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On distinguished representative domains

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Abstract. In previous papers, the class $C(B^4)$ of pseudoconformally equivalent domains B^4 has been considered. It has been shown that to an arbitrary point t, $t \in B^4$ there exists a representative domain $R(B^4, t)$. In the case of conformal mappings of bounded domains B^2 in the plane, $R(B^2, t)$ is the unit circle $\{|z-t| < 1\}$. In the present paper, we introduce interior distinguished sets, as follows: with the class $C(B)$ of pseudoconformally equivalent, one can associate an invariant

$$J(z, \bar{z}) = \frac{K}{T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2}, \quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

where K is the kernel function of B.

One can show that $J(z, \bar{z})$ has finitely many critical sets and we assume that some of these sets are points t_ν, $\nu = 1, 2, \ldots, N$. The domains $R(B, t_\nu)$ are called *distinguished representative domains.*

Some properties of $R(B, t_\nu)$ are indicated.

1. Introduction. A mapping of a domain (¹) $B \equiv B^4$, by a pair

$$z_k^* = w_k(z_1, z_2), \quad k = 1, 2, \tag{1}$$

of analytic functions (with a non-vanishing Jacobian in B) is called a PCT (*pseudoconformal transformation*). The totality of the domains which one obtains by PCT's from B forms the *class* $C(B)$ of pseudoconformally equivalent domains. A subset, say S, $S \in B$, which in every PCT of B preserves some characteristic properties, is called an *interior distinguished set.* One of the problems of the theory of PCT's is to determine in a class $C(B)$ a special domain $R(B, t)$, called a *representative domain with respect to an interior point* $t = (t_1, t_2)$ *of* B.

A PCT (1) is said to be *normalized at* $t = (t_1, t_2) \in B$, if

$$w_k(t_1, t_2) = t_k, \quad \left(\frac{\partial w_k(z_1, z_2)}{\partial z_p}\right)_{z_p = t_p} = \delta_{kp} = \begin{cases} 1 & \text{for } k = p, \\ 0 & \text{for } k \neq p, \end{cases} \quad k = 1, 2;\ p = 1, 2. \tag{2}$$

See (47), p. 188 of [2] (²).

(¹) B^4 is a domain of the $x_1y_1x_2y_2$-space; x_ν, y_ν are real.

(²) To avoid a repetition of various known considerations, we assume here that the reader is acquainted with [2].

Using the theory of the kernel function, see p. 188–198 of [2], a mapping normalized at t

$$z_1^* = v^{10}(z; t), \quad z_2^* = v^{01}(z; t), \quad z = (z_1, z_2) \tag{3}$$

was defined which maps B onto $R(B, t)$. It possesses the property that if B and B^* can be mapped pseudoconformally onto each other by a PCT normalized at t, then

$$v_{B^*}^{10}(z^*, t) = v_B^{10}(z, t), \quad v_{B^*}^{01}(z^*, t) = v_B^{01}(z, t). \tag{4}$$

The introduction of representative domains suggests the following problem: the determination of one (or finitely many) interior distinguished points t_ν in B, so that we can define *distinguished representative domains* $R(B, t_\nu)$, $\nu = 1, 2, \ldots, N$, $N < \infty$, in the class $C(B)$.

In the present paper we shall discuss this problem.

Remark. One obtains the mappings normalized at 0 from a PCT with a fixed point at 0 by a *linear* transformation, namely, it holds

$$\begin{aligned} v_{B^*}^{10}(z^*, 0) &= a_{11} v_B^{10}(z, 0) + a_{12} v_B^{01}(z, 0), \\ v_{B^*}^{01}(z^*, 0) &= a_{21} v_B^{10}(z, 0) + a_{22} v_B^{01}(z, 0), \end{aligned} \tag{5}$$

$$a_{mn} \text{ — constants.}$$

See (72), p. 191 of [2].

