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A functional-analytic approach to turbulent convection

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The present paper is devoted to turbulent convection of incompressible viscous fluid. The central role in the description of turbulent convection will be played by some characteristic functionals. The possibility of such handling of turbulence in general was first mentioned by Kolmogoroff [3]. A strict mathematical approach to turbulence by means of characteristic functionals was given by Hopf [1], Lewis and Kraichnan [5].

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1. Introduction. Let D denote the domain of a 3-dimensional Euclidean x-space, $x = (x_1, x_2, x_3)$, which is occupied by an incompressible fluid of constant density ϱ and of viscosity ν. The motion of this fluid will be described by the velocity vector field $\boldsymbol{u}(x, t) = [u_1(x, t), u_2(x, t), u_3(x, t)]$. Suppose that $\boldsymbol{f}(x, t) = [f_1(x, t), f_2(x, t), f_3(x, t)]$ represents the external force acting on the fluid, and suppose that the fluid is at rest. Then the temperature field $T(x, t)$ of the fluid cannot be arbitrary. If suitable conditions are not satisfied by $T(x, t)$ the fluid cannot be at rest. The flow obtained in this way is called *convection flow* or shortly *convection*. The particular case of convection arising in the fluid which moves in the unbounded domain between two horizontal planes is very important. Here the field of gravitation plays the role of the field of external force. Assume that the temperature T_1 of the lower plane is greater than the temperature T_2 of the higher plane. If the difference $T_1 - T_2$ is sufficiently small, then the fluid is at rest. If, however, the difference $T_1 - T_2$ is sufficiently large, then there arises a convection motion.

There exist two different kinds of convection flows: *laminar convection* and *turbulent convection*. The central role in the description of convection flows is played by the Grashof number. A convection is laminar if the Grashof number is small, and it is turbulent if the Grashof number is large. For the description of laminar convection the following equations are used:

$$\frac{\partial u_\alpha}{\partial t}+u_\beta\frac{\partial u_\alpha}{\partial x_\beta} = -\frac{\partial p}{\partial x_\alpha}+\nu\frac{\partial^2 u}{\partial x_\beta \partial x_\beta} -\gamma f_\alpha(x,t)T(x,t), \quad \alpha = 1,2,3; \tag{1}$$

$$\frac{\partial u_\beta}{\partial x_\beta} = 0, \tag{2}$$

$$\frac{\partial T}{\partial t}+u_\beta\frac{\partial T}{\partial x_\beta} = \chi\frac{\partial^2 T}{\partial x_\beta \partial x_\beta}. \tag{3}$$

Equation (1) is a slight generalization of equation (56,3) in [4], p. 267. Here $\boldsymbol{u}(x,t)$, $T(x,t)$ and $p(x,t)$ are functions representing the velocity, temperature and pressure of the fluid, respectively; the prescribed vector function $\boldsymbol{f}(x,t)$ represents the external force acting on the fluid and ν, γ, χ are positive constants representing the kinematic viscosity coefficient, the coefficient of thermal expansion and the coefficient of temperature conductivity, respectively. The density of the fluid is assumed to be equal to one. We employ the summation convention over repeated indices. (1) is the system of forced Navier-Stokes equations, (2) is the condition of incompressibility, (3) is the equation of heat conduction for the moving fluid. We assume that the field of force $\boldsymbol{f}(x,t)$ is an irrotational field. In the case $T(x,t)\equiv 0$ equations (1), (2) and (3) clearly become

$$\frac{\partial u_\alpha}{\partial t}+u_\beta\frac{\partial u_\alpha}{\partial x_\beta} = -\frac{\partial p}{\partial x_\alpha}+\nu\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}; \qquad \frac{\partial u_\beta}{\partial x_\beta} = 0.$$

In the case of laminar convection, from equations (1), (2) and (3), velocity and temperature may be uniquely determined for $t\geqslant t_0$ by specifying the initial conditions at $t = t_0$, and by appropriate boundary conditions. In the case of turbulent convection this property does not hold. It is typical for turbulent convection that different time developments $[\boldsymbol{u}(x,t), T(x,t)]$ can take place under the same initial and boundary conditions.

2. Mathematical approach to turbulent convection (cf. [2]). Let $M_1(x_1,t_1)$ be an arbitrary point in $D\times[t_0,\infty)$. One supposes that the vector $[\boldsymbol{u}(x_1,t_1), T(x_1,t_1)]$ is a random vector determined by its probability density. The vector field $[\boldsymbol{u}(x,t), T(x,t)]$ is regarded as a random vector field, i.e., one supposes that for every N (N arbitrary positive integer) points $M_1(x_1,t_1),\ldots,M_N(x_N,t_N)$ such that $(x_i,t_i)\in D\times[t_0,\infty)$, $i=1,\ldots,N$, there exists a probability density $p(M_1,\ldots,M_N)$ of the random vectors $[\boldsymbol{u}(M_1),T(M_1)],\ldots,[\boldsymbol{u}(M_N),T(M_N)]$. We say that the random vector field $[\boldsymbol{u}(M),T(M)]$ is known if the probability density

$p(M_1, \ldots, M_N)$ for any positive integer N and for all points $M_1, \ldots, M_N$ in $D\times[t_0, \infty)$ is known. One supposes that the vector $[\boldsymbol{u}(x, t), T(x, t)]$ satisfies equations (1), (2) and (3) (cf. [2], p. 297). Let us consider the initial random vector field $[\boldsymbol{u}(x, t_0), T(x, t_0)]$ such that $\partial u_\beta/\partial x_\beta = 0$ and suppose that the initial functions $p(M_1, \ldots, M_N)$ for any positive integer N and for arbitrary points $M_1 = (x_1, t_0), \ldots, M_N = (x_N, t_0)$ are given. Then the basic problem of turbulent convection is to find the time development of the random vector field $[\boldsymbol{u}(x, t), T(x, t)]$, i.e., to find the time development of the functions $p(M_1, \ldots, M_N)$ for any positive integer N and for all points $M_1, \ldots, M_N$ in $D\times[t_0, \infty)$. In the present paper some characteristic functionals are introduced and the basic problem of turbulent convection is expressed in terms of these functionals. We derive a functional differential equations for the characteristic functionals. Should it be possible to solve these equations, this would yield a complete solution of the basic problem of turbulent convection.

