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ON σ-DISCRETE COVERINGS CONSISTING OF CONNECTED SETS

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In this note it is proved that every open covering of a locally connected paracompact space admits a σ-discrete open refinement consisting of connected sets (¹). As a corollary we infer that every metric locally connected space has a σ-discrete base consisting of connected sets. This theorem gives the affirmative answer to Problems 688 and 689 (²) raised by Engelking and Lelek in [2]. It is not known whether every open covering of a locally connected paracompact space has a locally finite open refinement consisting of connected sets (³).

Definition. Let A be a subspace of a topological space X. Define $C(x, A, X)$ to be the component of A containing x if $x \in A$ and the empty set otherwise.

Definition. We say that a family of sets $\{A_t\}_{t \in T}$ is *directed* if it is directed with respect to the inclusion, i.e., for every $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $A_{t_1} \cup A_{t_2} \subset A_{t_3}$.

LEMMA. *If X is a connected and locally connected space and $\mathfrak{U} = \{U_t\}_{t \in T}$ is an open directed covering of X, then, for every $x \in X$,*

$$X = \bigcup_{t \in T} C(x, U_t, X).$$

Proof. For every $t \in T$, let $\mathfrak{V}_t = \{V_{t,s}\}_{s \in S_t}$ be the family of components of U_t. For every $x_0 \in X$, there exists a finite chain $V_{t_1,s_1}, \ldots, V_{t_n,s_n}$ consisting of sets belonging to $\bigcup_{t \in T} \mathfrak{V}_t$ such that $x \in V_{t_1,s_1}$, $x_0 \in V_{t_n,s_n}$ and $V_{t_i,s_i} \cap V_{t_{i+1},s_{i+1}} \neq 0$ for $1 \leqslant i < n$ (cf. [3], § 46, II, Theorem 8). If we choose $t_0 \in T$ such that $\bigcup_{i=1}^{n} U_{t_i} \subset U_{t_0}$, then $x_0 \in C(x, U_{t_0}, X)$ which completes the proof.

(¹) Our notations and terminology are as in [1].

(²) The positive solution of Problem 689 was first obtained in [4].

(³) The problem has been formulated by A. H. Stone.

COROLLARY 1. *Let X be a normal locally connected space and V an open subset of X. Then V is a connected F_σ if and only if $V = \bigcup_{n=1}^{\infty} V_n$, where V_n are open, connected and $\overline{V_n} \subset V_{n+1}$.*

Proof. Only the "only if" part needs the proof. Since X is normal and V is an F_σ, we can assume that $V = \bigcup_{n=1}^{\infty} U_n$, where U_n are open and $\overline{U_n} \subset U_{n+1}$. Let us choose $x \in V$. By the Lemma,

$$V = \bigcup_{n=1}^{\infty} C(x, U_n, V).$$

It suffices to put $V_n = C(x, U_n, V)$.

Remark. Corollary 1 is a generalization of Theorem 14 in § 49 of [3].

THEOREM. *Every σ-discrete (σ-locally finite) open covering of a locally connected normal and countably paracompact space X admits a σ-discrete (σ-locally finite) open refinement consisting of connected sets.*

Proof. Let $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$ be a σ-discrete (σ-locally finite) open covering of X, where $\mathfrak{V}_n = \{V_{n,t}\}_{t \in T_n}$ is a discrete (locally finite) family. We can find a closed covering $\{F_{n,t}\}_{n=1, t \in T_n}^{\infty}$ of the space X such that $F_{n,t} \subset V_{n,t}$ (cf. [1], Theorem 5.2.1 (iv) and Lemma 1 to Theorem 5.1.3). Since X is normal, there exist open F_σ-sets $G_{n,t}$ such that $F_{n,t} \subset G_{n,t} \subset V_{n,t}$. Let us assume that

$$G_{n,t} = \bigcup_{m=1}^{\infty} G_{n,t}^m,$$

where $G_{n,t}^m$ are open and $\overline{G_{n,t}^m} \subset G_{n,t}^{m+1}$. On the other hand,

$$G_{n,t} = \bigcup_{s \in S_{n,t}} U_{n,t,s},$$

where $\{U_{n,t,s}\}_{s \in S_{n,t}}$ is the family of components of $G_{n,t}$.

Let us put $U_{n,t,s}^m = U_{n,t,s} \cap G_{n,t}^m$. Choose an $x_{n,t,s} \in U_{n,t,s}$ if $U_{n,t,s} \neq 0$ and take an arbitrary $x_{n,t,s}$ if $U_{n,t,s} = 0$. By the Lemma,

$$U_{n,t,s} = \bigcup_{m=1}^{\infty} C(x_{n,t,s}, U_{n,t,s}^m, U_{n,t,s}).$$

Write

$$\mathfrak{C} = \bigcup_{n,m=1}^{\infty} \mathfrak{C}_n^m,$$

where $\mathfrak{C}_n^m = \{C(x_{n,t,s}, U_{n,t,s}^m, U_{n,t,s})\}_{t \in T_n, s \in S_{n,t}}$.

We easily see that $\mathfrak{C}$ is an open refinement of $\mathfrak{V}$ consisting of connected sets. It suffices to prove that, for every $n, m = 1, 2, \ldots$, $\mathfrak{C}_n^m$ is a discrete (locally finite) family.

Let us fix n, m and $x \in X$, and put $T_n(x) = \{t \in T_n\colon x \in G_{n,t}\}$. For every $t \in T_n(x)$, we can find precisely one $s(t) \in S_{n,t}$ such that $x \in U_{n,t,s(t)}$. Then

$$H = \bigcap_{t \in T_n(x)} U_{n,t,s(t)} \setminus \bigcup_{t \in T_n \setminus T_n(x)} \overline{G^m_{n,t}}$$

is an open neighbourhood of x which intersects at most one element (at most finitely many elements) of $\mathfrak{C}^m_n$ which completes the proof.

COROLLARY 2. *Every countable open covering of a countably paracompact, normal and locally connected space has a σ-discrete open refinement consisting of connected sets.*

COROLLARY 3. *Every open covering of a locally connected paracompact space has an open σ-discrete refinement consisting of connected sets.*

COROLLARY 4. *Every metric locally connected space has a σ-discrete base consisting of connected sets.*

Remark. Not every open covering of a strongly paracompact locally connected space admits a star-finite open refinement consisting of connected sets. The hedgehog with $\aleph_0$ prickles (cf. [1], Example 4.1.3), which is metric and separable, is an easy example.

On the other hand, every open covering of a weakly paracompact locally connected space obviously admits a point-finite open refinement, consisting of connected sets.

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REFERENCES

[1] R. Engelking, *Outline of general topology*, Amsterdam - Warszawa 1968.

[2] — and A. Lelek, *Metrizability and weight of inverses under confluent mappings*, Colloquium Mathematicum 21 (1970), p. 239-246.

[3] K. Kuratowski, *Topology II*, New York 1968.

[4] T. Przymusiński, *Metrizability of inverse images of metric spaces under open perfect and 0-dimensional mappings*, Colloquium Mathematicum 24 (1972), p. 175-180.

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