the region x_1 from the region $x_2 + x_3$, then there exists a line which cuts the regions x_1, x_2, x_3.*

4.2. THEOREM. *For any non-zero region there exists a line which cuts it.*

4.3. THEOREM. *For any two non-zero regions there exists a line which cuts them.*

4.4. THEOREM. *There exists a line.*

4.5. THEOREM. *For any line A there exists a line B which intersects it.*

Axiom 2.3 implies the following assertion:

4.6. THEOREM. *If $A \| B$ and $B \| C$, then $A \| C$.*

4.7. THEOREM. *The three distinct lines A, B, C are parallel if and only if the net (ABC) contains two vertical zero regions.*

Proof. It is easy to see that the condition of our theorem is necessary.

On the other hand, assuming that $a^+ \cdot (b^+ \cdot c^- + b^- \cdot c^+) = 0$, we infer that, by hypothesis and by Axiom 2.4, the subnets $a^+(B)$, $a^+(C)$ contain zero regions. Therefore, $A \| B$ and $A \| C$. Thus, our assertion follows by Theorem 4.6.

4.8. THEOREM. *If $A \| B$ and $B \| C$, then the lines A, B, C are tied.*

Proof. We assume that the sides of lines A, B, C are labelled in such a way that

(1) $$a^+ \cdot b^+ = 0$$

and also one of the following conditions is satisfied:

(2) $$a^- \cdot c^+ = 0,$$

(3) $$a^+ \cdot c^+ = 0.$$

Conditions (1) and (2) imply that $a^+ \cdot b^+ \cdot c^- + a^- \cdot b^- \cdot c^+ = 0$. From (1) and (3) it follows by 2.3 that $b^+ \cdot c^- = 0$ or $b^- \cdot c^+ = 0$. Thus, $a^+ \cdot b^- \cdot c^+ + a^- \cdot b^+ \cdot c^- = 0$ or $a^+ \cdot b^+ \cdot c^- + a^- \cdot b^- \cdot c^+ = 0$.

Theorems 4.7, 4.8 and Definition 3.7 imply the following assertion:

4.9. THEOREM. *If, in the net (ABC), two vertical regions are zero, then the lines A, B, C are tied.*

4.10. THEOREM. *There exists a net (ABC) which has at most one zero region.*

Proof. Theorems 4.4 and 4.5 imply the existence of two intersecting lines, A and B. Thus, the following regions are non-zero:

$$x_1 = a^+ \cdot b^+, \quad x_2 = a^- \cdot b^+, \quad x_3 = a^+ \cdot b^-, \quad x_4 = a^- \cdot b^-.$$

Assume that C is such a line that

$$(x_2 + x_3) \cdot c^+ = (a^- \cdot b^+ + a^+ \cdot b^-) \cdot c^+ = 0.$$

The last equality, Theorem 4.7, Definition 3.3 and the hypothesis imply a contradiction. Thus, by Theorem 4.1, there exists a line which cuts all the three regions x_1, x_2, x_3.

4.11. THEOREM. *There exists no line which cuts three pairwise vertical regions of the net* (ABC).

Proof. Consider the regions

$$x_1 = a^- \cdot b^+ \cdot c^+, \quad x_2 = a^+ \cdot b^- \cdot c^+, \quad x_3 = a^+ \cdot b^+ \cdot c^-.$$

If at least one of these regions is a zero region, our theorem holds. If all of them are non-zero, the line A separates the region x_1 from the region $x_2 + x_3$, the line B separates the region x_2 from the region $x_1 + x_3$, and the line C separates the region x_3 from the region $x_1 + x_2$. Thus, our theorem follows by Axiom 2.2.

4.12. THEOREM. *Every line cuts at most four regions of the net* (ABC).

Proof. We divide the regions of the net (ABC) into two classes, I and II, such that the regions belonging to the same class are pairwise vertical. This decomposition can be tabulated in the following way.

$$\text{I:} \quad a^+ \cdot b^+ \cdot c^+, \quad a^+ \cdot b^- \cdot c^-, \quad a^- \cdot b^+ \cdot c^-, \quad a^- \cdot b^- \cdot c^+,$$
$$\text{II:} \quad a^- \cdot b^- \cdot c^-, \quad a^- \cdot b^+ \cdot c^+, \quad a^+ \cdot b^- \cdot c^+, \quad a^+ \cdot b^+ \cdot c^-.$$

By Theorem 4.11, any line cuts at most two regions of the class I and at most two regions of the class II. Thus, it cuts at most four regions of the net (ABC).

The assertions given below are immediate consequences of the theorems already proved.

4.13. THEOREM. *If* $x_1 \neq 0$, $x_2 \neq 0$, *and if the line* A *cuts neither* x_1 *nor* x_2, *then exactly one of the following assertions holds*:

(i) *the regions* x_1, x_2 *lie on the same side of the line* A;

(ii) *the line* A *separates the region* x_1 *from the region* x_2.

4.14. THEOREM. *If* $x_1 = a^+ \cdot b^+ \cdot c^+$, $x_2 = a^- \cdot b^+ \cdot c^+$, *and if the line* D *cuts neither the region* x_1 *nor the region* x_2, *then exactly one of the following assertions holds*:

(i) *the regions* x_1, x_2 *lie on the same side of the line* D;

(ii) $D = A$.

4.15. THEOREM. *If* $x_1 = a^+ \cdot b^- \cdot c^+ \neq 0$, $x_2 = a^+ \cdot b^+ \cdot c^+ \neq 0$, $x_3 = a^- \cdot b^+ \cdot c^+ \neq 0$, *and if the line D, distinct from A, B and C, separates the region* x_1 *from the region* x_3, *then D cuts the region* x_2.

4.16. THEOREM. *Every net (ABC) contains at least one zero region.*

Proof. The conclusion of our theorem is obvious if (1) two of the lines A, B, C are parallel or if (2) the lines A, B, C are tied. So, we have only to consider the case where neither (1) nor (2) holds.

Let us introduce the following notation:

$$x_1 = a^+ \cdot b^+ \cdot c^+, \quad x_2 = a^+ \cdot b^- \cdot c^-, \quad x_3 = a^- \cdot b^+ \cdot c^-, \quad x_4 = a^- \cdot b^- \cdot c^+,$$

$$x_5 = a^- \cdot b^- \cdot c^-, \quad x_6 = a^- \cdot b^+ \cdot c^+, \quad x_7 = a^+ \cdot b^- \cdot c^+, \quad x_8 = a^+ \cdot b^+ \cdot c^-.$$

We suppose that all these regions are non-zero.

By Theorem 4.14, the line A is the only line separating the region x_6 from the region x_1, and C is the only line separating the region x_6 from the region x_3. Since the lines A and C are distinct by hypothesis, there exists no line separating the region x_6 from the region $x_1 + x_3$. Thus, by Axiom 2.2, there exists a line D which cuts every of the regions x_1, x_3, x_6. Clearly, the line D is different from the lines A, B, C. By Theorem 4.11, the line D does not cut the regions x_2, x_4, and cuts at most one of the regions x_5, x_7, x_8. It is easy to notice that, whichever of the regions x_5, x_7, x_8 is cut by the line D, and even in the case where the line D cuts none of them, it is possible to order the uncut regions into a sequence of consecutively adjacent regions. By Theorem 4.14, all these regions lie on the same side of the line D. Labelling that side with d^+, we notice that the following regions are non-zero:

$$a^+ \cdot b^+ \cdot c^+ \cdot d^-, \quad a^+ \cdot b^+ \cdot c^- \cdot d^+, \quad a^+ \cdot b^- \cdot c^+ \cdot d^+,$$

$$a^- \cdot b^+ \cdot c^+ \cdot d^-, \quad a^- \cdot b^+ \cdot c^- \cdot d^+, \quad a^- \cdot b^- \cdot c^+ \cdot d^+.$$

This implies that, in the net (BCD), the line A cuts pairwise vertical regions $b^+ \cdot c^+ \cdot d^-$, $b^+ \cdot c^- \cdot d^+$, $b^- \cdot c^+ \cdot d^+$, which, by Theorem 4.11, is impossible. Thus, at least one of the regions x_i, $i = 1, 2, \ldots, 8$, is zero.

The following assertion is a consequence of the definition of tied lines and of Theorem 4.7:

4.17. THEOREM. *If the lines A, B, C are not tied, and no two of them are parallel, the net (ABC) contains exactly one zero region.*

5. Properties of the net S_4.

5.1. THEOREM. *If* $(a^+ \cdot b^+ \cdot c^+ + a^- \cdot b^- \cdot c^-) \cdot d^+ = 0$, *the subnet* $d^+ \cdot (AB)$ *contains a zero region or* $a^+ \cdot b^+ \cdot c^+ + a^- \cdot b^- \cdot c^- = 0$.

Proof. Assuming that the subnet $d^+ \cdot (AB)$ contains no zero region, we infer that, by hypothesis and by Theorem 4.15, the line C cuts the regions $a^+\cdot b^-\cdot d^+$, $a^-\cdot b^+\cdot d^+$. Therefore, by Theorem 4.11, the line C does not cut the regions $a^+\cdot b^+\cdot d^-$, $a^-\cdot b^-\cdot d^-$, and it follows that, by Theorem 4.14, $(a^+\cdot b^+\cdot c^+ + a^-\cdot b^-\cdot c^-)\cdot d^- = 0$. The last condition and the hypothesis imply that $a^+\cdot b^+\cdot c^+ + a^-\cdot b^-\cdot c^- = 0$.

The following assertion is a corollary to Theorems 4.16 and 4.12:

5.2. THEOREM. *A net* $(ABCD)$ *has at least five zero regions.*

5.3. Definition. If, for any two regions x, y belonging to a subset Z of a net, there exists a sequence of regions belonging to Z such that x is the first term of the sequence, and y is the last one, and any two consecutive terms of the sequence are adjacent regions, then the set Z is said to be a *connected* subset of that net.

5.4. THEOREM. *In a net* $(ABCD)$, *the zero regions constitute a connected subset.*

Proof. (1) If two of the lines determinig the net $(ABCD)$ coincide and, e.g., $c^+\cdot d^- + c^-\cdot d^+ = 0$, then all regions of the subnets $c^+\cdot d^-\cdot(AB)$, $c^-\cdot d^+\cdot(AB)$ are zero. The net (ABC) has at least one zero region. Hence, we can label the sides of lines A, B, C so that $a^-\cdot b^-\cdot c^- = 0$. Therefore, all regions of the subnet $a^-\cdot b^-\cdot c^-\cdot(D)$ are zero. It is easy to see that the union Z of subnets $c^+\cdot d^-\cdot(AB)$, $c^-\cdot d^+\cdot(AB)$, $a^-\cdot b^-\cdot c^-\cdot(D)$ is a connected set, and that, for every region x of the net $(ABCD)$ not belonging to Z, there exists, in the set Z, a region y adjacent to x. And this implies the conclusion of our theorem.

(2) We assume that all lines of the net $(ABCD)$ are distinct. Definition 3.4 assigns to every pair x, y of regions in the net $(ABCD)$ a number $L_4(xy)$. It is clear that $L_4(xy) \leqslant 4$.

Since in the net $(ABCD)$ there exists, for every region x, only one region y such that $L_4(xy) = 4$, and, on the other hand, in the net $(ABCD)$, the number of zero regions is not less than 5, so, if x and y are zero regions and $L_4(xy) = 4$, there exists a zero region z such that $L_4(xz) \leqslant 3$.