3. A space-time characteristic functional. Let D be a bounded or unbounded domain in the Euclidean space R^3 (e.g. $D = R^3$). Consider a fixed vector field $[\boldsymbol{a}(x, t), b(x, t)]$ which is defined on the boundary $\partial D\times[t_0, \infty)$ of $D\times[t_0, \infty)$. Denote by K the space of vector fields $[\boldsymbol{u}(x, t), T(x, t)]$ which satisfy the system of equations (1), (2) and (3) in $D\times(t_0, \infty)$ and satisfy the boundary condition $[\boldsymbol{u}(x, t), T(x, t)] = [\boldsymbol{a}(x, t), b(x, t)]$ on $\partial D\times[t_0, \infty)$. If D is an unbounded domain with bounded or empty ∂D, we impose on the space K the following boundary condition at infinity

$$\lim_{|x|\to\infty} [\boldsymbol{u}(x, t), T(x, t)] = \boldsymbol{C}$$

for all $[\boldsymbol{u}, T]\in K$, where $\boldsymbol{C}$ is a given constant vector independent of the particular vector fields considered. It is assumed that $\boldsymbol{u}(x, t)$ and $T(x, t)$ are sufficiently smooth and that all quantities entering in (1), (2) and (3) are defined and continuous throughout $\bar{D}\times[t_0, \infty)$. In the space K we introduce a completely additive set function $P(A)$ which is defined for subsets A of K such that $P(A)\geqslant 0$ for all $A\subset K, P(K) = 1$. Denote by H the space of vector fields $[\boldsymbol{y}(x, t), s(x, t)], \boldsymbol{y}(x, t) = [y_1(x, t), y_2(x, t), y_3(x, t)]$, continuous on $D\times[t_0, \infty)$ and with compact support in $D\times[t_0, \infty)$. Let

$$\{\boldsymbol{y}, s; \boldsymbol{u}, T\} = \{\boldsymbol{y}(x, t), s(x, t); \boldsymbol{u}(x, t), T(x, t)\} = \int_{D\times[t_0, \infty)} [y_j(x, t)u_j(x, t)+s(x, t)T(x, t)]\,dx\,dt$$

and

$$\Gamma(\boldsymbol{y}, s) = \langle\exp(i\{\boldsymbol{y}, s; \boldsymbol{u}, T\})\rangle = \int\exp(i\{\boldsymbol{y}, s; \boldsymbol{u}, T\})\,dP.$$

Then $\Gamma(y, s)$ is a functional of (y, s) and will be called the *space-time characteristic functional* of the probability distribution P. For the sake of applications we extend the definition of $\Gamma(y, s)$. Let

$$
\tag{4}
\begin{aligned}
\tilde{y}_\beta(x, t) &= \sum_{j=1}^{N} \theta_j \delta_{a_j\beta} \delta(x-x_j)\delta(t-t_j),\\
\tilde{s}(x, t) &= \sum_{j=1}^{N} \theta_j \delta_{a_j4} \delta(x-x_j)\delta(t-t_j),
\end{aligned}
$$

where $\delta_{a_j\beta}$ is the Kronecker delta, $\delta(t)$ and $\delta(x)$ are the one-dimensional and three-dimensional Dirac delta functions, and $\theta_1, \ldots, \theta_N$ are real numbers; $a_j = 1, 2, 3, 4$; $\beta = 1, 2, 3$. We set

$$\Gamma(\tilde{y}, \tilde{s}) = \lim_{n\to\infty} \Gamma[y_n(x, t), s_n(x, t)],$$

where $\tilde{y}(x, t) = [\tilde{y}_1(x, t), \tilde{y}_2(x, t), \tilde{y}_3(x, t)]$ and $[y_n(x, t), s_n(x, t)]$ is a fundamental sequence which determines the distributions (4) (cf. [6], p. 10). For $\tilde{y}, \tilde{s}$ given by (4) we have

$$\{\tilde{y}, \tilde{s}; u, T\} = \sum_{\substack{j=1\\ a_j\leqslant 3}}^{N} \theta_j u_{a_j}(x_j, t_j) + \sum_{j=1}^{N} \theta_j \delta_{a_j4} T(x_j, t_j).$$

Hence

$$\Gamma(\tilde{y}, \tilde{s}) = \Big\langle \exp\Big[i \sum_{\substack{j=1\\ a_j\leqslant 3}}^{N} \theta_j u_{a_j}(x_j, t_j) + \sum_{j=1}^{N} \theta_j \delta_{a_j4} T(x_j, t_j)\Big]\Big\rangle.$$

Thus we see that $\Gamma(\tilde{y}, \tilde{s})$ is the characteristic function of the N-dimensional probability distribution of random variable $[\xi_{a_1}(x_1, t_1), \ldots$ $\ldots, \xi_{a_N}(x_N, t_N)]$, where $\xi_{a_k}(x_k, t_k) = u_{a_k}(x_k, t_k)$ for $a_k \leqslant 3$ and $\xi_{a_k}(x_k, t_k) = T(x_k, t_k)$ for $a_k = 4$. The probability density is given by

$$p(\xi_1, \ldots, \xi_N) = (2\pi)^{-N} \int \exp\Big(-i \sum_{j=1}^{N} \theta_j \xi_j\Big) \Gamma(\tilde{y}, \tilde{s}) d\theta_1 \ldots d\theta_N.$$

Thus we see that the probability density $p(\xi_1, \ldots, \xi_N)$ and its characteristic function can be obtained from the characteristic functional $\Gamma(y, s)$ by means of the special choice of y and s. The functional $\Gamma(y, s)$ furnishes a full description of turbulent convection.

4. Derivation of the functional differential equations. In this section we obtain equations for $\Gamma(y, s)$ from the basic flow equations (1), (2) and (3). First, however, let us recall some definitions. For simplicity we confine ourselves to the case where the functional Γ depends only on one function $y(x, t)$.

A functional $\Gamma(y(x,t))$ is said to be *differentiable* for a particular function $y = y(x,t)$ if there exists a function $A(y; x, t)$ which, besides being dependent on $y(x,t)$, is such that

$$\lim_{\substack{\sup|\delta y(x,t)| \to 0 \\ x \in D \\ t \geq t_0}} \frac{\Gamma(y(x,t)+\delta y(x,t)) - \Gamma(y(x,t)) - \int_D \int_{t_0}^{\infty} A(y; x, t)\,\delta y(x,t)\,dx\,dt}{\int_D \int_{t_0}^{\infty} |\delta y(x,t)|\,dx\,dt} = 0.$$

The function $A(y; x, t)$ will be called the *functional* or *Volterra derivative of* Γ with respect to $y(x,t)$ at the point (x,t). It will be denoted by

$$A(y; x, t) = \frac{\delta \Gamma(y(x,t))}{\delta y(x,t)}.$$

$A(y; x, t)$ is a functional of $y(x,t)$ and for fixed $y(x,t)$ $A(y; x, t)$ is a function of (x,t).

