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ON CHROMATIC NUMBER OF PRODUCTS OF TWO GRAPHS

BY

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Berge [2] and Miller [3] have proved a theorem on the chromatic number of the conjunction of graphs, and Aberth [1] improved the theorem of Berge [2] on the chromatic number of the cartesian product of two graphs.

In this paper we give new proofs of above theorems by applying Vitaver's theorem [5]. We shall also prove some new theorems on the chromatic number for some other operations on two graphs.

A *graph* G is a pair (X, R), where X is a finite set of elements (vertices) and R a relation for which the following two conditions hold:

$$x_1 R x_2 \Rightarrow x_2 R x_1, \tag{1}$$

$$\neg x R x. \tag{2}$$

If G is an undirected graph, we may assign to each edge of G a direction; the resulting directed graph $\vec{G} = (X, \vec{R})$ will be called an orientation of G.

Vitaver [5] proved the following

PROPOSITION 1. *Let $\vec{G}$ be a directed graph which contains no directed path of the length $\geqslant k$, where $k \geqslant 1$. Then G is k-colourable.*

Obviously, $\vec{G}$ is acyclic.

Remarks I. Let an undirected graph G be k-colourable. Then it is easy to see that G has an orientation $\vec{G}$ in which every directed sequence of edges has length $\leqslant k-1$.

II. Let $\chi(G)$ be the chromatic number of graph G. If H is a subgraph of G, then $\chi(H) \leqslant \chi(G)$.

Let $G_1 = (X, R')$, $G_2 = (Y, R'')$ and $x_i \in X$, $y_j \in Y$.

Definition 1. The *conjunction* $G = G_1 \wedge G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1) R (x_2, y_2)$ if and only if $x_1 R' x_2$ and $y_1 R'' y_2$.

THEOREM 1. $\chi(G_1 \wedge G_2) \leqslant \min\{\chi(G_1), \chi(G_2)\}$.

Proof. Let $\chi(G_1) = p$, $\chi(G_2) = q$, and $p \leqslant q$.

As follows from remark I, graph $\vec{G}_1$ contains no directed path of length $\geqslant p$. Let there be given a relation $\vec{R}$ in $\vec{G} = (\overrightarrow{G_1 \wedge G_2})$ by the condition: $(x_1, y_1)\vec{R}(x_2, y_2)$ if and only if $x_1\vec{R}'x_2$ in $\vec{G}_1$. Then $\overrightarrow{G_1 \wedge G_2}$ is acyclic and contains no path of length $\geqslant p$. By proposition 1, we have $\chi(G_1 \wedge G_2) \leqslant \min\{\chi(G_1), \chi(G_2)\}$.

Definition 2. The *cartesian product* $G = G_1 \times G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1R''y_2$ or $y_1 = y_2$ and $x_1R'x_2$.

THEOREM 2. $\chi(G_1 \times G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Proof. Let $\vec{G}_i$ be acyclic $(i = 1, 2)$. In $\vec{G}_1$ there exists the maximal set of vertices $V_0 = \{x^1, \ldots, x^m\}$ such that if $x^iR'x$ (in G_1) then $x^i\vec{R}'x$ for all x, and, similarily, in $\vec{G}_2$ there exists the maximal set of vertices $U_0 = \{y^1, \ldots, y^n\}$ such that if $y^jR''y$ (in G_2) then $y^j\vec{R}''y$ for all y.

Let V_r be the set of vertices of $\vec{G}_1$ such that $x \in V_r$ if and only if $\max\limits_i \varrho(x^i, x) = r < \infty$, and let $T(x) = r$ if and only if $x \in V_r$. Similarily, $y \in U_s$ if and only if $\max\limits_j \varrho(y^j, y) = s < \infty$ and $T(y) = s$ if and only if $y \in U_s$. Obviously, if $\chi(G_1) = p$ and $\chi(G_2) = q$, then $T(x) \leqslant p-1$ and $T(y) \leqslant q-1$.

