APPROXIMATION AND FUNCTION SPACES
BANACH CENTER PUBLICATIONS, VOLUME 22
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

UNIFORM BOUNDEDNESS OF TRIGONOMETRIC POLYNOMIALS IN SEVERAL VARIABLES OF A SPECIAL TYPE

S. A. TELYAKOVSKII

Steklov Mathematical Institute, Academy of Sciences of the U.S.S.R.
Moscow, U.S.S.R.

It is well known that the polynomials $\sum_{n=1}^{N} \frac{\sin nx}{n}$ are uniformly bounded, i.e. for all N and x we have the estimate

$$\left|\sum_{n=1}^{N} \frac{\sin nx}{n}\right| \leqslant C \tag{1}$$

with an absolute constant C. This fact has various applications in the theory of trigonometric series and in approximation theory.

Similar estimates may also be obtained for polynomials in several variables

$$\sum_{(n_1,\ldots,n_m)\in A} \frac{\sin n_1 x_1}{n_1} \cdots \frac{\sin n_m x_m}{n_m}, \tag{2}$$

where the summation is over all integer points of a "sufficiently regular" domain A of the m-dimensional space (we assume A to be contained in $(0, \infty)^m$).

Clearly, if A is any rectangular parallelepiped with edges parallel to the coordinate hyperplanes, then the polynomials (2) are bounded by a constant depending on m only. Other classes of domains with this property are also known. The author [1] proved, when studying the estimates of the derivatives of trigonometric polynomials in several variables, that if A is taken to consist of all $(n_1, \ldots, n_m)$ satisfying a condition of the form

$$n_1^{r_1} \ldots n_m^{r_m} \leqslant N,$$

where $r_1, \ldots, r_m$, N are arbitrary positive numbers, then the polynomials (2) are bounded by a constant depending on m only.

In this paper we exhibit a more general class of domains A with this property. For simplicity and better visualisation we restrict ourselves to polynomials in two variables, i.e. to the case $m = 2$.

We first recall from [1] the proof of the uniform boundedness of the polynomials

$$\sum_{(n_1, n_2) \in A} \frac{\sin n_1 x_1}{n_1} \frac{\sin n_2 x_2}{n_2},$$

where A consists of all points of the first quadrant of the (n_1, n_2) plane satisfying

$$n_1^{r_1} n_2^{r_2} \leqslant N, \quad r_1 > 0,\ r_2 > 0,\ N > 0. \tag{3}$$

It is sufficient to consider x_1, x_2 with $0 < x_1 \leqslant \pi$, $0 < x_2 \leqslant \pi$. For fixed x_1, x_2 we draw the line $n_1 x_1 = n_2 x_2$ in the (n_1, n_2) plane. Let (N_1, N_2) be its intersection point with the boundary of A (Fig. 1).

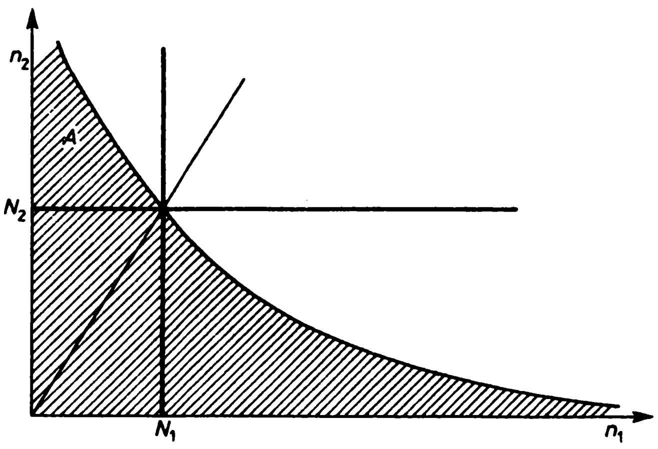

Fig. 1

The lines $n_1 = N_1$ and $n_2 = N_2$ divide the first quadrant into four parts; we will estimate separately the polynomials

$$S(A_j) = \sum_{(n_1, n_2) \in A_j} \frac{\sin n_1 x_1}{n_1} \frac{\sin n_2 x_2}{n_2},$$

where A_j, $j = 1$ to 4, are the respective subsets of A.

Let A_1 be those (n_1, n_2) for which $n_1 \leqslant N_1$ and $n_2 \leqslant N_2$. Then

$$S(A_1) = \left(\sum_{n_1 = 1}^{N_1} \frac{\sin n_1 x_1}{n_1} \right) \left(\sum_{n_2 = 1}^{N_2} \frac{\sin n_2 x_2}{n_2} \right). \tag{4}$$

Note that in general N_1 and N_2 are not integers; the sign $\sum_{n=1}^{N}$ denotes the summation over all n satisfying $1 \leqslant n \leqslant N$.