The higher order functional derivatives can be defined in the same way and we use here a similar notation, e.g.

$$\frac{\delta}{\delta y(x_2,t_2)}\left[\frac{\delta \Gamma(y(x,t))}{\delta y(x_1,t_1)}\right] = \frac{\delta^2 \Gamma(y(x,t))}{\delta y(x_2,t_2)\,\delta y(x_1,t_1)}$$

denotes the second order functional derivative.

We now return to the characteristic functional $\Gamma(\mathbf{y}, s)$. Observe that

$$\frac{\delta \Gamma(\mathbf{y}, s)}{\delta y_\alpha(x,t)} = \langle i u_\alpha(x,t) \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\rangle,$$

$$\frac{\delta \Gamma(\mathbf{y}, s)}{\delta s(x,t)} = \langle i T(x,t) \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\rangle,$$

$$\frac{\delta^2 \Gamma(\mathbf{y}, s)}{\delta y_\alpha(x,t)\,\delta y_\beta(x,t)} = -\langle u_\alpha(x,t)\,u_\beta(x,t) \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\rangle,$$

$$\frac{\delta^2 \Gamma(\mathbf{y}, s)}{\delta s^2(x,t)} = -\langle T^2(x,t) \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\rangle,$$

$$\frac{\delta^2 \Gamma(\mathbf{y}, s)}{\delta y_\alpha(x,t)\,\delta s(x,t)} = -\langle u_\alpha(x,t)\,T(x,t) \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\rangle.$$

Hence and from (1), (2) and (3) we have

$$\frac{\partial}{\partial t}\,\frac{\delta \Gamma(\mathbf{y}(x,t), s)}{\delta y_\alpha(x,t)} = \left\langle i\,\frac{\partial u_\alpha}{\partial t} \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\right\rangle$$

$$= \left\langle i\left[-\frac{\partial p}{\partial x_\alpha} - u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \nu\,\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta} - \gamma f_\alpha(x,t)\,T(x,t)\right] \exp(i\{\mathbf{y}, s; \mathbf{u}, T\})\right\rangle$$

and

$$\frac{\partial}{\partial t}\frac{\delta\Gamma\big(y(x,t),s\big)}{\delta s(x,t)} = \left\langle i\frac{\partial T}{\partial t}\exp(i\{y,s;u,T\})\right\rangle$$

$$= \left\langle i\left[-u_\beta\frac{\partial T}{\partial x_\beta}+\chi\frac{\partial^2 T}{\partial x_\beta\partial x_\beta}\right]\exp(i\{y,s;u,T\})\right\rangle.$$

In a similar manner we obtain

$$\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\Gamma(y,s)}{\delta y_\alpha(x,t)} = \left\langle i\frac{\partial^2 u_\alpha}{\partial x_\beta\partial x_\beta}\exp(i\{y,s;u,T\})\right\rangle.$$

According to (2) we have

$$\frac{\partial}{\partial x_\beta}\frac{\delta^2\Gamma(y,s)}{\delta y_\alpha\delta y_\beta} = -\left\langle\left(u_\beta\frac{\partial u_\alpha}{\partial x_\beta}+u_\alpha\frac{\partial u_\beta}{\partial x_\beta}\right)\exp(i\{y,s;u,T\})\right\rangle$$

$$= -\left\langle u_\beta\frac{\partial u_\alpha}{\partial x_\beta}\exp(i\{y,s;u,T\})\right\rangle,$$

$$\frac{\partial}{\partial x_\alpha}\frac{\delta^2\Gamma}{\delta y_\alpha\delta s} = -\left\langle\left(\frac{\partial u_\alpha}{\partial x_\alpha}T+u_\alpha\frac{\partial T}{\partial x_\alpha}\right)\exp(i\{y,s;u,T\})\right\rangle$$

$$= -\left\langle u_\alpha\frac{\partial T}{\partial x_\alpha}\exp(i\{y,s;u,T\})\right\rangle,$$

$$\frac{\partial^2}{\partial x_\alpha\partial x_\alpha}\frac{\delta\Gamma}{\delta s} = \left\langle i\frac{\partial^2 T}{\partial x_\alpha\partial x_\alpha}\exp(i\{y,s;u,T\})\right\rangle.$$

Hence we can get

$$\text{(5)}\qquad \frac{\partial}{\partial t}\frac{\delta\Gamma(y,s)}{\delta y_\alpha(x,t)} = i\frac{\partial}{\partial x_\beta}\frac{\delta^2\Gamma}{\delta y_\alpha\delta y_\beta}+\nu\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\Gamma}{\delta y_\alpha}-\gamma f_\alpha\frac{\delta\Gamma}{\delta s}-\frac{\partial\pi}{\partial x_\alpha},$$

$\alpha = 1,2,3$, where

$$\pi = i\langle p(x,t)\exp(i\{y,s;u,T\})\rangle$$

and

$$\text{(6)}\qquad \frac{\partial}{\partial t}\frac{\delta\Gamma}{\delta s} = i\frac{\partial}{\partial x_\alpha}\frac{\delta^2\Gamma}{\delta y_\alpha\delta s}+i\chi\frac{\partial^2}{\partial x_\alpha\partial x_\alpha}\frac{\delta\Gamma}{\delta s}.$$

From the incompressibility condition (2) it follows immediately that

$$\text{(7)}\qquad \frac{\partial}{\partial x_\alpha}\frac{\delta\Gamma}{\delta y_\alpha} = 0.$$

We now represent the property (7) in another form. The first argument of Γ is a continuous vector on $D\times[t_0,\infty)$ and for fixed t vanishes outside a bounded domain E_t such that $\bar{E}_t \subset D$. Let $\tilde{E}_t$ be a bounded domain such that $\tilde{E}_t \subset D$ and $\bar{E}_t \subset \tilde{E}_t$. For fixed t we represent the vector $\boldsymbol{y}(x, t)$ by the formula

$$\boldsymbol{y}(x, t) = \tilde{\boldsymbol{y}}(x, t) + \mathrm{grad}_x \varphi(x, t),$$

where $\partial \tilde{y}_\alpha / \partial x_\alpha = 0$ and the interior normal part $\tilde{y}_n$ of vector $\tilde{\boldsymbol{y}}(x, t)$ vanishes on $\partial \tilde{E}_t$; $\varphi(x, t)$ denotes a scalar which for arbitrarily fixed $t \in [t_0, \infty)$ vanishes outside $\tilde{E}_t$. Using Green's theorem we get

$$\int_{\tilde{E}_t} \mathrm{grad}_x \varphi(x, t) \cdot \boldsymbol{u}(x, t)\, dx = 0$$

for all functions $\boldsymbol{u}(x, t)$ which satisfy (2). Hence

$$(8) \qquad \int_{t_0}^{\infty} \int_D \boldsymbol{y}(x, t) \cdot \boldsymbol{u}(x, t)\, dx\, dt = \int_{t_0}^{\infty} \int_D (\tilde{\boldsymbol{y}} + \mathrm{grad}_x \varphi) \cdot \boldsymbol{u}(x, t)\, dx\, dt = \int_{t_0}^{\infty} \int_D \tilde{\boldsymbol{y}}(x, t)\, \boldsymbol{u}(x, t)\, dx\, dt.$$

By (8) we obtain the important relationship

$$(9) \qquad \Gamma(\boldsymbol{y}, s) = \Gamma(\tilde{\boldsymbol{y}}, s).$$

This relationship is another expression for (7) and for the incompressibility condition (2).