Let $t = \max\{p, q\}$ and let $T\big((x, y)\big)$ be a function into J_t, the group of integers $(\mathrm{mod}\, t)$, such that

$$(*) \qquad T\big((x, y)\big) = T(x) + T(y).$$

In $G_1 \times G_2$ there do not exist two vertices (x_1, y_1) and (x_2, y_2) such that $(x_1, y_1)R(x_2, y_2)$ and $T\big((x_1, y_1)\big) = T\big((x_2, y_2)\big)$.

Indeed, for, if $(x_1, y_1)R(x_2, y_2)$ and $T\big((x_1, y_1)\big) = T\big((x_2, y_2)\big)$, then, by definition 2,

(a) $T(x_1) = T(x_2)$ and $T(y_1) \neq T(y_2)$

or

(b) $T(y_1) = T(y_2)$ and $T(x_1) \neq T(x_2)$,

and from (a), (b) and (*) it would follow that $T\big((x_1, y_1)\big) \neq T\big((x_2, y_2)\big)$ — a contradiction.

Hence and from (*) it follows that

$$T\big((x, y)\big) \leqslant \max\{p-1, q-1\}.$$

Thus

$$\chi(G_1 \times G_2) \leqslant \max\{\chi(G_1), \chi(G_2)\}.$$

Since in $G_1 \times G_2$ there exist subgraphs H_1 and H_2 such that $H_1 \simeq G_1$ and $H_2 \simeq G_2$, the proof is complete by remark II.

Definition 3. The *disjunction* $G = G_1 \vee G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 R' x_2$ or $y_1 R'' y_2$.

THEOREM 3. $\max\{\chi(G_1), \chi(G_2)\} \leqslant \chi(G_1 \vee G_2) \leqslant \chi(G_1)\chi(G_2)$.

Proof. Clearly, for each graph G_i there exists an epimorphism f_i $(i = 1, 2)$ such that (see [3])

$$f_1: G_1 \to K_p \quad \text{and} \quad f_2: G_2 \to K_q,$$

where $p = \chi(G_1)$ and $q = \chi(G_2)$. If we define

$$f: G_1 \vee G_2 \to K_p \vee K_q$$

by

$$f(x_1, y_1) = \big(f_1(x_1), f_2(y_1)\big),$$

then f is also an epimorphism. Since $K_p \vee K_q \simeq K_{pq}$, we obtain $\chi(G_1 \vee G_2) \leqslant \chi(G_1)\chi(G_2)$. Obviously, $G_1 \vee G_2$ contains subgraphs H_1 and H_2 such that $H_i \simeq G_i$ $(i = 1, 2)$. Thus, by remark II, the theorem is proved.

Definition 4. *Symmetric difference* $G = G_1 \oplus G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if either $x_1 R' x_2$ or $y_1 R'' y_2$ (but not both).

Since $G_1 \oplus G_2$ is a subgraph of $G_1 \vee G_2$, we have, by remark II and definition 4,

THEOREM 4. $\max\{\chi(G_1), \chi(G_2)\} \leqslant \chi(G_1 \oplus G_2) \leqslant \chi(G_1)\chi(G_2)$.

Definition 5. *Joint negation* $G = G_1 \downarrow G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $\neg x_1 R' x_2$ and $\neg y_1 R'' y_2$.

Let $\bar{G}$ be the complement of G. It is clear that $G_1 \downarrow G_2$ is a subgraph of $\bar{G}_1 \vee \bar{G}_2$ and that $G_1 \downarrow G_2$ contains subgraphs H_1 and H_2 such that $H_i \simeq \bar{G}_i$ $(i = 1, 2)$.

By remark II we then obtain

THEOREM 5. $\max\{\chi(\bar{G}_1), \chi(\bar{G}_2)\} \leqslant \chi(G_1 \downarrow G_2) \leqslant \chi(\bar{G}_1)\chi(\bar{G}_2)$.

Definition 6. *Alternative negation* $G = G_1 | G_2$ is a graph $(X \times Y, R)$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $\neg x_1 R' x_2$ or $\neg y_1 R'' y_2$.

Let $|X| = p_1$ and $|Y| = p_2$.

LEMMA 1. *If K_n (a complete graph) is a subgraph of $\bar{G}_2$, then K_{np_1} is a subgraph of $G_1 | G_2$.*

Proof. Let $V(K_n) = \{y_1, \ldots, y_n\}$.