From (4) and (1) we obtain the estimate

$$|S(A_1)| \leqslant C^2.$$

Let A_2 be those points of A for which $n_2 \leqslant N_2$ and $n_1 > N_1$. We rewrite the double sum $S(A_2)$ as an iterated one:

$$S(A_2) = \sum_{n_2=1}^{N_2} \frac{\sin n_2 x_2}{n_2} \left(\sum_{n_1=N_1+1}^{N(n_2)} \frac{\sin n_1 x_1}{n_1} \right),$$

where $N(n_2)$ is the greatest integer n_1 such that (n_1, n_2) belongs to A. Hence the estimate $|\sin n_2 x_2| \leqslant n_2 x_2$ yields

$$|S(A_2)| \leqslant \sum_{n_2=1}^{N_2} x_2 \left| \sum_{n_1=N_1+1}^{N(n_2)} \frac{\sin n_1 x_1}{n_1} \right|. \tag{5}$$

We now use the well-known estimate

$$\left| \sum_{n=p}^{q} \frac{\sin nx}{n} \right| \leqslant \frac{C_1}{px}, \qquad 0 < x \leqslant \pi,\ p > 0,\ q \leqslant \infty, \tag{6}$$

with an absolute constant C_1. This is easily obtained by applying the Abel transformation.

Applying the estimate (6) to (5), we obtain

$$|S(A_2)| \leqslant \sum_{n_2=1}^{N_2} x_2 \frac{C_1}{N_1 x_1} \leqslant C_1 \frac{N_2 x_2}{N_1 x_1}.$$

But $N_1 x_1 = N_2 x_2$, and therefore

$$|S(A_2)| \leqslant C_1.$$

For the part of A where $n_1 \leqslant N_1$, $n_2 > N_2$, the estimate is similar, and there are no points in A with $n_1 > N_1$ and $n_2 > N_2$.

We have thus proved the estimate

$$\left| \sum_{(n_1, n_2) \in A} \frac{\sin n_1 x_1}{n_1} \frac{\sin n_2 x_2}{n_2} \right| \leqslant C_2$$

with an absolute constant C_2 for domains A described by (3).

The above considerations carry over without change to the so-called normal domains, i.e. to domains which, along with each point (n_1^0, n_2^0), contain the whole rectangle $0 < n_1 \leqslant n_1^0$, $0 < n_2 \leqslant n_2^0$. However, the fact that in the process of decomposition of the domain we have used its boundary does not permit the same reasoning to be carried out for more general classes of domains.

This defect is eliminated in the proof given below, where the decomposition of the domain is independent of its particular shape.

THEOREM. *Let A be any domain contained in the first quadrant of the (n_1, n_2) plane with the property that every line parallel to a coordinate axis has at most two points in common with the boundary of A. Then we have the estimate*

$$\left|\sum_{(n_1,n_2)\in A} \frac{\sin n_1 x_1}{n_1} \frac{\sin n_2 x_2}{n_2}\right| \leqslant C_3, \tag{7}$$

where C_3 is an absolute constant and the summation is over all integer points (n_1, n_2) in A.

Clearly, any convex domain satisfies the assumptions of the theorem. Note that A can contain infinitely many integer points.

We will assume x_1 and x_2 to satisfy $0 < x_1 \leqslant \pi$, $0 < x_2 \leqslant \pi$. For fixed x_1, x_2 we draw the line $n_1 x_1 = n_2 x_2$. It contains the point with the coordinates $M_1 = 1/x_1$, $M_2 = 1/x_2$.

We divide the first quadrant of the (n_1, n_2) plane into five parts, and denote by B_j the respective subsets of A (Fig. 2):

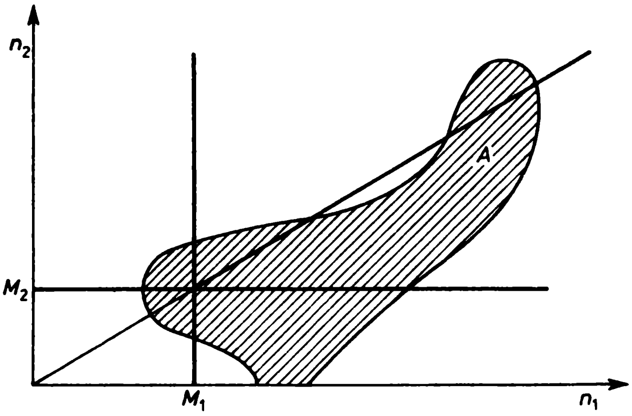

Fig. 2

$$B_1 = \{(n_1, n_2)\in A:\ n_1 \leqslant M_1,\ n_2 \leqslant M_2\},$$

$$B_2 = \{(n_1, n_2)\in A:\ n_1 > M_1,\ n_2 \leqslant M_2\},$$

$$B_3 = \{(n_1, n_2)\in A:\ n_1 \leqslant M_1,\ n_2 > M_2\},$$

$$B_4 = \{(n_1, n_2)\in A:\ n_1 > M_1,\ n_2 > M_2,\ n_1 x_1 \geqslant n_2 x_2\},$$