Various devices for eliminating the pressure term $\partial \pi / \partial x_\alpha$ from (5) are possible. The method we shall use here is to introduce the testing field

$$\boldsymbol{\eta}(x, t) = \big(\eta_1(x, t), \eta_2(x, t), \eta_3(x, t)\big),$$

which vanishes sufficiently rapidly at spatial infinity and satisfies the condition

$$(10) \qquad \frac{\partial \eta_\alpha}{\partial x_\alpha} = 0.$$

Then

$$\int_D \eta_\alpha(x, t) \frac{\partial \pi}{\partial x_\alpha}\, dx = -\int \frac{\partial \eta_\alpha}{\partial x_\alpha} \pi\, dx = 0,$$

and from (5) we obtain

$$(11) \qquad \int_D \eta_\alpha(x, t) \left[\frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y_\alpha(x, t)} - i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Gamma}{\delta y_\alpha \delta y_\beta} - \nu \frac{\partial^2}{\partial x_\beta \partial x_\beta} \frac{\delta \Gamma}{\delta y_\alpha} + \gamma f_\alpha \frac{\delta \Gamma}{\delta s} \right] dx\, dt = 0.$$

This must be satisfied by Γ for all testing fields $\boldsymbol{\eta}(x, t)$ which satisfy (10). Equations (6) and (11) are the required equations for the functional $\Gamma(\boldsymbol{y}, s)$.

Three further conditions on $\Gamma(\boldsymbol{y}, s)$ are

$$(12) \qquad \Gamma(\mathbf{0}, 0) = 1, \quad \Gamma^*(\boldsymbol{y}, s) = \Gamma(-\boldsymbol{y}, -s), \quad |\Gamma(\boldsymbol{y}, s)| \leqslant 1.$$

Here $*$ denotes the complex conjugate. These conditions follow immediately from the definition of $\Gamma(\boldsymbol{y}, s)$.

Let $P_0(A)$ be a probability distribution defined for subsets of the space of vector fields $[\boldsymbol{u}(x, t_0), T(x, t_0)]$ such that $\partial u_a/\partial x_a = 0$. The basic problem of turbulent convection can be formulated as follows: find the solution of (6) and (11) which satisfies conditions (12) and the following initial condition

$$\Gamma\big(\boldsymbol{y}(x)\,\delta(t-t_0), s(x)\,\delta(t-t_0)\big) = \Gamma_0\big(\boldsymbol{y}(x), s(x)\big),$$

where $\Gamma_0(\boldsymbol{y}, s)$ denotes a characteristic functional of the probability distribution $P_0(A)$.

5. A space characteristic functional. Consider a fixed vector field $[\boldsymbol{a}(x), b(x)]$ which is defined on $D \cup \partial D$ and is such that $\partial a_a/\partial x_a = 0$ on D. Denote by K_0 the space of vector fields $[\boldsymbol{u}(x), T(x)]$ which satisfy the incompressibility condition $\partial u_a/\partial x_a = 0$ in D and satisfy the boundary condition $[\boldsymbol{u}(x), T(x)] = [\boldsymbol{a}(x), b(x)]$ on ∂D. If D is an unbounded domain with bounded (or empty) ∂D, we impose on the space K_0 the boundary condition at infinity

$$\lim_{|x|\to\infty} [\boldsymbol{u}(x), T(x)] = (\boldsymbol{a}, b)$$

for all $[\boldsymbol{u}(x), T(x)] \in K_0$, where $(\boldsymbol{a}, b)$ is a given constant vector independent of the particular vector fields considered. We assume that the solution $[\boldsymbol{u}(x, t), T(x, t)]$ of system (1), (2) and (3) is uniquely determined in $D \times (t_0, \infty)$ by the initial conditions

$$\boldsymbol{u}(x, t_0) = \boldsymbol{a}(x), \qquad T(x, t_0) = b(x)$$

for $x \in D$ and the boundary conditions $\boldsymbol{u}(x, t) = \boldsymbol{a}(x)$, $T(x, t) = b(x)$ for $t \geqslant t_0$ and $x \in \partial D$.

Let $P(\Delta)$ denote the probability that the point of K_0 falls into the part Δ of K_0. $P(\Delta)$ is a completely additive set function which is defined for all subsets of K_0 such that $P(\Delta) \geqslant 0$, $P(K_0) = 1$. Denote by H_0 the space of vector fields $[\boldsymbol{y}(x), s(x)]$ continuous on D and with a compact support in D. Let

$$[\boldsymbol{y}, s; \boldsymbol{u}, T; t] = \int_D \big(y_a(x)\, u_a(x, t) + s(x)\, T(x, t)\big)\, dx$$

and consider the expression

$$\Gamma(y(x), s(x); t) = \int \exp(i[y, s; u, T; t])dP = \langle \exp(i[y, s; u, T; t])\rangle,$$

where $[u(x, t_0), T(x, t_0)] \in K_0$. Then $\Gamma(y(x), s(x); t)$ is a functional of $(y(x), s(x))$ and a function of t and will be called the *spatial characteristic functional* of the probability distribution of the random vector $[u(x, t), T(x, t)]$ for fixed t. Then just in the same way as in the case of the space-time characteristic functional we evaluate the characteristic functional for the special argument $\tilde{y}(x), \tilde{s}(x)$ whose components are given by

$$\tilde{y}_\beta(x) = \sum_{j=1}^{N} \theta_j \delta_{\alpha_j\beta} \delta(x - x_j), \qquad \tilde{s}(x) = \sum_{j=1}^{N} \theta_j \delta_{\alpha_j 4} \delta(x - x_j).$$

Thus, we obtain the description of turbulent convection for fixed t.

Similarly to our deduction of property (9) it is easy to prove that

$$\Gamma(y(x), s(x); t) = \Gamma(\tilde{\tilde{y}}(x), s(x); t), \tag{9'}$$

where $\tilde{\tilde{y}}(x)$ is the solenoidal part of $y(x)$.