Obviously, in G_2 we have

$$\text{(a)} \qquad \neg y_i R'' y_j \quad \text{for } i, j = 1, \ldots, n.$$

From (a), by definition 6, we infer that $(x_k, y_i)R(x_r, y_j)$ for $k, r = 1, \ldots, p_1$ and $i, j = 1, \ldots, n$. Hence the lemma is proved.

Let I_{G_i} be the isolated set of G_i $(i = 1, 2)$.

LEMMA 2. *If* $I_{G_1} = \{x_1', \ldots, x_s'\}$, *then* K_{sp_2} *is a subgraph of* $G_1|G_2$.

Proof. By definition 6, $(x_i', y_k)R(x_j', y_r)$ for $i, j = 1, \ldots, s$ and $k, r = 1, \ldots, p_2$. Hence K_{sp_2} is a subgraph of $G_1|G_2$.

LEMMA 3. *If* K_n *is a subgraph of* $\bar{G}_2$ *and* $I_{G_1} = \{x_1', \ldots, x_s'\}$, *then* K_d, *where* $d = n(p_1 - s) + p_2 s$, *is a subgraph of* $G_1|G_2$.

Proof. It follows from lemma 1 that K_{d_1}, where $d_1 = n(p_1 - s)$, is a subgraph of $G_1|G_2$, and from lemma 2 that K_{d_2}, where $d_2 = sp_2$, is a subgraph of $G_1|G_2$.

Now it is easy to see that $V(K_{d_1}) \cap V(K_{d_2}) = \emptyset$, and that if $x \in V(K_{d_1})$ and $y \in V(K_{d_2})$, then xRy. Thus the lemma is proved.

By $\{a\}$ we mean the least integer p which $p \geqslant a$.

Nordhaus and Gaddum [4] proved

PROPOSITION 2. $\{2\sqrt{n}\} \leqslant \chi(G) + \chi(\bar{G}) \leqslant n+1$, *where* $n = |V(G)|$.

THEOREM 6. *Let* K_n *be a maximal complete subgraph of* $\bar{G}_2$ *and* K_m *be a maximal complete subgraph of* $\bar{G}_1$. *Then*

$$\max\{d_1, d_2\} \leqslant \chi(G_1|G_2) \leqslant p_1 p_2 + 1 - \chi(G_1 \wedge G_2),$$

where

$$d_1 = n(p_1 - |I_{G_1}|) + p_2 |I_{G_1}|, \qquad d_2 = m(p_2 - |I_{G_2}|) + p_1 |I_{G_2}|.$$

Proof. It follows from lemma 3 that each K_{d_i} $(i = 1, 2)$ is a subgraph of $G_1|G_2$. Hence $\max\{d_1, d_2\} \leqslant \chi(G_1|G_2)$.

Since $G_1|G_2 \simeq \overline{G_1 \wedge G_2}$, we have, by proposition 2,

$$\chi(G_1|G_2) \leqslant p_1 p_2 + 1 - \chi(G_1 \wedge G_2).$$

Thus the theorem is proved.

PROBLEM. Prove or disprove (**P 783**):

$$\chi(G_1|G_2) = \max\{d_1^*, d_2^*\},$$

where

$$d_1^* = \chi(\bar{G}_2)(p_1 - |I_{G_1}|) + p_2 |I_{G_1}|, \qquad d_2^* = \chi(\bar{G}_1)(p_2 - |I_{G_2}|) + p_1 |I_{G_2}|.$$

REFERENCES

[1] O. Aberth, *On the sum of graphs*, Revue Française d'Informatique et de Recherche Opérationnelle 33 (1964), p. 353-358.

[2] C. Berge, *The theory of graphs and its applications*, New York 1962.

[3] D. J. Miller, *The categorial product of graphs*, Canadian Journal of Mathematics 20 (1968), p. 1511-1521.

[4] E. A. Nordhaus and J. W. Gaddum, *On complementary graphs*, American Mathematical Monthly 63 (1956), p. 175-177.

[5] Л. М. Витавер, *Нахождение минимальных раскрасок вершин графа с помощью булевых степеней матрицы смежности*, Доклады Академии Наук СССР 147 (1962), p. 758-759.

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