2. Interior distinguished sets. The kernel function $K_B(z, \bar{z})$ is a *relative invariant*, see (25) p. 180 of [2]. Using K_B, one can get various *absolute invariants* (with respect to PCT's). For instance,

$$J^{(2)} \equiv J_B(z, \bar{z}) = \frac{K_B}{T_{1\bar{1}} T_{2\bar{2}} - |T_{1\bar{2}}|^2} = \frac{K^4}{\begin{vmatrix} K & K_{001\bar{0}} & K_{000\bar{1}} \\ K_{100\bar{0}} & K_{101\bar{0}} & K_{100\bar{1}} \\ K_{010\bar{0}} & K_{011\bar{0}} & K_{010\bar{1}} \end{vmatrix}}, \tag{6}$$

$$T_{m\bar{n}} = \frac{\partial^2 \log K_B}{\partial z_m \partial \bar{z}_n}, \qquad K \equiv K_B,$$

is an (absolute) invariant involving K_B and the first and second derivatives,

$$K_{mn\overline{MN}} = \frac{\partial^{m+n+M+N} K}{\partial z_1^m \partial z_2^n \partial \bar{z}_1^M \partial \bar{z}_2^N}, \qquad m + n + M + N \leqslant 2,$$

of K, see (37a), p. 183 of [2].

LEMMA. *Using the relation*

$$J^{(\nu+1)} = \frac{K}{j_{1\bar{1}}^{(\nu)} j_{2\bar{2}}^{(\nu)} - |j_{1\bar{2}}^{(\nu)}|^2}, \quad j_{m\bar{n}}^{(\nu)} = \frac{\partial^2 \log J_{(\nu)}}{\partial z_m \partial \bar{z}_n}, \tag{7}$$

one can get further absolute invariants.

Proof. Let

$$z_k^* = z_k^*(z_1, z_2), \quad k = 1, 2, \tag{8}$$

be a PCT of B,

$$d^{(\nu)} = j_{11}^{(\nu)} j_{22}^{(\nu)} - |j_{12}^{(\nu)}|^2 = \begin{vmatrix} \dfrac{\partial^2 j^{(\nu)}}{\partial z_1^* \partial \bar{z}_1^*} & \dfrac{\partial^2 j^{(\nu)}}{\partial z_2^* \partial \bar{z}_1^*} \\ \dfrac{\partial^2 j^{(\nu)}}{\partial z_1^* \partial \bar{z}_2^*} & \dfrac{\partial^2 j^{(\nu)}}{\partial z_2^* \partial \bar{z}_2^*} \end{vmatrix} \tag{9}$$

is a relative invariant.

Since

$$d^{(\nu)}(z_1^*, z_2^*) = d^{(\nu)}(z_1, z_2) \left| \frac{\partial(z_1, z_2)}{\partial(z_1^*, z_2^*)} \right|^2 \tag{10}$$

holds, and $z_k(z_1^*, z_2^*)$, $k = 1, 2$, are analytic functions of two complex variables z_1^*, z_2^*,

$$\frac{\partial^2 \log \boldsymbol{J}^{(\nu)}(z_1^*, z_2^*)}{\partial z_m^* \partial \bar{z}_n^*} = \frac{\partial^2 \log \boldsymbol{J}^{(\nu)}(z_1, z_2)}{\partial z_m^* \partial \bar{z}_n^*},$$

Therefore,

$$\begin{vmatrix} \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1^* \partial \bar{z}_1^*} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2^* \partial \bar{z}_1^*} \\ \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1^* \partial \bar{z}_2^*} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2^* \partial \bar{z}_2^*} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1 \partial \bar{z}_1^*} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2 \partial \bar{z}_1^*} \\ \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1 \partial \bar{z}_2^*} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2 \partial \bar{z}_2^*} \end{vmatrix} \frac{\partial(z_1, z_2)}{\partial(z_1^*, \bar{z}_2^*)} \tag{11}$$

$$= \begin{vmatrix} \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1 \partial \bar{z}_1} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2 \partial \bar{z}_1} \\ \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_1 \partial \bar{z}_2} & \dfrac{\partial^2 \log \boldsymbol{J}^{(\nu)}}{\partial z_2 \partial \bar{z}_2} \end{vmatrix} \left| \frac{\partial(z_1, z_2)}{\partial(z_1^*, z_1^*)} \right|^2 .$$

See p. 207 of [10].