From equations (1), (2) and (3) we now derive a functional differential equation for $\Gamma(y, s; t)$. Observe that

$$\frac{\partial \Gamma(y, s; t)}{\partial t} = \left\langle i \int_D \left[y_\alpha(x) \frac{\partial u_\alpha(x, t)}{\partial t} + s(x) \frac{\partial T}{\partial t} \right] dx \exp(i[y, s; u, T; t]) \right\rangle.$$

From (1) and (3) we have

$$\frac{\partial \Gamma}{\partial t} = \left\langle i \int_D \left\{ y_\alpha(x) \left[-u_\beta \frac{\partial u_\alpha}{\partial x_\beta} - \frac{\partial p}{\partial x_\alpha} + \nu \frac{\partial^2 u}{\partial x_\beta \partial x_\beta} - \gamma f_\alpha T \right] + \right.\right.$$
$$\left.\left. + s(x) \left[-u_\beta \frac{\partial T}{\partial x_\beta} + \chi \frac{\partial^2 T}{\partial x_\beta \partial x_\beta} \right] \right\} dx \exp(i[y, s; u, T; t]) \right\rangle.$$

Then in the same manner as we deduced (5) and (6) we obtain

$$\frac{\partial \Gamma}{\partial t} = \int_D \left\{ y_\alpha(x) \left[i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Gamma}{\delta y_\alpha \delta y_\beta} + \nu \frac{\partial^2}{\partial x_\beta \partial x_\beta} \frac{\delta \Gamma}{\delta y_\alpha} - \gamma f_\alpha \frac{\delta \Gamma}{\delta s} \right] + \right.$$
$$\left. + s(x) \left[i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Gamma}{\delta y_\beta \delta s} + \chi \frac{\partial^2}{\partial x_\beta \partial x_\beta} \frac{\delta \Gamma}{\delta s} \right] \right\} dx. \tag{13}$$

Thus equation (13) is the required equation for the functional $\Gamma(y, s; t)$.

6. Oseen's equations, the case of the space characteristic functional. The basic problem of turbulent convection was expressed (in Section 5) in terms of solutions of (13). We used the assumptions that the random vector $[\boldsymbol{u}(x,t), T(x,t)]$ satisfies equations (1), (2) and (3). The finding of the solution of the initial problem for equation (13) involves great mathematical difficulties (cf. [2], p. 299). In the present section a simpler problem is considered. Suppose that $D = R^3$ and the random vector $[\boldsymbol{u}(x,t), T(x,t)]$ satisfies the following equations of Oseen:

$$(14)\qquad \frac{\partial u_m(x,t)}{\partial t} = -a_j(t)\frac{\partial u_m(x,t)}{\partial x_j} - \frac{\partial p(x,t)}{\partial x_m} + \nu\frac{\partial^2 u_m(x,t)}{\partial x_j \partial x_j} - \gamma f_m(t) T(x,t),$$

$$(15)\qquad \frac{\partial T}{\partial t} = -a_j(t)\frac{\partial T}{\partial x_j} + \chi\frac{\partial^2 T(x,t)}{\partial x_j \partial x_j}$$

and the continuity equation (2), where $m = 1, 2, 3$ and $\boldsymbol{a}(t) = [a_1(t), a_2(t), a_3(t)]$ is a non-random continuous vector field defined for $t \geqslant t_0$. In this case we obtain an equation analogous to Hopf's equation (13) and we investigate the time developement of characteristic functionals satisfying this equation.

Suppose that for fixed t the random vector $[\boldsymbol{u}(x,t), T(x,t)]$ vanishes outside a compact domain D_t. Let

$$u_m(x,t) = \int e^{ikx} v_m(k,t)\,dk, \qquad p(x,t) = \int e^{ikx} q(k,t)\,dk,$$
$$T(x,t) = \int e^{ikx} w(k,t)\,dk,$$

where $m = 1, 2, 3$; $k = (k_1, k_2, k_3) \in R^3$, $kx = k_m x_m$, $v_m^*(k,t) = v_m(-k,t)$, $q^*(k,t) = q(-k,t)$, $w^*(k,t) = w(-k,t)$. Then we have

$$\int e^{ikx}\frac{\partial v_m(k,t)}{\partial t}\,dk = -a_j(t)\,i\int k_j e^{ikx} v_m(k,t)\,dk - i\int k_m e^{ikx} q(k,t)\,dk - \nu\int |k|^2 e^{ikx} v_m(k,t)\,dk - \gamma f_m(t)\int e^{ikx} w(k,t)\,dk,$$

$$\int e^{ikx}\frac{\partial w(k,t)}{\partial t}\,dk = -a_j(t)\,i\int k_j e^{ikx} w(k,t)\,dk - \chi\int |k|^2 e^{ikx} w(k,t)\,dk,$$

$$\int k_m e^{ikx} v_m(k,t)\,dk = 0.$$

Hence

$$(16)\qquad \frac{\partial v_m(k,t)}{\partial t} = -ia_j(t)k_j v_m(k,t) - ik_m q(k,t) - \nu |k|^2 v_m(k,t) - \gamma f_m(t) w(k,t), \qquad m = 1, 2, 3,$$

$$(17)\qquad \frac{\partial w(k,t)}{\partial t} = -ia_j(t)k_j w(k,t) - \chi |k|^2 w(k,t),$$

$$k_m v_m(k, t) = 0. \tag{18}$$

Consider at first a simpler case where

$$w(k, t) \equiv 0. \tag{19}$$

From (16), (18) and (19) we find $q(k, t) \equiv 0$. Hence

$$\frac{\partial v_m(k, t)}{\partial t} = \left(-ia_j(t)k_j - \nu|k|^2\right)v_m(k, t). \tag{20}$$

Let

$$v_m(k, t) = \exp[-iA_j(t)k_j - \nu|k|^2(t-t_0)]\,b_m(k, t), \tag{21}$$

where $A_j(t) = \int\limits_{t_0}^{t} a_j(s)\,ds$. Remark that $v_m(k, t_0) = b_m(k, t_0)$ and

$$\frac{\partial v_m(k, t)}{\partial t} \tag{22}$$
$$= \left(-ia_j(t)k_j - \nu|k|^2\right)\exp[-iA_j(t)k_j - \nu|k|^2(t-t_0)]\,b_m(k, t) +$$
$$+ \exp[-iA_j(t)k_j - \nu|k|^2(t-t_0)]\frac{\partial b_m(k, t)}{\partial t}.$$