If x, y are zero regions of the net $(ABCD)$ and $L_4(xy) = 3$, then, by Theorem 5.1, either (i) there exists a zero region z such that both $L_4(xz) \leqslant 2$ and $L_4(yz) \leqslant 2$ or (ii) three of the lines determining the net $(ABCD)$ are tied, In the case (ii), assume that

$$x = a^+. b^+\cdot c^+\cdot d^+, \quad y = a^+\cdot b^-\cdot c^-\cdot d^-.$$

Thus, by Theorem 5.1, just the lines B, C, D are tied, and the identity $b^+\cdot c^+\cdot d^+ + b^-\cdot c^-\cdot d^- = 0$ holds. Therefore, all regions of subnets $b^+\cdot c^+\cdot d^+\cdot(A)$, $b^-\cdot c^-\cdot d^-\cdot(A)$ are zero, and hence, $x' = a^-\cdot b^+\cdot c^+\cdot d^+$, $y' = a^+\cdot b^-\cdot c^-\cdot d^-$ are zero regions. It is easy to see that every region z of the net $(ABCD)$ satisfies either the conditions $L_4(xz) \leqslant 2$ and $L_4(yz) \leqslant 2$ or the conditions $L_4(x'z) \leqslant 2$ and $L_4(y'z) \leqslant 2$. Since, by Theorem 5.2, there exists a zero region z which is different from the regions x, x', y, y', either the sequence x, z, y or the sequence x, x', z, y', y consists of such zero regions that any two consecutive terms are two adjacent or two vertical regions.

If x, y are zero regions of the net $(ABCD)$ and $L_4(xy) = 2$, then there exists, by the hypothesis and by Axiom 2.4, such a zero region z that $L_4(xz) = 1$ and $L_4(yz) = 1$. Thus, the conclusion of our theorem follows.

5.5. THEOREM. *If the lines A, B, C are not tied and no two of them are parallel, there exists no line which simultaneously cuts four regions constituting two pairs of opposite regions.*

Proof. From Theorem 4.17 it follows that exactly one of the regions in the net (ABC) is zero. Let $a^- \cdot b^- \cdot c^-$ be the zero region of that net. In contradiction to the conclusion, let us suppose that the line D cuts the regions

$$x_1 = a^+ \cdot b^+ \cdot c^-, \quad x_2 = a^+ \cdot b^- \cdot c^+, \quad x_3 = a^- \cdot b^- \cdot c^+, \quad x_4 = a^- \cdot b^+ \cdot c^-.$$

By Theorem 4.11, the line D does not cut the regions

$$x_5 = a^+ \cdot b^+ \cdot c^+, \quad x_6 = a^- \cdot b^+ \cdot c^+, \quad x_7 = a^+ \cdot b^- \cdot c^-.$$

By Theorem 4.14, the regions x_5, x_6 lie on the same side of the line D. If we denote this side by d^-, then the zero regions of the net $(ABCD)$ are

$$a^- \cdot b^- \cdot c^- \cdot d^-, \quad a^- \cdot b^- \cdot c^- \cdot d^+, \quad a^+ \cdot b^+ \cdot c^+ \cdot d^+, \quad a^- \cdot b^+ \cdot c^+ \cdot d^+,$$

and one of the regions $a^+ \cdot b^- \cdot c^- \cdot d^+$ and $a^+ \cdot b^- \cdot c^- \cdot d^-$. It is easy to see that the zero regions of the net $(ABCD)$ do not constitute a connected set, which is not possible by Theorem 5.4. Thus the conclusion follows.

5.6. Definition We shall say that the net $(ABCD)$ *belongs to the class R* if the lines determining the net, A, B, C, D, satisfy the following conditions:
(1) $a^- \cdot c^- + b^- \cdot d^- = 0$;
(2) the lines A, C are distinct;
(3) the lines B, D are distinct;
(4) the line A intersects the lines B, D;
(5) the line C intersects the lines B, D

5.7. THEOREM. *If the net $(ABCD)$ belongs to the class R, then every region of the subnets $a^- \cdot c^- \cdot (BD)$, $b^- \cdot d^- \cdot (AC)$ is zero, and all other regions are non-zero.*

Proof. By condition (1) of Definition 5.6, all regions of the subnets $a^- \cdot c^- \cdot (BD)$, $b^- \cdot d^- \cdot (AC)$ are zero. We shall show that the net $(ABCD)$ contains no zero region which does not belong to the union of subnets $a^- \cdot c^- \cdot (BD)$, $b^- \cdot d^- \cdot (AC)$, but which is adjacent to one which does. Since the argument is quite similar in all possible cases, we shall consider only one of them.

If we suppose that $a^- \cdot b^+ \cdot c^+ \cdot d^+ = 0$, it follows that, by hypothesis, $a^- \cdot b^+ \cdot d^+ = 0$ and $a^- \cdot b^- \cdot d^- = 0$. Since two vertical regions of the net (ABD) are zero, we infer that, by Theorem 4.7, $A \| B \| D$, which contradicts condition (4) of Definition 5.6.

It is easy to see that the union of subnets $a^- \cdot c^- \cdot (BD)$, $b^- \cdot d^- \cdot (AC)$ consists of seven regions, and that the regions, which are adjacent to the former, are the following:

$$a^- \cdot b^+ \cdot c^+ \cdot d^+, \quad a^+ \cdot b^- \cdot c^+ \cdot d^+, \quad a^+ \cdot b^+ \cdot c^- \cdot d^+, \quad a^+ \cdot b^+ \cdot c^+ \cdot d^-,$$
$$a^- \cdot b^- \cdot c^+ \cdot d^+, \quad a^- \cdot b^+ \cdot c^+ \cdot d^-, \quad a^+ \cdot b^- \cdot c^- \cdot d^+, \quad a^+ \cdot b^+ \cdot c^- \cdot d^-.$$

Thus, we have only to consider the region $x_0 = a^+ \cdot b^+ \cdot c^+ \cdot d^+$. But, by Theorem 5.4, we infer that $x_0 \neq 0$.

Axiom 2.4 implies the following assertion:

5.8. THEOREM. *If* (1) $a^- \cdot c^- = 0$, (2) *the lines* A, C *are distinct,* (3) *the lines* A, B *intersect,* (4) $a^- \cdot b^- \cdot d^+ \neq 0$ *and* $b^- \cdot c^- \cdot d^+ \neq 0$, *then* $a^+ \cdot b^- \cdot c^+ \cdot d^+ \neq 0$.

5.9. THEOREM. *If the net* $(ABCD)$ *belongs to the class* R, *and the line* E *cuts the regions* $x_1 = a^- \cdot b^+ \cdot c^+ \cdot d^+$ *and* $x_2 = a^+ \cdot b^- \cdot c^+ \cdot d^+$, *then the regions* $a^+ \cdot b^+ \cdot c^-$ *and* $a^+ \cdot b^+ \cdot d^-$ *lie on the same side of the line* E.

Proof. If the line E cuts the regions x_1, x_2, it cuts the regions $a^- \cdot b^+ \cdot c^+$, $a^+ \cdot b^- \cdot c^+$ in the net (ABC) and the regions $a^- \cdot b^+ \cdot d^+$, $a^+ \cdot b^- \cdot d^+$ in the net (ABD). Thus, by Theorem 4.11, the line E does not cut the regions $a^+ \cdot b^+ \cdot c^-$ and $a^+ \cdot b^+ \cdot d^-$. If these regions would lie on different sides of the line E, which we could write

$$a^+ \cdot b^+ \cdot c^- \cdot e^+ + a^+ \cdot b^+ \cdot d^- \cdot e^- = 0,$$

choosing a convenient notation for the sides of E, then the following should hold:

$$a^+ \cdot b^+ \cdot c^- \cdot d^- = a^+ \cdot b^+ \cdot c^- \cdot d^- \cdot (e^+ + e^-) = 0.$$

But the latter relationship fails by Theorem 5.7. Thus, the conclusion of our theorem follows.

6. Pseudopoints.

6.1. Definition. A net (AB) is said to be a *pseudopoint*, if all its regions are non-zero.

6.2. Definition. The pseudopoint (AB) is said *to lie on the line* C, or the line C is said *to pass through the pseudopoint* (AB), if the lines A, B, C are tied.

It is a simple matter to prove the following assertions.

6.3. THEOREM. *If the pseudopoint* (AB) *lies on the line* C, *then the net* (AC) *is a pseudopoint and* (AC) *lies on the line* B, *or the net* (BC) *is a pseudopoint and* (BC) *lies on the line* A.

6.4. THEOREM. *The pseudopoint* (AB) *lies on the lines* A, B.

6.5. THEOREM. *If the net* (AB) *is a pseudopoint, if the lines* A, B, C *are tied, and if the lines* A, B, D *are tied, then the lines* A, C, D *are also tied.*

The equivalence of pseudopoints is introduced by the following condition:

6.6. Definition. The pseudopoint (AB) is *equivalent to the pseudopoint* (CD), if (AB) lies on the lines C, D.

The following assertions can be proved.

6.7. THEOREM. *The equivalence of pseudopoints is an equivalence relation.*

6.8. THEOREM. *If* $A \| B$, *there is no pseudopoint which lies both on the line* A *and* B.

6.9. THEOREM. *If the pseudopoint* (AB) *does not lie on the line* C, *there exists such a line* D *parallel to* C *that the pseudopoint* (AB) *lies on the line* D.

Proof. If the line C is parallel to A or to B, the conclusion follows. So, we have only to consider the case where the line C is not parallel either to A or to B.

By Theorem 4.17, the net (ABC) has exactly one zero region. Let us assume that

$$x_0 = a^- \cdot b^- \cdot c^- = 0.$$

We introduce the following notation:

$$x_1 = a^+ \cdot b^+ \cdot c^+, \quad x_2 = a^+ \cdot b^- \cdot c^-, \quad x_3 = a^- \cdot b^+ \cdot c^-, \quad x_4 = a^- \cdot b^- \cdot c^+,$$
$$x_5 = a^- \cdot b^+ \cdot c^+, \quad x_6 = a^+ \cdot b^- \cdot c^+, \quad x_7 = a^+ \cdot b^+ \cdot c^-.$$

By Theorem 4.11, there exists no line which cuts all the three regions x_1, x_3, x_4.

From Theorems 4.15 and 5.5 if follows that there exists no line which separates the regions x_1, x_2 and cuts the regions x_3, x_4. By Theorem 4.11, no line cuts all the three regions x_2, x_3, x_4.

Thus, a line, which cuts every of the regions x_2, x_3, x_4, does not exist. By Axiom 2.2, there exists a line D which separates the region x_4 from the region $x_1 + x_2 + x_3$. If we let d^+ stand for that side of the line D which contains the region x_4, then

$$(1) \qquad (a^+ \cdot b^+ \cdot c^+ + a^+ \cdot b^- \cdot c^- + a^- \cdot b^+ \cdot c^-) \cdot d^+ + a^- \cdot b^- \cdot c^+ \cdot d^- = 0.$$

By Theorem 4.15, the line D cuts the regions x_5, x_6. Therefore, by

Theorem 4.11, it does not cut the region x_7. The regions x_1 and x_7 are adjacent, and hence, by Theorem 4.14, they lie on the same side of the line D. This implies the condition

(2) $$a^+ \cdot b^+ \cdot c^- \cdot d^+ = 0 .$$

Conditions (1) and (2) and the hypothesis give the following relationships:

$$c^- \cdot d^+ = 0 \quad \text{and} \quad a^+ \cdot b^+ \cdot d^+ + a^- \cdot b^- \cdot c^- = 0 ,$$

which show that the pseudopoint (AB) lies on the line D, and that the lines C and D are parallel.

6.10. THEOREM. *There exists only one line, parallel to the line C, that passes through the pseudopoint (AB).*

Proof. Suppose that two lines E, D parallel to C pass through the pseudopoint (AB). By Theorem 4.6, the lines E and D are parallel. From Theorem 6.3, it follows that (1) the net (AD) is a pseudopoint or (2) the net (AE) is a pseudopoint.