Remark. For certain domains (for instance for a hypersphere or a bicylinder) $\boldsymbol{J}_B^{(2)}$ are constant. In general the $\boldsymbol{J}_B^{(\nu)}(z, \bar{z})$, $\nu = 1, 2, \ldots, N$, obtained using procedure (7) are *not* independent from each other.

We suppose that the boundary hypersurface ∂B is at every point twice continuously differentiable (Hypothesis I). Then by a conveniently chosen PCT, see p. 12 of [3], in the neighborhood of every boundary point ∂B can be brought in the form

(12) $$2x_1 - ay_1^2 - \sigma|z_2|^2 + \dots = 0.$$

We assume (Hypothesis II) that at every point of ∂B it holds

(13) $$\sigma > 0.$$

Then, as shown in [3], it holds at every point of ∂B

(14) $$J_B(z, \bar{z}) = \frac{2}{9\pi^2},$$

i.e., $J_B^{(2)}$ assumes constant boundary values on ∂B.

We make now an additional Hypothesis III: $J^{(2)}(z, \bar{z})$ has monotone values, when approaching the boundary ∂B, namely, we assume that it decreases for $z \to Q \varepsilon \partial B$, if we approach along the interior normal to ∂B.

$J^{(2)}(z, \bar{z})$ is an analytic function of four real variables in B. We assume that all critical sets of $J^{(2)}$ are non-degenerate (Hypothesis IV), and therefore by Morse's results the following relations between the numbers M^μ of critical points of index μ of $J^{(2)}$

(15) $$M^0 - M^1 + M^2 - M^3 + M^4 = N$$

and

(16) $$M^k \geqslant R^k$$

hold. Here N is the Euler characteristic, k the connectivity number and R^k the k-th Betti number of B. See (2) and (3), p. 29 of [11].

In the following we shall assume (Hypothesis IV) that among the critical sets of $J_B^{(2)}$ there is one (and only one) which is a critical point, say t, of $J_B^{(2)}$ and where $J_B^{(2)}$ assumes a value which is smaller than the value of $J_B^{(2)}$ at other critical sets. Since $J_B^{(2)}(z, \bar{z})$ is an invariant with respect to PCT's, in a PCT of B onto B^*, $B^* \varepsilon C(B)$, t goes over into the critical point t^* of $J_{B^*}^{(2)}(z^*, \bar{z}^*)$; t^* possesses the same properties as t, in particular relation (4) holds. $R(B, t)$ is denoted as the *distinguished representative domain* in the class $C(B)$.

3. Distinguished representative domains. While in the case of the theory of functions of *one* complex variable, the simply connected, bounded domains B^2 are homogeneous, i.e., by a conformal transformation an arbitrary point $z^{(1)}$, $z^{(1)} \epsilon B^2$, can be transformed into another point $z^{(2)}$, $z^{(2)} \epsilon B^2$, this is no longer the case for domains of two (and n, $n > 2$) complex variables. In this connection, it seems of interest to introduce and investigate the "distinguished sets," i.e., subsets of domain B^4, which in PCT's of B^4 go over in subsets possessing the same characteristic properties. The introduction of the "distinguished boundary surface" represented the first step in this direction. If one attempts to develop the theory of PCT's using the method of the kernel function, it is natural to determine absolute invariants and then to study their critical sets (which we denote as *interior distinguished sets*). The theory of the kernel function leads in a natural way to the notion of representative domains $R(B^n, t)$ with respect to an arbitrary point t, $t \in B^n$, and it seems natural to choose a distinguished point for t. Of course, instead of a critical point of the invariant $J^{(2)}$, one could use some other distinguished points or some other invariants. In the present paper, we use the invariant which seems to be "simplest" in an approach based on the theory of the kernel function: $J^{(2)}(z, \bar{z})$ depends only on K and second derivatives of K.

The second important question is the study of geometrical properties of the representative domains. As shown in [2], [1], [12] and [9], p. 308, the Reinhardt domains, general circular domains and certain (m, p) circular domains are representative domains with respect to the center. On the other hand, as the example of a doubly connected domain (in the case of one complex variable) shows, the representative domains are not necessarily schlicht, see [2], p. 105–107.

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