If we insert (21) and (22) in (20), we obtain

$$\frac{\partial b_m(k, t)}{\partial t} = 0. \tag{23}$$

Consider the functional

$$\varphi[z(k), t] = \Gamma\left[(2\pi)^{-3}\int e^{ikx} z(k)\,dk, 0, t\right], \tag{24}$$

where $z(k) = z^*(-k)$. It has the form

$$\varphi[z(k), t] = \left\langle \exp\left[i\int z_j(k)\,v_j(k, t)\,dk\right]\right\rangle, \tag{25}$$

where the averaging $\langle\ \rangle$ is over the probability distribution of $v(k, t_0)$. Let

$$\psi[z(k), t] = \left\langle \exp\left[i\int z_j(k)\,b_j(k, t)\,dk\right]\right\rangle. \tag{26}$$

From (21), (25) and (26) we see that

$$\varphi[z(k), t] = \psi\left[z(k)\exp\left(-iA_j(t)k_j - \nu|k|^2(t-t_0)\right), t\right].$$

Let

$$\psi[z(k), t, s] = \left\langle \exp\left[i\int z_j(k)\exp\left(-ik_mA_m(t) - \nu|k|^2(t-t_0)\right)b_j(k, s)\,dk\right]\right\rangle.$$

Remark that

$$\psi[z(k), t, t] = \varphi[z(k), t].$$

From (23) we find

$$\frac{\partial\psi[z(k), t, s]}{\partial s} = \Big\langle \exp\Big[i\int z_j(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\cdot b_j(k, s)\,dk\Big]\times \times i\int z_j(k)\exp[-ik_m A_m(t) - \nu|k|^2(t-t_0)]\cdot\frac{\partial b_j(k, s)}{\partial s}\,dk\Big\rangle = 0.$$

Thus $\psi[z(k), t, s]$ is independent of s. Hence

$$\begin{aligned}\psi[z(k), t, s] &= \psi[z(k), t, t_0]\\ &= \Big\langle \exp\Big[i\int z_j(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\times \times b_j(k, t_0)\,dk\Big]\Big\rangle\\ &= \Big\langle \exp\Big[i\int z_j(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\times \times v_j(k, t_0)\,dk\Big]\Big\rangle\\ &= \varphi\big[z(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big), t_0\big].\end{aligned}$$

In particular,

$$\begin{aligned}\psi[z(k), t, t] &= \varphi[z(k), t]\\ &= \varphi\big[z(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big), t_0\big].\end{aligned}$$

Writing

$$\varphi[z(k), t_0] = \varphi_0[z(k)],$$

we get

$$\varphi[z(k), t] = \varphi_0\big[z(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\big].$$

From (24) we see that

$$\begin{aligned}\varphi[z(k), t] &= \varphi_0\big[z(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\big]\\ &= \Gamma_0\Big[(2\pi)^{-3}\int e^{ikx}z(k)\exp\big(-ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\,dk\Big]\\ &= \Gamma_0\Big[(2\pi)^{-3}\int \exp\big(ikx - ik_m A_m(t) - \nu|k|^2(t-t_0)\big)\times \times\Big(\int e^{-ikx'}y(x')\,dx'\Big)\,dk\Big]\\ &= \Gamma_0\Big[(2\pi)^{-3}\int\Big\{\int \exp\big(ik(x-x')\big)\exp\big(-ik_m A_m(t) - \nu|k|^2\times \times(t-t_0)\big)\,dk\Big\}\,y(x')\,dx'\Big],\end{aligned}$$

where Γ_0 is defined at the end of section 4. Since

$$\int_{-\infty}^{\infty} e^{-s^2}\cos ms\,ds = \sqrt{\pi}\,e^{-m^2/4},$$

we finally find

$$\Gamma[y(x), 0, t] = \Gamma_0\left[(4\pi\nu t)^{-3/2}\int \exp\left(-\frac{|x'-x+A(t)|^2}{4\nu t}\right)y(x')\,dx'\right],$$

where $A(t) = [A_1(t), A_2(t), A_3(t)]$.

Consider now the case where $w(k, t) \not\equiv 0$. Let

$$\varphi[z(k), \mu(k), t] = \Gamma\left[(2\pi)^{-3}\int e^{ikx}z(k)\,dk,\ (2\pi)^{-3}\int e^{ikx}\mu(k)\,dk,\ t\right]$$
$$= \left\langle \exp\left\{i\int[z_j(k)v_j(k, t)+\mu(k)w(k, t)]\,dk\right\}\right\rangle,$$

where the averaging $\langle\ \rangle$ is over the joint probability distribution of $v(k, t_0)$ and $w(k, t_0)$. From (16), (17) and (18) it follows that $\varphi[z(k), \mu(k), t]$ satisfies the equation

$$\frac{\partial\varphi}{\partial t} = \int z_m(k)\left(-ia_j(t)k_j-\nu|k|^2\right)\frac{\delta\varphi}{\delta z_m(k)}\,dk+ \tag{27}$$
$$+\gamma\int f_s(t)z_m(k)\left(|k|^{-2}k_mk_s-\delta_{ms}\right)\frac{\delta\varphi}{\delta\mu(k)}\,dk-$$
$$-\int\left(ia_j(t)k_j+\chi|k|^2\right)\mu(k)\frac{\delta\varphi}{\delta\mu}\,dk.$$

Let $\varphi_0[z(k), \mu(k)]$ be a given characteristic functional of a distribution of random vector $v(k, t_0), w(k, t_0)$. Consider

$$\varphi[z(k), \mu(k), t] = \varphi_0[\sigma(k, t), \varkappa(k, t)], \tag{28}$$

where

$$\sigma_s(k, t) = \exp\left(-ik_mA_m(t)-\nu|k|^2(t-t_0)\right)\left(|k|^{-2}k_jk_s-\delta_{js}\right)z_j(k),$$
$$\varkappa(k, t) = \exp\left(-ik_mA_m(t)-\nu|k|^2(t-t_0)\right)\times$$
$$\times\left(|k|^{-2}k_jk_s-\delta_{js}\right)\int_{t_0}^{t} f_s(\tau)e^{(\nu-\chi)|k|^2\tau}\,d\tau\, z_j(k)+$$
$$+\exp\left(-ik_mA_m(t)-\chi|k|^2(t-t_0)\right)\mu(k),\qquad s = 1, 2, 3.$$

THEOREM. *Functional* (28) *satisfies equation* (27) *and the condition* $\varphi[z(k), \mu(k), t_0] = \varphi_0[z(k), \mu(k)]$.