$$B_5 = \{(n_1, n_2)\in A:\ n_1 > M_1,\ n_2 > M_2,\ n_1 x_1 < n_2 x_2\}.$$

Let us note that a decomposition of the domain into parts satisfying conditions of the form $n_1 x_1 \geqslant n_2 x_2$ was used in [1] in proving the estimates for polynomials in three or more variables.

We now prove the uniform boundedness of each of the sums

$$S(B_j) = \sum_{(n_1,n_2)\in B_j} \frac{\sin n_1 x_1}{n_1}\,\frac{\sin n_2 x_2}{n_2}.$$

We have

$$|S(B_1)| \leqslant \sum_{(n_1,n_2)\in B_1} \frac{|\sin n_1 x_1|}{n_1}\,\frac{|\sin n_2 x_2|}{n_2} \leqslant \sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2} x_1 x_2 \leqslant 1.$$

For B_2, we rewrite the double sum as an iterated one:

$$S(B_2) = \sum_{n_2} \frac{\sin n_2 x_2}{n_2}\left(\sum_{n_1}{}^{(n_2)} \frac{\sin n_1 x_1}{n_1}\right), \tag{8}$$

where the sum over n_2 is taken over all integers $n_2 \leqslant M_2$ such that (n_1, n_2) is in B_2 for some integer n_1, and the sum over n_1 is taken over all integers n_1 with $(n_1, n_2)\in B_2$.

The representation (8) yields

$$|S(B_2)| \leqslant \sum_{n_2} x_2 \left|\sum_{n_1}{}^{(n_2)} \frac{\sin n_1 x_1}{n_1}\right|. \tag{9}$$

By the assumptions on A, the integers n_1 in the inner sum satisfy conditions of the form $p < n_1 < q < \infty$ or $p < n_1 < \infty$. But $p > M_1$ in both cases, and therefore (6) and (9) show that

$$|S(B_2)| \leqslant \sum_{n_2} x_2 \frac{C_1}{M_1 x_1} \leqslant C_1 \sum_{n_2=1}^{M_2} x_2 \leqslant C_1.$$

For $S(B_3)$ the reasoning is similar.

Consider now $S(B_4)$. We again use the representation (8), but now the summation over n_2 is taken over all $n_2 > M_2$ such that (n_1, n_2) is in B_4 for some integer n_1, and the summation over n_1 is taken over all integers n_1 with $(n_1, n_2)\in B_4$.

Applying (8) gives

$$|S(B_4)| \leqslant \sum_{n_2} \frac{1}{n_2}\left|\sum_{n_1}{}^{(n_2)} \frac{\sin n_1 x_1}{n_1}\right|. \tag{10}$$

Here the n_1 also satisfy conditions of the form $p < n_1 < q < \infty$ or $p < n_1 < \infty$. But now $p \geqslant n_2 x_2/x_1$, and so (10) and (6) yield

$$|S(B_4)| \leqslant \sum_{n_2} \frac{1}{n_2}\,\frac{C_1}{\frac{n_2 x_2}{x_1} x_1} \leqslant \frac{C_1}{x_2} \sum_{n_2=M_2+1}^{\infty} \frac{1}{n_2^2} \leqslant \frac{C_1}{x_2 M_2} = C_1.$$

Since the estimate for $S(B_5)$ is analogous, the proof of the theorem is complete.

The corresponding theorem for $m > 2$ is proved similarly. We also assume that each line parallel to a coordinate axis has at most two intersection points with the boundary of A, and we conclude that the polynomials (2) are bounded by a constant depending on m only.

The present paper was written in Warsaw during the activities of the semester on approximation theory at the Banach Center in March 1986. The author expresses his gratitude to Z. Ciesielski and P. Binev for discussing the subject.

References

[1] S. A. Telyakovskii, *On the estimates of the derivatives of trigonometric polynomials in several variables*, Sibirsk. Mat. Zh. 4 (1963), 1404–1411 (in Russian).

Translated from the Russian
by Jerzy Trzeciak

Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986