Let us assume that the net (AD) is a pseudopoint. Consider the net (ADE). We infer that, by Theorem 6.5, the lines A, D, E are tied. Taking account of the fact that the lines D and E are parallel, we infer that the net (ADE) has at least three zero regions. It is easy to see that two of them are vertical. This implies that, by Theorem 4.7, the lines A and D are parallel, which contradicts the hypothesis. In case (2), we derive the contradiction in a similar way.

6.11. THEOREM. *For every two pseudopoints there exists a line passing through them.*

Proof. We assume that (AB) is a pseudopoint. If the other pseudopoint lies on the line A or on the line B, our theorem holds. So let us agree that the other pseudopoint does not lie either on A or on B. By Theorems 6.7 and 6.9, the other pseudopoint can be determined by lines C, D such that $C \| A$ and $D \| B$. By hypothesis, $A \neq C$ and $B \neq D$.

We can label the sides of the lines A, B, C, D so that

(1) $$a^- \cdot c^- + b^- \cdot d^- = 0 .$$

By Theorem 4.6, the nets (AD) and (BC) are pseudopoints. Thus, it follows that the net $(ABCD)$ belongs to the class R. By Theorem 5.7, the following regions of the net $(ABCD)$ are non-zero:

$$x_1 = a^- \cdot b^+ \cdot c^+ \cdot d^+ , \quad x_2 = a^+ \cdot b^- \cdot c^+ \cdot d^+ , \quad x_3 = a^+ \cdot b^+ \cdot c^- \cdot d^+ ,$$
$$x_4 = a^+ \cdot b^+ \cdot c^+ \cdot d^- , \quad x_5 = a^- \cdot b^- \cdot c^+ \cdot d^+ , \quad x_6 = a^+ \cdot b^- \cdot c^- \cdot d^+ ,$$
$$x_7 = a^+ \cdot b^+ \cdot c^- \cdot d^- , \quad x_8 = a^- \cdot b^+ \cdot c^+ \cdot d^- , \quad x_9 = a^+ \cdot b^+ \cdot c^+ \cdot d^+ .$$

By Theorem 5.9, no line cuts all the three regions x_1, x_4, $x_2 + x_3 + x_6$.

Therefore, by Axiom 2.2, there exists a line E which separates the regions x_1+x_4 and $x_2+x_3+x_6$. Conveniently labelling the sides of the line E, we infer that

(2) $$(a^-\cdot d^+ + a^+\cdot d^-)\cdot b^+\cdot c^+\cdot e^+ + (b^-\cdot c^+ + b^+\cdot c^-)\cdot a^+\cdot d^+\cdot e^- = 0.$$

Conditions (1) and (2) imply the following ones:

(3) $$(a^-\cdot b^+\cdot e^+ + a^+\cdot b^-\cdot e^-)\cdot d^+ = 0,$$

(4) $$(c^+\cdot d^-\cdot e^+ + c^-\cdot d^+\cdot e^-)\cdot a^+ = 0.$$

Since, by hypothesis, all regions of the subnets $d^+\cdot(AB)$, $a^+\cdot(CD)$ are non-zero, it follows by Theorem 5.1 that

$$a^-\cdot b^+\cdot e^+ + a^+\cdot b^-\cdot e^- = 0 \quad \text{and} \quad c^+\cdot d^-\cdot e^+ + c^-\cdot d^+\cdot e^- = 0.$$

And hence, the conclusion follows.

Theorem 6.7 implies the following one:

6.12. THEOREM. *At most one line can pass through two non-equivalent pseudopoints.*

On the other hand, Theorem 4.10 gives the following assertion:

6.13. THEOREM. *On the given line, there lie at least two non-equivalent pseudopoints, and there exists a pseudopoint which does not lie on the given line.*

7. The ordering of pseudopoints.

7.1. Definition. The pseudopoint (AB) is said *to lie in the half-plane* c^+, written $(AB)\epsilon c^+$, if all regions of the subnet $c^+\cdot(AB)$ are different from zero.

7.2. Definition. The line E is said *to lie between the pseudopoints* (AB) *and* (CD), written $(AB)|E|(CD)$, if the sides of E can be labelled so that $(AB)\epsilon e^+$ and $(CD)\epsilon e^-$.

7.3. Definition. The pseudopoint (AD) is said *to lie between the pseudopoints* (BD), (CD), written $(BD)|(AD)|(CD)$, if the line A lies between the pseudopoints (BD) and (CD).

These definitions and the theorems of section 4 imply the following assertions:

7.4. THEOREM. *If* $(AB)\epsilon c^+$, *if the lines* A, B, D *are tied, and if the net* (AD) *is a pseudopoint, then* $(AD)\epsilon c^+$.

7.5. THEOREM. *If* $(AB)\epsilon c^+$, *then lines* A, B, C *are not tied.*

7.6. THEOREM. *If* $(AD)|B|(AC)$, *then the net* (AB) *is a pseudopoint.*

7.7. THEOREM. *If the lines* A, B, C *are not tied, and* (AB) *is a pseudopoint, then either* $(AB)\epsilon c^+$ *or* $(AB)\epsilon c^-$.

7.8. THEOREM. *If*

(i) $(AB)|D|(AC)$,

(ii) *the lines* A, D, E *are tied,*

(iii) *the lines* A *and* E *are distinct,*

then $(AB)|E|(AC)$.

Proof. By Theorem 7.4, we must consider only the case where the lines B, C, D are parallel. Assume the sides of these lines are labelled so that

(1) $$b^+ \cdot d^- + c^- \cdot d^+ = 0.$$

By Theorem 7.6, the net (AD) is a pseudopoint. By Theorem 7.5, neither the pseudopoints (AB) and (AD) nor the pseudopoints (AC) and (AD) are equivalent. This fact and condition (i) imply that the lines B, C, D are distinct. If the line E were tied with the lines A, B, then E would pass through two non-equivalent pseudopoints (AB), (AD) and, by Theorem 6.12, it would coincide with the line A, which is impossible by condition (iii). Thus, the lines A, B, E are not tied. Therefore, by Theorem 7.7, the sides of the line E can be labelled so that

(2) $$(AB) \epsilon e^+,$$

and that one of the following assertions holds:

(3) $$(AC) \epsilon e^+$$

or

(4) $$(AC) \epsilon e^-.$$

If conditions (2) and (4) are satisfied, our theorem is true. We shall show that conditions (2) and (3) lead to a contradiction.

Conditions (2) and (3) imply the following relationships:

(5) $$a^+ \cdot b^+ \cdot e^+ \neq 0,$$

(6) $$a^- \cdot b^+ \cdot e^+ \neq 0,$$

(7) $$a^+ \cdot c^- \cdot e^+ \neq 0,$$

(8) $$a^- \cdot c^- \cdot e^+ \neq 0.$$

From (1) we infer that $b^+ \cdot d^+ = b^+$, and hence, conditions (5) and (6) imply

(9) $$a^+ \cdot d^+ \cdot e^+ \neq 0$$

and

(10) $$a^- \cdot d^+ \cdot e^+ \neq 0.$$

By Theorem 5.8, conditions (7) and (9) imply that $a^+ \cdot c^+ \cdot d^- \cdot e^+ \neq 0$. Thus

(11) $$a^+ \cdot d^- \cdot e^+ \neq 0.$$

In a similar way, we derive from (8) and (10) that

(12) $$a^- \cdot d^- \cdot e^+ \neq 0,$$

and conditions (9), (10), (11) and (12) imply that the lines A, D, E are not tied, which contradicts condition (ii).

It is rather a simple matter to prove the following assertions:

7.9. THEOREM. *If (AB) and (AC) are two non-equivalent pseudopoints, then there exists a line D such that $(AB)|(AC)|(AD)$.*

7.10. THEOREM. *If $(AB)|(AC)|(AD)$, then the relation $(AC)|(AB)|(AD)$ fails.*

7.11. THEOREM. *If $(AB)|D|(AC)$, and if the net (BC) is a pseudopoint, then one of the following conditions holds:*
(i) *the lines B, C, D are tied,*
(ii) $(AC)|D|(BC)$,
(iii) $(AB)|D|(BC)$.

8. The points.

8.1. Definition. An equivalence class of pseudopoints will be called a *point.*

By this definition, every pseudopoint determines a point, and two points are distinct if and only if they are determined by non-equivalent pseudopoints.

The symbol $\{AB\}$ will stand for the point determined by the pseudopoint (AB).

8.2. Definition. The incidence relation is said *to hold between the point $\{AB\}$ and the line C,* written $\{AB\}\epsilon C$, if the pseudopoint (AB) lies on the line C.

The theorems of preceding paragraphs imply the following ones:

8.3. THEOREM. *$\{AB\}\epsilon C$ if and only if the pseudopoint (AB) lies on the line C.*

8.4. THEOREM. *For every line A there exist two different points $\{AB\}$, $\{AC\}$ such that $\{AB\}\epsilon A$ and $\{AC\}\epsilon A$.*

8.5. THEOREM. *For every two points $\{AB\}$, $\{CD\}$ there exists a line E such that $\{AB\}\epsilon E$ and $\{CD\}\epsilon E$.*

8.6. THEOREM. *If the points $\{AB\}$, $\{CD\}$ are distinct, then there exists at most one line E such that $\{AB\}\epsilon E$ and $\{CD\}\epsilon E$.*

8.7. THEOREM. *There exist three points $\{AB\}$, $\{CD\}$, $\{EF\}$ such that the relation $\{EF\}\epsilon G$ fails if the relations $\{AB\}\epsilon G$ and $\{CD\}\epsilon G$ hold.*

8.8. THEOREM. *For every line A and for every point $\{BC\}$ there exists one and only one line D such that the conditions $\{BC\}\epsilon D$ and $A\|D$ hold.*

Theorems 8.4-8.7 are identical with the incidence axioms for points and lines. Theorem 8.8 is the Euclid's axiom.

Thus, we have shown that the incidence axioms of plane geometry and the Euclid's axiom are satisfied in the theory based on the axioms explained in Sections 1 and 2.

In order to show that the betweenness axioms are also satisfied in that theory, we give the following definitions:

8.9. **Definition.** The point $\{AB\}$ is said *to lie in the half-plane* c^+, if every pseudopoint equivalent to (AB) lies in the half-plane c^+.

8.10. **Definition.** The line E is said *to lie between the points* $\{AB\}$, $\{CD\}$, written $\beta(\{AB\}E\{CD\})$, if the points $\{AB\}$, $\{CD\}$ lie in different half-planes determined by the line E.

8.11. **Definition.** The point $\{AB\}$ is said *to lie between the points* $\{AC\}$, $\{AD\}$, written $\beta(\{AC\}\{AB\}\{AD\})$, if every line E, tied with the lines A, B and different from A, lies between the points $\{AC\}$, $\{AD\}$.

These definitions and the theorems of Section 7 imply the following assertions:

8.12. THEOREM. *The following two conditions are equivalent*:

(1) $$\beta(\{AB\}E\{CD\}),$$

(2) $$(AB)|E|(CD).$$

8.13. THEOREM. *The following two conditions are equivalent*:

(1) $$\beta(\{AB\}\{AC\}\{AD\}),$$

(2) $$(AB)|(AC)|(AD).$$

8.14. THEOREM. *If* $\beta(\{AB\}\{CD\}\{EF\})$, *then the points* $\{AB\}$, $\{CD\}$, $\{EF\}$ *are collinear and distinct.*

8.15. THEOREM. *If* $\beta(\{AB\}\{AC\}\{AD\})$, *then* $\beta(\{AD\}\{AC\}\{AB\})$.

8.16. THEOREM. *If* $\beta(\{AB\}\{AC\}\{AD\})$ *is true, then* $\beta(\{AC\}\{AB\}\{AD\})$ *is false.*

8.17. THEOREM. *If* $\{AB\}$ *and* $\{AC\}$ *are different points, then there exists a point* $\{AD\}$ *such that* $\beta(\{AB\}\{AC\}\{AD\})$.