Indeed, we have

$$\frac{\partial\varphi}{\partial t} = \int\left\{\frac{\delta\varphi_0}{\delta\sigma_s}\left(-ia_m(t)k_m-\nu|k|^2\right)\left(|k|^{-2}k_jk_s-\delta_{js}\right)z_j(k)\times\right.$$
$$\times\exp\left(-ik_mA_m(t)-\nu|k|^2(t-t_0)\right)+$$
$$+\frac{\delta\varphi_0}{\delta\varkappa}\left[\left(-ik_ma_m(t)-\nu|k|^2\right)\exp\left(-ik_mA_m(t)-\nu|k|^2(t-t_0)\right)\times\right.$$

$$\times(|k|^{-2}k_jk_s-\delta_{js})\int_{t_0}^{t}f_s(\tau)\exp\big((\nu-\chi)|k|^2\tau\big)d\tau z_j(k)+$$
$$+\exp\big(-ik_mA_m(t)-\nu|k|^2(t-t_0)\big)(|k|^{-2}k_jk_s-\delta_{js})\times$$
$$\times z_j(k)f_s(t)\exp\big((\nu-\chi)|k|^2t\big)+$$
$$+\big(-ik_mA_m(t)-\chi|k|^2t\big)\exp\big(-ik_mA_m(t)-\chi|k|^2(t-t_0)\big)\mu(k)\Big]\Big\}dk\,.$$

On the other hand,

$$\frac{\delta\varphi}{\delta z_m(k)}=\int\left[\frac{\delta\varphi_0}{\delta\sigma_a(a,t)}\,\frac{\delta\sigma_a(a,t)}{\delta z_m(k)}+\frac{\delta\varphi_0}{\delta\varkappa(a,t)}\,\frac{\delta\varkappa(a,t)}{\delta z_m(k)}\right]da$$

and

$$\frac{\delta\varphi}{\delta\mu(k)}=\int\frac{\delta\varphi_0}{\delta\varkappa(a,t)}\,\frac{\delta\varkappa(a,t)}{\delta\mu(k)}\,da.$$

Since

$$\frac{\delta\sigma_j(a,t)}{\delta z_m(k)}=\exp\big(-ia_nA_n(t)-\nu|a|^2(t-t_0)\big)(|a|^{-2}a_ma_j-\delta_{mj})\,\delta(a-k),$$

$$\frac{\delta\varkappa(a,t)}{\delta z_m(k)}=\exp\big(-ia_nA_n(t)-\nu|a|^2(t-t_0)\big)\times$$
$$\times\int_{t_0}^{t}f_s(\tau)\exp\big((\nu-\chi)|a|^2\tau\big)d\tau\,\delta(a-k)(|a|^{-2}a_ma_s-\delta_{ms}),$$

$$\frac{\delta\varkappa(a,t)}{\delta\mu(k)}=\exp\big(-ia_nA_n(t)-\chi|a|^2(t-t_0)\big)\,\delta(a-k),$$

we find

$$\frac{\delta\varphi}{\delta z_m(k)}=\frac{\delta\varphi_0}{\delta\sigma_j(k,t)}\exp\big(-ik_nA_n(t)-|k|^2(t-t_0)\big)(|k|^{-2}k_mk_j-\delta_{mj})+ \tag{29}$$
$$+\frac{\delta\varphi_0}{\delta\varkappa(k,t)}\exp\big(-ik_nA_n(t)-\nu|k|^2(t-t_0)\big)\times$$
$$\times\int_{t_0}^{t}f_s(\tau)\exp\big((\nu-\chi)|k|^2\tau\big)d\tau\,(|k|^{-2}k_mk_s-\delta_{ms})$$

and

$$\frac{\delta\varphi}{\delta\mu(k)}=\frac{\delta\varphi_0}{\delta\varkappa(k,t)}\exp\big(-ik_nA_n(t)-\chi|k|^2(t-t_0)\big). \tag{30}$$

From (29) and (30) we see that functional (28) satisfies (27).
Remark that

$$\varphi[z(k),\mu(k),t_0]=\varphi_0[\sigma(k,t_0),\varkappa(k,t_0)],$$

where

$$\sigma_s(k, t_0) = (|k|^{-2} k_j k_s - \delta_{js}) z_j(k), \qquad \varkappa(k, t_0) = \mu(k).$$

Hence and from (9′) we have established the theorem.

7. Oseen's equations, the case of the space-time characteristic functional. In the present section we also consider the case where $D = R^3$ and the random vector $[u(x, t), T(x, t)]$ satisfies equations (2), (14) and (15). We shall deal here with the characteristic functionals $\Gamma(y(x, t), s(x, t))$. From equations (2), (14) and (15) the equations which are simpler than Lewis and Kraichnan's equations (6), (11) can be obtained. We obtain an explicit solution of these equations.

Similarly to our deduction of equations (6) and (11) we get

$$(11') \qquad \int \eta_a(x, t) \left\{ \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y_a} + a_j(t) \frac{\partial}{\partial x_j} \frac{\delta \Gamma}{\delta y_a} - \nu \frac{\partial^2}{\partial x_j \partial x_j} \frac{\delta \Gamma}{\delta y_a} - \gamma f_a(t) \frac{\delta \Gamma}{\delta s} \right\} dx dt = 0,$$

$$(6') \qquad \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta s} = -a_j(t) \frac{\partial}{\partial x_j} \frac{\delta \Gamma}{\delta s} + \chi \frac{\partial^2}{\partial x_j \partial x_j} \frac{\delta \Gamma}{\delta s}.$$

We shall apply the Fourier transform to equations (11′) and (6′). Let

$$y_a(x, t) = (2\pi)^{-3} \int z_a(k, t) e^{ikx} dk,$$

$$\eta_a(x, t) = (2\pi)^{-3} \int \lambda_a(k, t) e^{ikx} dk,$$

$$s(x, t) = (2\pi)^{-3} \int \mu(k, t) e^{ikx} dk,$$

and

$$\varphi[z(k, t), \mu(k, t)] = \Gamma\left[(2\pi)^{-3} \int z(k, t) e^{ikx} dk, (2\pi)^{-3} \int \mu(k, t) e^{ikx} dk\right].$$

By simple calculation (cf. [5], pp. 402-403) we get

$$(31) \qquad \int \lambda_a(k, t) \left\{ \frac{\partial}{\partial t} \frac{\delta \varphi}{\delta z_a(k, t)} + [-i a_j(t) k_j + \nu |k|^2] \frac{\delta \varphi}{\delta z_a(k, t)} - \gamma f_a(t) \frac{\delta \varphi}{\delta \mu(k, t)} \right\} dk dt = 0,$$

$$(32) \qquad \frac{\partial}{\partial t} \frac{\delta \varphi}{\delta \mu(k, t)} = \left(i a_j(t) k_j - \chi |k|^2\right) \frac{\delta \varphi}{\delta \mu(k, t)}$$

and by (2) we find

$$(33) \qquad k_a \frac{\delta \varphi}{\delta z_a(k, t)} = 0.$$

Remark that

$$k_\alpha \lambda_\alpha(k, t) = 0.$$

Consider at first equation (32). Let $A_j(t) = \int_{t_0}^{t} a_j(s)\,ds$. Then

$$\frac{\delta\varphi}{\delta\mu(k,t)} = \exp\big(iA_j(t)k_j - \chi|k|^2(t-t_0)\big)B[\boldsymbol{z}(k,t), \mu(k,t), k], \tag{34}$$

where $B[\boldsymbol{z}(k,t), \mu(k,t), k]$ is an arbitrary functional independent of t. In view of the lemma of [5] there exists a scalar function $p(k,t)$ such that

$$\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta z_\alpha(k,t)} + \big(-ia_j(t)k_j + \nu|k|^2\big)\frac{\delta\varphi}{\delta z_\alpha(k,t)} - \gamma f_\alpha(t)\frac{\delta\varphi}{\delta\mu(k,t)} \equiv p(k,t)k_\alpha. \tag{35}$$

From (35) and (33) we have

$$p(k,t) = -\gamma f_\beta k_\beta |k|^{-2}\frac{\delta\varphi}{\delta\mu(k,t)}. \tag{36}$$

From (34), (35) and (36) we get

$$\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta z_\alpha(k,t)} + \big(-ia_j(t)k_j + \nu|k|^2\big)\frac{\delta\varphi}{\delta z_\alpha(k,t)} -$$
$$-\exp\big(iA_j(t)k_j - \chi|k|^2(t-t_0)\big)B[\boldsymbol{z}(k,t), \mu(k,t), k]\gamma[f_\alpha - (|k|^{-2}k_\alpha k_\beta f_\beta)] = 0.$$

Hence

$$\frac{\delta\varphi}{\delta z_\alpha(k,t)} = \exp\big(iA_j(t)k_j - \nu|k|^2(t-t_0)\big)\{C_\alpha[\boldsymbol{z}(k,t), \mu(k,t), k] +$$
$$+ (\delta_{\alpha\beta} - |k|^{-2}k_\alpha k_\beta)\}B[\boldsymbol{z}(k,t), \mu(k,t), k]\gamma\int_{t_0}^{t} f_\beta(\tau)e^{(\nu-\chi)k^2\tau}\,d\tau, \tag{37}$$

where $k_\alpha C_\alpha[\boldsymbol{z}(k,t), \mu(k,t), k] = 0$.

Let $\varphi_0[\boldsymbol{z}(k), \mu(k)]$ be a given characteristic functional of a distribution of random vector $[\boldsymbol{v}(k,t_0), w(k,t_0)]$. Consider

$$\varphi[\boldsymbol{z}(k,t), \mu(k,t)] = \varphi_0[\boldsymbol{r}(k,t), \zeta(k,t)], \tag{38}$$

where $\boldsymbol{r}(k,t) = \big(r_1(k,t), r_2(k,t), r_3(k,t)\big)$,

$$r_\beta(k,t) = \int_{t_0}^{\infty} \exp\big(ik_m A_m(\tau) - \nu|k|^2(\tau-t_0)\big)\big(\delta_{\alpha\beta} - |k|^{-2}k_\alpha k_\beta\big)z_\alpha(k,\tau)\,d\tau,$$
$$\beta = 1, 2, 3,$$

$$\zeta(k,t) = \int_{t_0}^{\infty} \exp\big(ik_m A_m(\tau) - \nu|k|^2(\tau-t_0)\big)(\delta_{\alpha\beta} - |k|^{-2}k_\alpha k_\beta)\times$$
$$\times\gamma\int_{t_0}^{\tau} f_\beta(s)\exp\big((\nu-\chi)|k|^2 s\big)\,ds\,z_\alpha(k,\tau)\,d\tau +$$
$$+\int_{t_0}^{\infty} \exp\big(ik_m A_m(\tau) - \chi|k|^2(\tau-t_0)\big)\mu(k,\tau)\,d\tau.$$

Formula (37) suggests that the following theorem holds:

THEOREM. *Functional* (38) *satisfies equations* (31), (32), (33) *and the condition*

$$\varphi[z(k)\delta(t-t_0), \mu(k)\delta(t-t_0)] = \varphi_0[z(k), \mu(k)]. \tag{39}$$

Proof. Since

$$\frac{\delta\varphi}{\delta z_a(k,t)} = \int\left[\frac{\delta\varphi_0}{\delta r_\beta(k',t)}\frac{\delta r_\beta(k',t)}{\delta z_a(k,t)} + \frac{\delta\varphi_0}{\delta\zeta(k',t)}\frac{\zeta\delta(k',t)}{\delta z_a(k,t)}\right]dk'$$

and

$$\frac{\delta r_\beta(k',t)}{\delta z_a(k,t)} = \exp\left(ik'_m A_m(t) - \nu|k'|^2(t-t_0)\right)\left(|k'|^{-2}k'_a k'_\beta - \delta_{a\beta}\right)\delta(k'-k),$$

$$\frac{\delta\zeta(k',t)}{\delta z_a(k,t)} = \exp\left(ik'_m A_m(t) - \nu|k'|^2(t-t_0)\right)\left(|k'|^{-2}k'_a k'_\beta - \delta_{a\beta}\right)\times$$
$$\times\gamma\int_{t_0}^{t} f_\beta(s)\exp\left((\nu-\chi)|k'|^2 s\right)ds\,\delta(k'-k),$$

we have

$$\frac{\delta\varphi}{\delta z_a(k,t)} = \frac{\delta\varphi_0}{\delta r_\beta(k,t)}\left(|k|^{-2}k_a k_\beta - \delta_{a\beta}\right)\exp\left(ik_m A_m(t) - \nu|k|^2(t-t_0)\right) +$$
$$+\frac{\delta\varphi_0}{\delta\zeta(k,t)}\left(|k|^{-2}k_a k_\beta - \delta_{a\beta}\right)\exp\left(ik_m A_m(t) - \nu|k|^2(t-t_0)\right)\times$$
$$\times\gamma\int_{t_0}^{t} f_\beta(s)\exp\left((\nu-\chi)k^2 s\right)ds.$$

Similarly we find

$$\frac{\delta\varphi}{\delta\mu(k,t)} = \frac{\delta\varphi_0}{\delta\zeta(k,t)}\exp\left(ik_m A_m(t) - \chi|k|^2(t-t_0)\right).$$

Thus, we see that functional (38) satisfies (33), (34) and (37). Hence it follows that (38) also satisfies (35) and (31). From (9) it follows that condition (39) holds.

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