8.18. THEOREM. *If the points* $\{AB\}$, $\{CD\}$, $\{EF\}$ *are distinct and non-collinear, and* $\beta(\{AB\}G\{CD\})$, *then one of the following conditions is satisfied*:

(1) $$\beta(\{AB\}G\{EF\}),$$

(2) $$\beta(\{CD\}G\{EF\}),$$

(3) $$\{EF\}\in G.$$

8.19. THEOREM. *The condition* $\beta(\{AB\}G\{CD\})$ *is equivalent to the pair of following conditions:*

(1) *neither* $\{AB\}\epsilon G$ *nor* $\{CD\}\epsilon G$,

(2) *there exists a point* $\{EF\}$ *such that* $\{EF\}\epsilon G$ *and*

$$\beta(\{AB\}\{EF\}\{CD\}).$$

Theorems 8.14-8.18 coincide with the betweenness axioms.

Thus, we have shown that the betweenness axioms are satisfied in the theory based on the axioms explained in Sections 1 and 2.

Therefore, it has been established that the axioms of Section 2 and the definitions introduced in next sections determine, in the body of Boolean algebra, a model of non-Desarguean geometry (without the continuity axiom).

Our objective in the next paragraph is a derivation of the continuity axiom from the complete additivity axiom of complete Boolean algebra.

9. Continuity axiom. The complete additivity axiom of complete Boolean algebra can be given the following from:

9.1. AXIOM. For every family of regions, K, there exists a region $\bigcup K$, called the *union* of regions in the family K, and such that the following conditions are satisfied:

(i) For every region y, $y\cdot\bigcup K = 0$ if and only if, for every region $x\epsilon K$, $x\cdot y = 0$.

(ii) For every region y if there exists a region x belonging to the family K and such that $y\cdot x^- = 0$, then $y\cdot(\bigcup K)^- = 0$.

Theorems 4.2, 4.3 and 6.9 imply the following assertion:

9.2. THEOREM. *For every region y, and for every line A, there exists a line B parallel to A and cutting the region y.*

9.3. THEOREM. *If b_0 is any half-plane, and K_1, K_2 are the sets of half-planes such that*

(1) *for every half-plane* $a\epsilon K_1$, $a\cdot b_0 = 0$,

(2) *a half-plane b belongs to K_2 if and only if, for every half-plane a belonging to K_1,* $a\cdot b = 0$,

then $\bigcup K_1 = (\bigcup K_2)^-$.

Proof. Let x and y stand for $\bigcup K_1$ and $\bigcup K_2$ respectively. Condition (2) and Axiom 9.1 imply that

$$x\cdot y = 0. \tag{3}$$

Let us suppose that

$$x^-\cdot y^- \neq 0. \tag{4}$$

By Theorem 9.2, there exists a line D parallel to the line B_0 and cutting the region $x^- \cdot y^-$. Therefore, the following conditions are satisfied:

$$x^- \cdot y^- \cdot d^+ \neq 0, \tag{5}$$

$$x^- \cdot y^- \cdot d^- \neq 0. \tag{6}$$

Condition (5) implies that $y^- \cdot d^+ \neq 0$, which gives, by Axiom 9.1, the condition $b_0 \cdot d^+ \neq 0$. Similarly, condition (6) implies that $b_0 \cdot d^- \neq 0$. Let us label the sides of the line D so that

$$b_0^- \cdot d^- = 0. \tag{7}$$

Conditions (7) and (1) imply that, for every a in K_1, $a \cdot d^- = 0$, and we infer that, by condition (2), $d^- \epsilon K_2$. Since $d^+ \cdot d^- = 0$, it follows that, by condition (ii) of Axiom 9.1, $y^- \cdot d^- = 0$, which is impossible by (6). Thus, $x = y^-$.

9.4. THEOREM. *If $a_0 \epsilon K$ and if $a^+ \cdot a_0^- = 0$ for every $a \epsilon K$, then $a_0^+ = \bigcup K$.*

The foregoing theorem is an immediate consequence of Axiom 9.1.

Also the proofs of the following assertions are straightforward.

9.5. THEOREM. *If, for a finite sequence of half-planes a_i^+, $i = 1, 2, \ldots$ $\ldots, n$, $a_i^+ \cdot b^+ = 0$, then there exists a positive integer k not greater than n and such that $a_i^+ \cdot a_k^- = 0$ for every $i = 1, 2, \ldots, n$.*

9.6. THEOREM. *If b^+ is a half-plane, if K is a non-empty set of half-planes a^+ such that $a^+ \cdot b^- = 0$, and if there exists a line C which cuts the regions $y \cdot \bigcup K$ and $x \cdot \bigcup K$, then there exists a half-plane $a_0^+ \epsilon K$ such that the line C cuts the regions $x \cdot a_0^+$ and $y \cdot a_0^+$.*

9.7. THEOREM. *If the net (ABC) has exactly two zero regions and $a^+ \cdot d^- \cdot (b^+ \cdot c^- + b^- \cdot c^+) + b^- \cdot c^- = 0$, then $a^+ \cdot d^- = 0$.*

9.8. THEOREM. *If b_0^+ is a half-plane, if K_1 is a set of half-planes, and if $a^+ \cdot b_0^+ = 0$ for every half-plane $a \epsilon K_1$, then $\bigcup K_1$ is a half-plane.*

Proof. Let K_2 denote the set of half-planes such that $b^+ \epsilon K_2$ if and only if $a^+ \cdot b^+ = 0$ for every half-plane $a^+ \epsilon K_1$.

The set K_2 is not empty, because $b_0^+ \epsilon K_2$.

Let x, y stand for $\bigcup K_1$ and $\bigcup K_2$ respectively. By Theorem 9.3 we infer that

$$x = y^- \quad \text{and} \quad x^- = y. \tag{1}$$

Let C and D be two different parallel lines intersecting the line B_0. We assume that

$$c^- \cdot d^- = 0. \tag{2}$$

By Theorem 4.8, the lines C and D intersect every line A determined by half-planes $a^+ \epsilon K_1$. According to Theorem 4.7, the following regions of the net (ACD) are non-zero:

$$x_1 = a^+ \cdot c^+ \cdot d^+, \quad x_2 = a^+ \cdot c^+ \cdot d^-, \quad x_3 = a^+ \cdot c^- \cdot d^+,$$

$$x_4 = a^- \cdot c^+ \cdot d^+, \quad x_5 = a^- \cdot c^+ \cdot d^-, \quad x_6 = a^- \cdot c^- \cdot d^+.$$

In the net (B_0CD), the following regions are non-zero:

$$x_1' = b_0^+ \cdot c^+ \cdot d^+, \quad x_2' = b_0^+ \cdot c^+ \cdot d^-, \quad x_3' = b_0^+ \cdot c^- \cdot d^+,$$

$$x_4' = b_0^- \cdot c^+ \cdot d^+, \quad x_5' = b_0^- \cdot c^+ \cdot d^-, \quad x_6' = b_0^- \cdot c^- \cdot d^+.$$

Consider the regions

$$x_7 = x^- \cdot c^+ \cdot d^+, \quad x_8 = x^+ \cdot c^+ \cdot d^-, \quad x_9 = x^+ \cdot c^- \cdot d^+.$$

Condition (ii) of Axiom 9.1 implies that $x^+ \cdot a^+ = a^+$. And condition (i) of Axiom 9.1 shows that $x^- \cdot b_0^+ = b_0^+$. Therefore

$$x_1' = x_7 \cdot b^+, \quad x_2 = x_8 \cdot a^+, \quad x_3 = x_9 \cdot a^+.$$

It follows that $x_7 \neq 0$, $x_8 \neq 0$, and $x_9 \neq 0$.

Suppose there exists a line E which cuts all the three regions x_7, x_8, x_9. By condition (ii) of Axiom 9.1, the line E cuts the region x_4 for every $a^+ \epsilon K_1$, and, by Theorem 9.6, there exists a half-plane $a^+ \epsilon K_1$ such that the line E cuts the regions x_2, x_3. Thus, we infer that there exists a half-plane $a^+ \epsilon K_1$ such that the line E cuts all the three regions x_2, x_3, x_4, which is impossible by Theorem 4.11. Therefore, by Axiom 2.2, there exists a line E separating the region x_7 from the region $x_8 + x_9$. Hence, labelling properly the sides of the line E, we can claim that

$$x^- \cdot c^+ \cdot d^+ \cdot e^+ + (c^+ \cdot d^- + c^- \cdot d^+) \cdot e^- \cdot x = 0. \tag{3}$$

By condition (i) of Axiom 9.1 it follows that, for every half-plane $a^+ \epsilon K_1$,

$$(c^+ \cdot d^- + c^- \cdot d^+) \cdot a^+ \cdot e^- = 0. \tag{4}$$

Conditions (2) and (4) imply that, by Theorem 9.7, $a^+ \cdot e^- = 0$. Thus, by definition of the set K_2, it follows that

$$e^- \epsilon K_2. \tag{5}$$

We shall show that, for every $b^+ \epsilon K_2$,

$$b^+ \cdot e^+ = 0. \tag{6}$$

Let us suppose that condition (6) fails; thus, there exists a half-plane $b_1 \epsilon K_2$ such that

$$b_1^+ \cdot e^+ \neq 0. \tag{7}$$

By definition of the set K_2 it follows that, for every $b_1, b_2 \epsilon K_2$, either $b_1^+ \cdot b_2^- = 0$ or $b_1^- \cdot b_2^+ = 0$. Thus, according to (5) and (7), it follows that

(8) $$b_1^- \cdot e^- = 0.$$

By construction of the lines C, D, by Theorem 4.6 and by conditions (2) and (8), it follows that the net (CB_1DE) belongs to the class R, and thus we infer that, by Theorem 5.7,

(9) $$c^+ \cdot d^+ \cdot e^+ \cdot b_1^+ = 0.$$

By (9) and by condition (i) of Axiom 9.1 it follows that

$$c^+ \cdot d^+ \cdot e^+ \cdot y \neq 0;$$

thus, by (1),

$$c^+ \cdot d^+ \cdot e^+ \cdot x^- \neq 0,$$

which is impossible according to (3). Therefore, condition (6) holds.

Conditions (5) and (6) imply that, by Theorem 9.4,

$$e^- = y^+.$$

This relationship and condition (1) show that $e^+ = x^+$, which completes the proof.

In order to establish the continuity axiom, we shall need the following theorem which is an immediate corollary of 7.7:

9.9. THEOREM. *For every point $\{AB\}$ and for every line C, at least one of the following conditions holds:*

(i) *the lines A, B, C are tied,*

(ii) $\{AB\} \epsilon c^+$,

(iii) $\{AB\} \epsilon c^-$.

9.10. THEOREM (The axiom of continuity). *If*

(1) *Z_1 and Z_2 are non-empty sets of points lying on the line A,*

(2) *a line B intersects the line A at a point M such that, for every point $P \epsilon Z_1$ and for every point $Q \epsilon Z_2$, the relation $\beta(MPQ)$ holds,*

then there exists a point S such that the relation $\beta(PSQ)$ is satisfied for every point $P \epsilon Z_1$ and for every point $Q \epsilon Z_2$ which are different from S.

Proof. Since the set Z_2 is not empty, we can choose in it a point Q. By hypothesis and by Theorem 9.9, we can label the sides of the line B so that

(3) $$Q \epsilon b^-.$$

By hypothesis, it follows that, for every point $P \epsilon Z_1$,

(4) $$P \epsilon b^-.$$

For each point $P \epsilon Z_1$ let us draw a line C passing through P and parallel to B. By (4), we can label the sides of the line C so that $b^+ \cdot c^- = 0$. That way we obtain a set of half-planes c^+ which we denote by K_1. It follows that, for every half-plane $c^+ \epsilon K_1$,

(5) $$b^+ \cdot c^- = 0.$$

In a similar way, we draw lines D parallel to B through points $Q \epsilon Z_2$. We label the sides of the lines D so that $M \epsilon d^+$; this is possible by hypothesis. And let K_2 stand for the set of all half-planes d^+. It follows that, for every half-plane $d^+ \epsilon K_2$,

(6) $$M \epsilon d^+.$$

The assumption $\beta(MPQ)$ implies that, by (5), for every point $P \epsilon Z_1$ and for every point $Q \epsilon Z_2$,

(7) $$c^+ \cdot d^- = 0$$

or

(8) $$c^+ \cdot d^+ = 0.$$

Since $P \epsilon d$ by hypothesis and by (6), we have $c^+ \cdot d^+ \neq 0$. And hence, condition (7) follows.

In particular, condition (7) holds for some fixed half-plane d^+, thus, by Theorem 9.8, there exists a half-plane $e^+ = \bigcup K_1$.

Condition (7) implies that, by condition (i) of Axiom 9.1,

(9) $$e^+ \cdot d^- = 0.$$

According to condition (ii) of Axiom 9.1, it follows that

(10) $$e^- \cdot c^+ = 0.$$

If we let stand S for the point $\{AE\}$, conditions (9) and (10) imply that $\beta(PSQ)$.

Summing up, we have shown that the addition of Axiom 9.1 to the set of axioms given in Sections 1 and 2 provides us with a theory which is, together with some later definitions, a model of non-Desarguean geometry containing the continuity axiom.

PART II

1. Axioms. By the non-Desarguean geometry I mean a theory, whose primitive notions are points and lines, denoted in the sequel by lower case and capital letters respectively, and relations of incidence and betweenness, denoted by lower case Greek letters ε, β respectively.

The symbol $\sim\varepsilon$ will stand for the negation of the relation ε, and $\sim\beta$ will stand for the negation of the relation β.

I assume the following system of axioms for the non-Desarguean geometry (in fact, this is an inessential variation of the axioms introduced by Borsuk and Szmielew in [1]):

1.1. AXIOM. For every line A, there exist two different points a and b such that $a, b \varepsilon A$.

1.2. AXIOM. For every two points a and b, there exists at least one line A such that $a, b \varepsilon A$.

1.3. AXIOM. If the points a and b are distinct, then there exists at most one line A such that $a, b \varepsilon A$.

1.4. AXIOM. There exist three points a, b, c such that if $a, b \varepsilon A$, then $c \sim \varepsilon A$.

1.5. AXIOM. If $\beta(abc)$, then the points a, b, c are distinct and there exists a line A such that $a, b, c \varepsilon A$.

1.6. AXIOM. If $\beta(abc)$, then $\beta(cba)$.

1.7. AXIOM. If $\beta(abc)$, then $\sim \beta(bca)$.

1.8. AXIOM. If the points a and b are distinct, then there exists a point c such that $\beta(abc)$.

1.9. Definition. The line B is said *to lie between the points a and c*, written $\beta(aBc)$, if and only if $a, c \sim \varepsilon B$ and $\beta(abc)$ for some point $b \varepsilon B$.

1.10. AXIOM. If the points a, b, c are distinct and $\beta(aLb)$, one of the following conditions holds:

(1) $$\beta(bLc),$$

(2) $$\beta(cLa),$$

(3) $$c \varepsilon L.$$

1.11. AXIOM. If $b \sim \varepsilon A$, there exists at most one line B such that $b \varepsilon B$ and $x \sim \varepsilon A$ for every point $x \varepsilon B$.

1.12. AXIOM. If Z_1 and Z_2 are non-empty sets of points such that, for every point $p \epsilon Z_1$ and for every point $q \epsilon Z_2$, the relation $\beta(apq)$ holds, then there exists a point c such that the relation $\beta(pcq)$ is satisfied for every point $p \epsilon Z_1$ and for every point $q \epsilon Z_2$ which are different from c.

Axioms 1.1-1.4 are called *incidence axioms*, Axioms 1.5-1.8 are called the *axioms of betweenness*, Axiom 1.10 is called the *Pasch's axiom*, Axiom 1.11 is called the *Euclid's axiom*, and Axiom 1.12 is called the *axiom of continuity*.

2. Definitions and corollaries. This paragraph contains a number of definitions and corollaries to our axioms, which will be used in the next paragraphs.

2.1. Definition. The points a and b are said *to lie on the same side of the line L* if and only if $\sim \beta(aLb)$.

2.2. Definition. The points a and b are said *to lie on different sides of the line* L if and only if $\beta(aLb)$.

2.3. Definition. The line L is said *to cut the point set* Z if and only if there exist points $a, b \epsilon Z$ such that $\beta(aLb)$.

2.4. Definition. The point set Z is said to be *convex* if the conditions (1) $a, c \epsilon Z$ and (2) $\beta(abc)$ imply that $b \epsilon Z$.

2.5. Definition. A convex point set Z is said to be *open* if the conditions (1) $a \epsilon Z$ and (2) $a \varepsilon L$ imply that the line L cuts the set Z.

In particular, the empty set is said to be open.

2.6. Definition. An arbitrary set Z is said to be *open* if it is the union of open convex sets.

2.7. COROLLARY. *If* $\beta(aLb)$, *then exactly one of the following conditions holds:*

(1) $$\beta(bLc),$$

(2) $$\beta(cLa),$$

(3) $$c \varepsilon L.$$

2.8. COROLLARY. *If* $\sim \beta(aLb)$ *and* $\sim \beta(bLc)$, *then* $\sim \beta(cLa)$.

In the sequel, Corollaries 2.7 and 2.8, which are derived from the Pasch's axiom, will also be referred to as *Pasch's axiom*.

3. Convex of a set.

3.1. Definition. For an arbitrary set x, the symbol Sx will denote such a set that $p \epsilon Sx$ if and only if $p \epsilon x$ or if there exist points $a, b \epsilon x$ such that $\beta(apb)$.

3.2. Definition. The set SSx is referred to as the convex of the set x.

E.g., if x is the set containing just three points a, b, c which are not collinear, Sx is the set which is called the *boundary* of the triangle determined by the vertices a, b, c, and SSx is the set which is called the *closed triangle* determined by the vertices a, b, c.

3.3. THEOREM. *If the point* p *satisfies the following conditions:*

(i) $$p \varepsilon A,$$

(ii) $$p \epsilon SSx,$$

(iii) $$p \sim \epsilon Sx,$$

then there exist points a *and* b *such that*

(iv) $$a, b \varepsilon A,$$

(v) $$\beta(apb),$$

(vi) $$a, b \epsilon Sx,$$

(vii) *at most one of the points* a, b *belongs to* x.

Proof. Condition (ii) implies that, by Definition 3.2, there exist points $c, d \in Sx$ such that $\beta(cpd)$. If $c, d \varepsilon A$, then conditions (iv), (v) and (vi) are satisfied.

If only one of the points c, d is incident with the line A, then either (1) one of the points c, d belongs to x or (2) none of the points c, d belongs to x.

In the case (1), if $c \in x$ and $d \sim\in x$, then, by Definition 3.1, there exist points $e, f \in x$ such that $\beta(edf)$. The line A intersects the common side cd of the triangles cde, cdf at the point p. Therefore, by Pasch's axiom, A intersects the boundary of the triangle cef at two different points a and b. Since $c, e, f \in x$, hence $a, b \in Sx$. Since a, b belong to different half-planes determined by the line A, we have $\beta(apb)$.

In the case (2), there exist points $e, f, g, h \in x$ such that $\beta(ecf)$ and $\beta(gdh)$. If one of the points e, f, g, h lies on the line cd, the case (2) reduces to the case (1). If none of these points lies on the line cd, let us assume that the points e, g lie on the same side of that line. If the line A intersects the side cd of the triangle cde, then A intersects the polygonal line cde by Pasch's axiom. If A does not intersect the side ce, it passes through the point e or intersects the side ed. Hence, in the latter case, A intersects the polygonal line egd in the triangle egd by Pasch's axiom. Therefore, A certainly intersects at least one of the segments ce, eg, gd or passes through one of the points e, g. Let a denote the point of intersection or a point of intersection, if there are many of them.

A similar argument shows that the line A intersects one of the segments cf, fh, hd or passes through one of the points f, h. Let b denote the point of intersection or a point of intersection, if there are many of them. Since the points e, f, g, h belong to x, we get $a, b \in Sx$. The points a, b belong to different half-planes determined by the line cd, and therefore, $\beta(apb)$. Thus, we have proved that there exist points a, b which satisfy conditions (iv), (v) and (vi). Condition (vii) immediately follows from (iii), (v) and (vi).

By Theorem 3.3, it is easy to prove the following assertions:

3.4 THEOREM. *The convex of an arbitrary set is a convex set.*

3.5 THEOREM. *The convex of an open set is an open set.*

3.6 THEOREM. *If x is an open set and if the line A cuts the convex SSx, then A cuts the set x.*

4. Properties of relations of betweenness and of being parallel.

4.1. Definition. The symbol $L(ab)$ will stand for the line determined by the points a and b. $\boldsymbol{H}(ab)$ will denote the half-line lying on the line $L(ab)$, with origin at the point a, and passing through the point b; on the other hand, $\boldsymbol{H}^*(ab)$ will denote the half-line lying on the line $L(ab)$, with origin at the point a, and which does not pass through the point b. It follows that

$$L(ab) = \boldsymbol{H}(ab) + a + \boldsymbol{H}^*(ab).$$

We introduce the following definition of parallel half-lines:

4.2. Definition. The half-lines $\boldsymbol{H}(ab)$ and $\boldsymbol{H}(cd)$ are said to be *parallel* if one of the following conditions is satisfied:

(1) the half-lines $\boldsymbol{H}(ab)$, $\boldsymbol{H}(cd)$ lie on the same line and $\boldsymbol{H}(ab) \subset \boldsymbol{H}(cd)$ or $\boldsymbol{H}(cd) \subset \boldsymbol{H}(ab)$;

(2) the half-lines lie on parallel lines and $\sim\beta\big(bL(ac)d\big)$.

In the sequel we give some theorems implied by axioms of Euclid and Pasch. There will be no proofs, but most of the theorems will be illustrated with appropriate figures.

4.3. THEOREM. *If* $L(ab)\|L(cd)$ *and* $L(cd)\|L(ef)$, *then* $L(ab)\|L(ef)$.

4.4. THEOREM. *If one of conditions* (1) *and* (2) *is satisfied,*

(1) $$L(ab)\|L(cd),$$

(2) $$\sim\beta(cL(ab)d) \quad \textit{and} \quad \sim\beta(aL(cd)b),$$

and the following conditions are satisfied

(3) $$e \varepsilon L(cb),$$

(4) $$\beta(aed),$$

then

(5) $$\beta(bec).$$

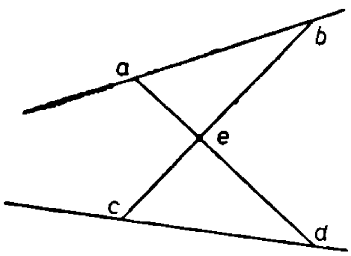

Fig. 1

4.5. THEOREM. *If* $\boldsymbol{H}(ab)\|\boldsymbol{H}(cd)$, *then* $\boldsymbol{H}^*(ab)\|\boldsymbol{H}^*(cd)$.

4.6. THEOREM. *If* $\beta(abc)$ *and the half-line* $\boldsymbol{H}(pq)$ *is parallel to one of the half-lines* $\boldsymbol{H}(ab)$, $\boldsymbol{H}(ac)$, $\boldsymbol{H}(bc)$, *then it is parallel to both of the remaining two half-lines.*

4.7. THEOREM. *If* $\beta(abc)$, *if* $e \varepsilon L(df)$, *and if one of conditions* (1), (2) *is satisfied,*

(1) $$L(ad)\|L(be)\|L(cf),$$

(2) *the lines $L(ad)$, $L(be)$, $L(cf)$ belong to the same pencil with the vertex p and, at the same time, either $d\varepsilon \boldsymbol{H}(pa)$, $e\varepsilon \boldsymbol{H}(pb)$ and $f\varepsilon \boldsymbol{H}(pc)$ or $d\varepsilon \boldsymbol{H}^*(pa)$, $e\varepsilon \boldsymbol{H}^*(pb)$ and $f\varepsilon \boldsymbol{H}^*(pc)$, then $\beta(def)$.*

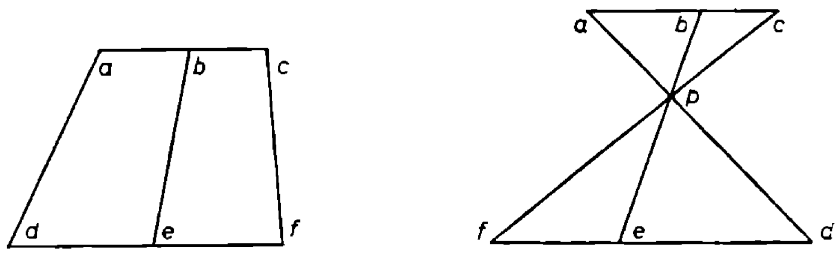

Fig. 2

4.8. THEOREM. *If*

$$c\varepsilon L(ab), \quad c\varepsilon L(de), \quad g, f\varepsilon L(ae),$$

$$\boldsymbol{H}(ab) \parallel \boldsymbol{H}(fd), \quad \boldsymbol{H}(af) \parallel \boldsymbol{H}(bd), \quad \boldsymbol{H}(bg) \parallel \boldsymbol{H}(de),$$

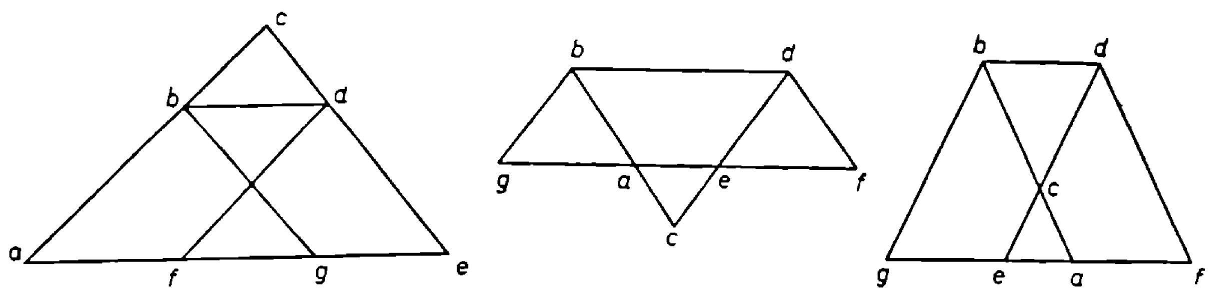

Fig. 3

then the following implications hold:

$$\beta(abc) \rightarrow \beta(age) \text{ and } \beta(afe),$$

$$\beta(bac) \rightarrow \beta(aef) \text{ and } \beta(gae),$$

$$\beta(acb) \rightarrow \beta(aeg) \text{ and } \beta(gaf).$$

4.9. THEOREM. *If*

$$L(ab) \parallel L(fd), \quad L(af) \parallel L(bd), \quad A \parallel B,$$

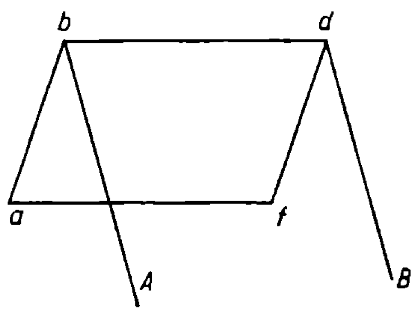

Fig. 4

$$b \varepsilon A, \quad a \sim \varepsilon A, \quad d \varepsilon B, \quad f \sim \varepsilon B,$$

then exactly one of the following conditions holds:

(1) $$\beta(aAd),$$

(2) $$\beta(bBf).$$

4.10. THEOREM. *If*

$$L(ab) \| L(de), \quad L(ac) \| L(df), \quad L(ad) \| L(be), \quad L(ad) \| L(cf),$$

$$A \| B, \quad a \varepsilon A, \quad d \varepsilon B, \quad \textit{and} \quad \beta(bAc),$$

then $\beta(cBf)$ (see Fig. 5).

4.11. THEOREM. *If*

$$\boldsymbol{H}(pa) \| \boldsymbol{H}(qd), \quad \boldsymbol{H}(pc) \| \boldsymbol{H}(qe), \quad A \| B, \quad \beta(abc),$$

$$p, b \varepsilon A, \quad q \varepsilon B,$$

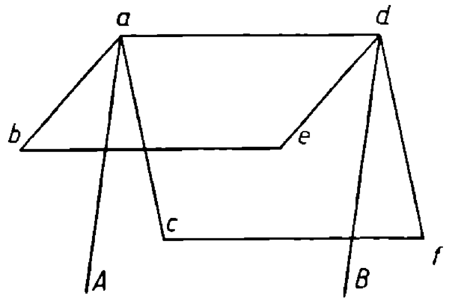

Fig. 5

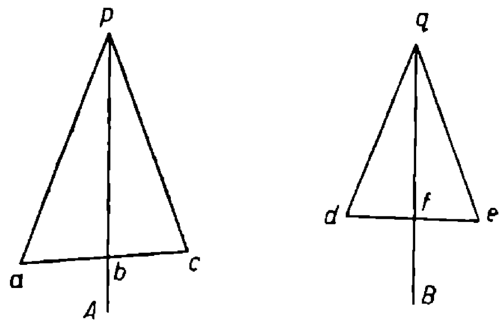

Fig. 6

then there exists a point f such that $f \varepsilon B$ and $\beta(dfe)$.

4.12. THEOREM. *If*

$$L(ab) \| L(cd), \quad L(ac) \| L(bd),$$

then $\beta(bL(ad)c)$ (see Fig. 7).

4.13. THEOREM. *If*

$$L(ab) \| L(cd), \quad L(ac) \| L(bd), \quad L(ab) \| L(ef), \quad L(ae) \| L(bf),$$

then $\sim \beta(dL(ce)f)$.

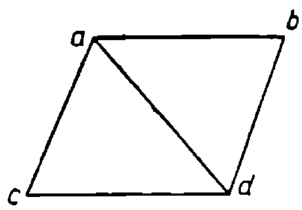

Fig. 7

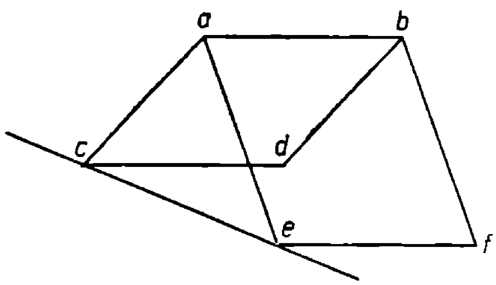

Fig. 8

4.14. THEOREM. *The relation of being parallel between half-lines is transitive.*

We introduce the betweenness relation for parallel lines by the following definition:

4.15. Definition. The line B is said *to lie between the lines A and C*, written $\beta(ABC)$, if and only if the lines A, B, C are distinct and parallel, and $\beta(abc)$ holds for every line intersecting the lines A, B, C at the points a, b, c respectively.

The following assertions can be proved:

4.16. THEOREM. *If the lines A, B, C are distinct and parallel, then the relation $\beta(ABC)$ holds if and only if there exists a line intersecting the lines A, B, C at the points a, b, c respectively so that $\beta(abc)$.*

4.17. THEOREM. *If the lines A, B, C are distinct and parallel, then exactly one of the following conditions holds:*

(1) $$\beta(ABC),$$

(2) $$\beta(BAC),$$

(3) $$\beta(ACB).$$

5. Hodograph.

5.1. Definition. A set Z of half-lines $\boldsymbol{H}(pa)$ with the fixed origin p is said to be *convex* if and only if, for every two half-lines $\boldsymbol{H}(pa)$, $\boldsymbol{H}(pc)$ belonging to the set Z and not lying on the same line, and for every point b satisfying the condition $\beta(abc)$, the half-line $\boldsymbol{H}(pb)$ belongs to the set Z.

5.2. Definition. A set Z of half-lines $\boldsymbol{H}(pb)$ with the fixed origin p is said to be *open* if, for every half-line $\boldsymbol{H}(pb)$ belonging to the set Z, there exist half-lines $\boldsymbol{H}(pa)$ and $\boldsymbol{H}(pc)$ belonging to the set Z and such that $\beta(aL(pb)c)$.

5.3. Definition. By the *hodograph* of the set x_1 on the set x_2 with the pole q we mean such a set of half-lines, written $H(qx_1x_2)$, that $\boldsymbol{H}(qp) \in H(qx_1x_2)$ if and only if there exist points $a_1 \in x_1$ and $a_2 \in x_2$ such that $\boldsymbol{H}(qp) \| \boldsymbol{H}(a_1a_2)$.

5.4. THEOREM. *If $\boldsymbol{H}(qp) \in H(qx_1x_2)$ and if the sets x_1, x_2 are open, convex and non-empty, then there exist arbitrary many half-lines $\boldsymbol{H}(a_1a_2)$, $\boldsymbol{H}(b_1b_2)$, $\boldsymbol{H}(c_1c_2)$, ... lying on different lines and such that $a_1, b_1, c_1, \ldots \in x_1$, $a_2, b_2, c_2, \ldots \in x_2$,* and $\boldsymbol{H}(a_1a_2) \| \boldsymbol{H}(b_1b_2) \| \boldsymbol{H}(c_1c_2) \ldots \| \boldsymbol{H}(qp)$.

Proof. By definition of hodograph, there exists a half-line $\boldsymbol{H}(a_1a_2)$ parallel to the half-line $\boldsymbol{H}(qp)$ and such that $a_1 \in x_1$ and $a_2 \in x_2$. Since the sets x_1, x_2 are open, there exist points $b_1 \in x_1$ and $b_2 \in x_2$ such that $\sim\beta(b_1L(a_1a_2)b_2)$. Using the included figures, we can easily show that the conclusion follows by Theorems 4.16, 4.15, 4.5 and by Pasch's axiom.

5.5. THEOREM. *If* $\boldsymbol{H}(qp)$, $\boldsymbol{H}^*(qp) \in \boldsymbol{H}(qx_1x_2)$, *and if the sets* x_1, x_2 *are open and convex, then* x_1 *and* x_2 *have a common point.*

Proof. Since the relation of being parallel between half-lines is transitive, it follows that, by Theorem 5.4, there exist points $a_1, b_1 \in x_1$

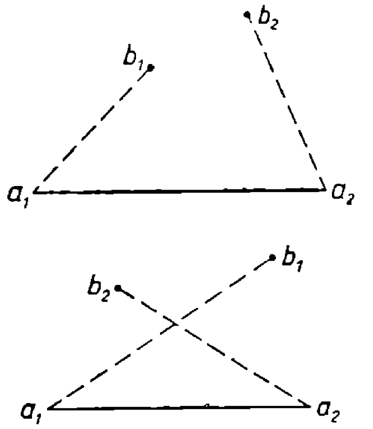

Fig. 9

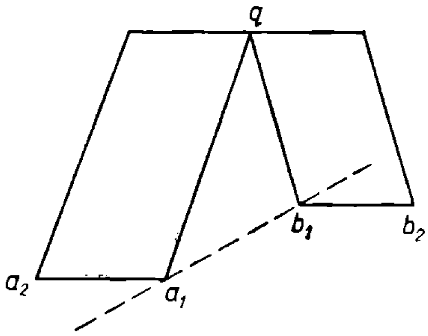

Fig. 10

and $a_2, b_2 \in x_2$ such that $\boldsymbol{H}(a_1a_2) \| \boldsymbol{H}(qp)$, $\boldsymbol{H}(b_1b_2) \| \boldsymbol{H}^*(qp)$, $L(a_1a_2) \neq L(b_1b_2)$ and $\beta\big(a_2L(a_1b_1)b_2\big)$. By these conditions, there exists a point $c \in x_2$ such that $c \in L(a_1b_1)$. By Theorem 4.4, it follows that $\beta(a_1cb_1)$. Hence, $c \in x_1$.

5.6. THEOREM. *If the sets* x_1, x_2, *having no common point, are open, convex and non-empty, then the hodograph is an open convex set of half-lines.*

Proof. Let us assume that $\boldsymbol{H}(qp_1)$, $\boldsymbol{H}(qp_2) \in H(qx_1x_2)$ and that $\beta(p_1pp_2)$. By Theorem 5.5, $L(qp_1) \neq L(qp_2)$. There exist points $a_1, b_1 \in x_1$ and $a_2, b_2 \in x_2$ such that $\boldsymbol{H}(a_1a_2) \| \boldsymbol{H}(qp_1)$, $\boldsymbol{H}(b_1b_2) \| \boldsymbol{H}(qp_2)$ and the lines $L(a_1a_2)$, $L(b_1b_2)$ intersect at the point c. By Theorem 5.4, it can be assumed that the points a_1, a_2, b_1, b_2, c are distinct, and that none of them lies on any of the lines $L(qp_1)$, $L(qp_2)$, $L(qp)$.

Only the following cases presented in the figures below are possible:

(i)	$\beta(a_1ca_2)$	and	$\beta(b_1cb_2)$,	(ii)	$\sim\beta(a_1ca_2)$	and	$\sim\beta(b_1cb_2)$,
(iii)	$\beta(a_1ca_2)$	and	$\beta(b_2b_1c)$,	(iv)	$\beta(b_1cb_2)$	and	$\beta(a_2a_1c)$,
(v)	$\beta(a_1ca_2)$	and	$\beta(b_1b_2c)$,	(vi)	$\beta(b_1cb_2)$	and	$\beta(a_1a_2c)$.

In the case (i) it follows that, by Theorem 4.6,

(1) $$\boldsymbol{H}(ca_2) \| \boldsymbol{H}(qp_1), \quad \boldsymbol{H}(cb_2) \| \boldsymbol{H}(qp_2),$$

and, by Theorem 4.11, there exists a point d_2 such that

(2) $$\beta(a_2d_2b_2)$$

and

(3) $$\boldsymbol{H}(qp) \| \boldsymbol{H}(cd_2).$$

By Theorem 4.7, conditions (i) and (2) imply that the lines $L(a_1b_1)$, $L(cd_2)$ intersect at such a point d_1 that

(4) $$\beta(a_1d_1b_1).$$

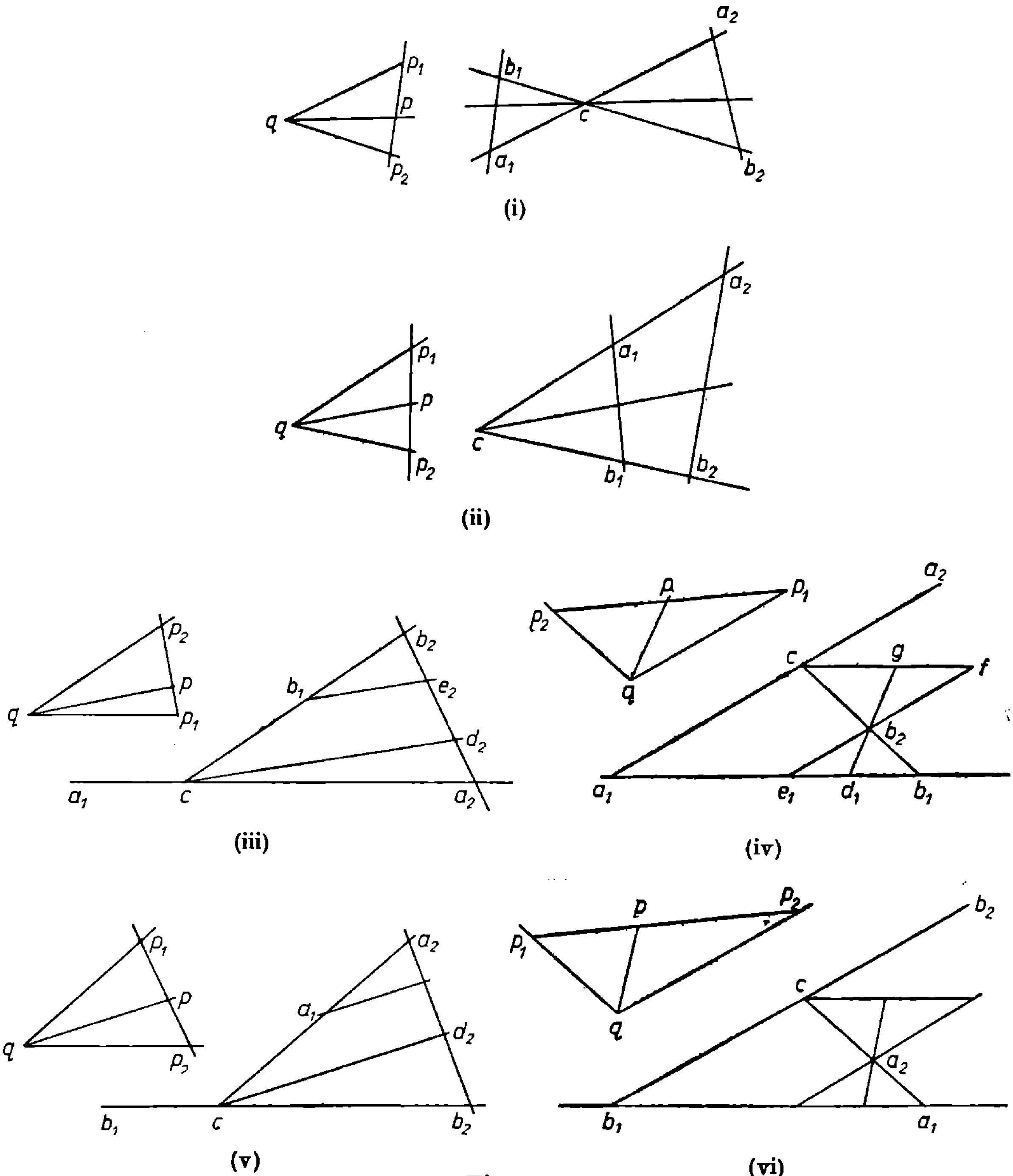

Fig. 11

By hypothesis, the segments a_1b_1, a_2b_2 are disjoint, therefore, the segments a_2d_2, a_1d_1 are disjoint, and, by Theorem 4.4, it follows that

(5) $$\beta(d_1cd_2).$$

From conditions (2) and (4) it follows that $d_1 \in x_1$, $d_2 \in x_2$, and from conditions (3) and (5) we infer that, by Theorem 4.6, $\boldsymbol{H}(d_1 d_2) \| \boldsymbol{H}(qp)$, and hence, $\boldsymbol{H}(qp) \in H(qx_1x_2)$.

In the case (ii) the argument is quite similar.

In the case (iii), it follows that, by Theorem 4.6, $\boldsymbol{H}(ca_2) \| \boldsymbol{H}(qp_1)$ and $\boldsymbol{H}(cb_2) \| \boldsymbol{H}(qp_2)$.

According to Theorem 4.11, there exists a point d_2 such that

(1) $$\beta(a_2 d_2 b_2)$$

and

(2) $$\boldsymbol{H}(cd_2) \| \boldsymbol{H}(qp).$$

Through the point b_1 we draw a line parallel to $L(cd_2)$. From the triangle b_2d_2c, it follows that, by condition (iii) and by Pasch's axiom, this line intersects the side d_2b_2 at a point e_2. Since $\beta(b_2e_2d_2)$, it follows that $e_2 \in x_2$ by condition (1) and by convexity of the set x_2. The points d_2, e_2 lie on the same side of the line $L(cb_2)$, and hence, $\boldsymbol{H}(cd_2) \| \boldsymbol{H}(b_1e_2)$. This implies that, by Theorem 4.14, $\boldsymbol{H}(qp) \| \boldsymbol{H}(b_1e_2)$. Since $b_1 \in x_1$ and $e_2 \in x_2$, it follows that $\boldsymbol{H}(qp) \in \boldsymbol{H}(qx_1x_2)$.

In the case (iv), we argue in the similar way.

In the case (v), we draw lines parallel to $L(a_1b_1)$ and $L(a_1a_2)$ through the points c and b_2 respectively. By conditions (v), by Pasch's axiom and by Theorem 4.4, we infer that the point f, at which the constructed lines intersect, and the point e_1, at which the latter of the constructed lines intersects the line $L(a_1b_1)$, satisfy the conditions

(1) $$\beta(e_1 b_2 f)$$

and

(2) $$\beta(a_1 e_1 b_1).$$

By Theorem 4.6 it follows that

(3) $$\boldsymbol{H}(qp_1) \| \boldsymbol{H}(a_1c).$$

By conditions (v) and by Theorem 4.6 it follows that

(4) $$\boldsymbol{H}(e_1b_2) \| \boldsymbol{H}(a_1c).$$

From conditions (4) and (1) and from Theorem 4.6 we derive

(5) $$\boldsymbol{H}(a_1c) \| \boldsymbol{H}(b_2f).$$

From (3) and (5) and from Theorem 4.14 it follows that

(6) $$\boldsymbol{H}(b_2f) \| \boldsymbol{H}(qp_1).$$

Conditions (v) and Theorem 4.6 imply that

(7) $$\boldsymbol{H}(b_2c) \| \boldsymbol{H}(qp_2).$$

According to (6) and (7), it follows that, by Theorem 4.9, there exists a point g lying on the line $L(cf)$ such that

(8) $$\beta(cgf),$$

(9) $$\boldsymbol{H}(b_2g) \| \boldsymbol{H}(qp).$$

We infer from Theorem 4.7 that there exists a point d_1 such that

(10) $$\beta(e_1d_1b_1) \quad \text{and} \quad d_1 \epsilon L(b_2g).$$

By Theorem 4.4 it follows that

(11) $$\beta(d_1b_2g).$$

Conditions (10) and (2) imply that $d_1 \epsilon x_1$, and conditions (11) and (9) imply that $\boldsymbol{H}(d_1b_2) \| \boldsymbol{H}(qp)$. Therefore, $\boldsymbol{H}(qp) \epsilon H(qx_1x_2)$.

In the case (vi) the proof is similar.

Thus, we have established the convexity of the hodograph. And it is an open set by the fact that x_2 is open and by Theorem 4.11.

6. Jaśkowski's theorem. Theorem 3.3 implies the following assertion:

6.1. THEOREM. *If a is any point belonging to the convex of the union of two open convex, non-empty sets, there exists a line which passes through a and cuts both of these sets.*

6.2. THEOREM. *If the open convex, non-empty sets x_1, x_2 have no point in common, then there exists a line A passing through a fixed point q and such that none of the half-lines determined by q on the line A belongs to the hodograph $H(qx_1x_2)$.*

Proof. Theorem 5.6 implies the following properties:

(1) the set $H(qx_1x_2)$ is open;

(2) the set $H(qx_1x_2)$ is convex.

Therefore, there exist half-lines $\boldsymbol{H}(qp_1), \boldsymbol{H}(qp_2) \epsilon H(qx_1x_2)$ such that $L(qp_1) \neq L(qp_2)$.

It is known that there exists a point p_3 such that $\beta(p_2pp_3)$. By Theorem 5.5, it follows that $\boldsymbol{H}(qp_3) \sim\epsilon H(qx_1x_2)$. Consider two sets of points lying on the line $L(p_1p_3)$ which are defined in the following way.

The point $p_1 \epsilon Z_1$. A point $p \epsilon Z_1$ if the two conditions are satisfied, $\beta(p_1pp_3)$ and $\boldsymbol{H}(qp) \epsilon H(qx_1x_2)$. The point $p_3 \epsilon Z_2$ and a point $p \epsilon Z_2$ if $\beta(p_1pp_3)$ and $\boldsymbol{H}(qp) \sim \epsilon \boldsymbol{H}(qx_1x_2)$.

If $p \epsilon Z_1$, $s \epsilon Z_2$, and if $\beta(p_1sp)$, then it follows that, by Theorem 5.6, $\boldsymbol{H}(qs) \epsilon H(qx_1x_2)$, which contradicts the definition of the set Z_2. Thus, for every $p \epsilon Z_1$ and for every point $s \epsilon Z_2$, it follows that $\beta(p_1ps)$.

The sets Z_1, Z_2 determine a Dedekind cut. By the axiom of continuity, there exists a point p_0 determined by this cut and such that

(3) conditions $p \neq p_1$ and $p \neq p_0$ imply that $p \in Z_1$ if and only if $\beta(p_1 p p_0)$,

(4) conditions $p \neq p_0$ and $p \neq p_3$ imply that $p \in Z_2$ if and only if $\beta(p_0 p p_3)$.

Suppose that $\boldsymbol{H}(qp_0) \in H(qx_1x_2)$. Therefore, $p_0 \neq p_3$.

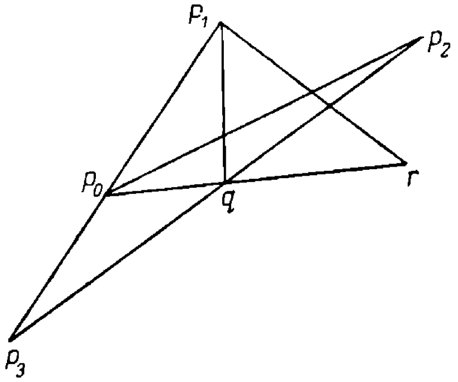

Fig. 12

Condition (1) implies that the hodograph contains a half-line $\boldsymbol{H}(qp_4)$ such that $\sim\beta(p_3 L(qp_0) p_4)$. The line $L(qp_4)$ intersects the side p_2p_3 of the triangle $p_0p_2p_3$. Therefore, it intersects, by Pasch's axiom, either (i) the side p_0p_3 or (ii) side p_0p_2.

In the case (i), the intersection point s of $L(qp_4)$ with the side p_0p_3 lies on the same side of the line $L(qp_0)$ as the point p_3, and hence, on the same side as the point p_4. In that case, $\boldsymbol{H}(qs) = \boldsymbol{H}(qp_4) \in H(qx_1x_2)$, and simultaneously $\beta(p_0 s p_3)$, which contradicts (4).

In the case (ii), if s stands for the intersection point of the line $L(qp_4)$ with the side p_0p_2, condition (2) implies that $\boldsymbol{H}(qs) \in H(qx_1x_2)$. Since $\boldsymbol{H}(qp_4) \in H(qx_1x_2)$ by hypothesis, and since $\beta(sqp_4)$, the sets x_1, x_2 have a point in common by Theorem 5.5, which contradicts the hypothesis.

Therefore, $\boldsymbol{H}(qp_0) \sim \in H(qx_1x_2)$.

Let r denote any point such that $\beta(p_0 q r)$. By (1), there exists in the hodograph $H(qx_1x_2)$ a half-line $\boldsymbol{H}(qp_5)$ such that $\beta(p_1 L(qp_0) p_5)$. The line $L(qp_5)$ intersects the side rp_0 of the triangle rp_0p_1, and hence, by the Pasch's axiom, one of the following three conditions holds: $p_1 \varepsilon L(qp_5)$, $\beta(p_0 L(qp_5) p_1)$, $\beta(p_1 L(qp_5) r)$.

In every case, the assumption about the line $L(qp_5)$ and condition (2) imply that the hodograph $H(qx_1x_2)$ contains two half-lines which lie on the same line and have opposite orientations, which is impossible by Theorem 5.5.

Therefore, no half-line lying on the line $L(qp_0)$ belongs to the hodograph $H(qx_1x_2)$.

6.3. Definition. The line A is said to be a *supporting line of an open convex, non-empty set x*, if the following conditions are satisfied:

(1) the line A does not cut the set x;

(2) for every point a belonging to the set x, for every point b which is incident with the line A, and for every point c such that $\beta(acb)$, the line C, passing through the point c and parallel to the line A, cuts the set x.

Using Pasch's axiom, we can easily prove the following assertion:

6.4. THEOREM. *The line A is a supporting line of a set x if and only if the following conditions are satisfied*:

(1) *the line A does not cut the set x;*

(2) *there exist a point $a\varepsilon A$ and a point $b \epsilon x$ such that, for every point c satisfying condition $\beta(acb)$, the line C passing through the point c and parallel to the line A, cuts the set x.*

6.5. Definition. If, for every point $a \epsilon x_1$ and for every point $b \epsilon x_2$, the line C satisfies condition $\beta(aCb)$, then C is said *to separate the sets x_1 and x_2*.

6.6. THEOREM. *If the line A does not cut the open convex set x and is not a supporting line of it, there exists a line B which is parallel to A, supports the set x and separates the set x from the line A.*

Proof. Let p be an arbitrary point of the set x, and let a be an arbitrary point of the line A.

We consider the points of the closed segment pa and divide them into two classes K_1, K_2. We include in the class K_1 the points of the segment pa such that the lines, passing through them and parallel to the line A, cut the set x. The class K_1 is not void, because $p \epsilon K_1$. We include in the class K_2 all other points of the segment ap. The class K_2 is not void, because $a \epsilon K_2$. By assumption that the set x is convex, it follows that, for every point $q \epsilon K_1$ and for every point c such that $\beta(pcq)$, there holds the relation $c \epsilon K_1$. Therefore, it follows that, for every $q \epsilon K_1$ and for every $c \epsilon K_2$, $\beta(pqc)$.

By the continuity axiom, there exists such a point b that $\beta(pba)$ and that the following implications hold:

(1) if $\beta(pqb)$, then $q \epsilon K_1$,

(2) if $\beta(bca)$, then $c \epsilon K_2$.

Through the point b we draw the line B parallel to the line A. If the line B cuts the set x, there exists a point $q \epsilon x$ such that $\beta(pBq)$. There are only three possibilities,

(3) $\beta(pAq)$,

(4) $q\varepsilon A$,

(5) there exists a point $a_1 \varepsilon A$ such that $\beta(a_1 qb)$.

In the case (3) or (4), the line A cuts the set x, which contradicts the hypothesis. In the case (5), line Q, passing through the point q and parallel to the line A, intersects the segment ba at the point $c \in K_1$ by the Pasch's axiom, which is impossible by condition (2). Thus, the line B does not cut the set x.

On the other hand, every line, passing through the points of the segment pb and parallel to the line A, cuts the set x by condition (1), and hence, B is a supporting line of the set x.

6.7. THEOREM. *If no line cuts all the three open sets x_1, x_2, x_3, there exists a line separating the set x_1 from the set x_2+x_3.*

Proof. Let y_1, y_2, y_3 be the convexes of the sets x_1, x_2, x_3 respectively. By Theorems 3.4 and 3.5, the sets y_1, y_2, y_3 are open and convex. By Theorem 3.6, no line cuts all the three sets y_1, y_2, y_3.

Let z denote the convex of the set y_2+y_3.

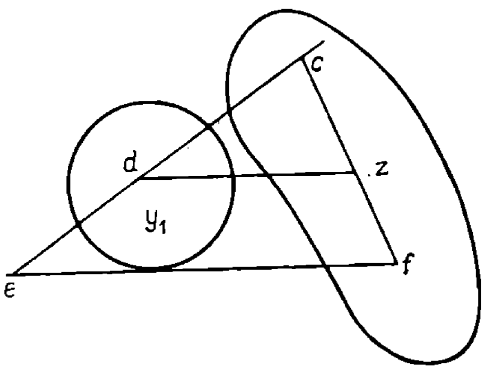

Fig. 13

By Theorem 6.2, there exists a line K passing through an arbitrary point q and such that no half-line, with the origin q and lying on the line K, belongs to the hodograph $H(qy_1z)$.

Let A be the line parallel to K and passing through a point which belongs to the set z. According to the condition that no half-line of the line K belongs to the hodograph $H(qy_1z)$, it follows that the line A does not cut the set y_1. By Theorem 6.6, there exists a line B which supports the set y_1 and which separates it from the line A.

Let us suppose that the line B cuts the set z. Thus, there exists a point c in the set z such that, for an arbitrary but fixed point $d \in y_1$, the relation $\sim\beta(cBd)$ holds. Since the half-lines of the line K do not belong to the hodograph $H(qy_1z)$, the lines $L(cd)$ and B intersect at a point e and one of the following conditions is satisfied:

(1) $$\beta(ecd),$$

(2) $$\beta(edc).$$

In the case (1), by the definition of the supporting line, the line, parallel to the line B and passing through the point c cuts the set y_1, which is impossible by the definition of the line K.

In the case (2), there exists a point $f \in z$ such that $f \in B$. The line parallel to B and passing through the point d cuts the side cf of the triangle ecf by the Pasch's axiom, which is impossible by the definition of the line K. Thus, the line B does not cut the set z, and since it separates some points of the considered sets, it separates the sets themselves.

The following assertion is obvious:

6.8. THEOREM. *If the line A separates the sets x_1, x_2, then it separates their convexes.*

6.9. THEOREM. *If there exists a line A cutting all the three sets x_1, x_2, x_3, then at least one of these sets cannot be separated by any line from the sum of the other two sets.*

Proof. Let y_1, y_2, y_3 be the convexes of the sets x_1, x_2, x_3 respectively. Since the line A cuts the sets y_1, y_2, y_3, there exist points $a_1 \in y_1$, $a_2 \in y_2$, $a_3 \in y_3$ lying on the line A. If at least two of the points a_1, a_2, a_3 coincide, e.g., $a_1 = a_2$, then no line separates the sets y_1, y_2, and by that reason, no line separates the set x_1 from the set $x_2 + x_3$ according to the foregoing theorem. If the points a_1, a_2, a_3 are distinct, one of them lies between the two others. Assuming that $\beta(a_1 a_2 a_3)$, we infer that every line separating the sets y_1, y_2 intersects the segment $a_1 a_2$, and every line separating the sets y_2, y_3 intersects the segment $a_2 a_3$. Since two different lines have at most one point in common, there exists no line separating the set y_2 from the set $y_1 + y_3$, and hence, there exists no line separating the set x_2 from the set $x_1 + x_3$.

Theorems 6.8 and 6.9 imply the following one:

6.10. THEOREM. *If the sets x_1, x_2, x_3 are open and non-empty, then exactly one of the following conditions holds:*

(1) *there exists a line cutting all the three sets x_1, x_2, x_3;*

(2) *there exist three lines such that every of them separates one of the sets, different at each turn, from the sum of the other two.*

Thus, we have reached the goal of Part II establishing, in the body of non-Desarguean geometry, the theorem stated by Jaśkowski, which was assumed as Axiom 2.2 in Part I.

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