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On transformations of self-adjoint linear differential systems and their reciprocals

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Abstract. There are investigated transformations of self-adjoint linear differential system

$$(*) \qquad u' = A(x)u + B(x)v, \quad v' = -C(x)u - A^T(x)v,$$

where $A(x)$, $B(x)$, $C(x)$ are $n \times n$ matrices of real-valued functions. The main result (Theorem 1) presents a new method of investigation of system $(*)$ by means of the so-called trigonometric matrices. Using this method, oscillation properties of $(*)$ and related systems are investigated.

1. Introduction. Consider a self-adjoint linear differential system

$$(1.1) \qquad u' = A(x)u + B(x)v, \quad v' = -C(x)u - A^T(x)v,$$

where A, B, C are $n \times n$ matrices of real-valued continuous functions, the matrices B, C are symmetric (i.e. $B^T = B$, $C^T = C$) and u, v are n-dimensional vector-solutions of (1.1). Simultaneously with (1.1), we shall consider the matrix system

$$(1.2) \qquad U' = A(x)U + B(x)V, \quad V' = -C(x)U - A^T(x)V,$$

where U, V are $n \times n$ matrices.

The aim of the paper is to study transformations and oscillations of (1.1). The main result of the paper (Theorem 1) gives a new method of investigation of (1.1) and (1.2) by means of the so-called trigonometric matrices. Using this method, the relationships between oscillation properties of (1.1) and the reciprocal system

$$u' = -A^T(x)u + C(x)v, \quad v' = -B(x)u - A^T(x)v$$

are investigated (Section 4), and some new oscillation criteria for (1.1) and related systems are derived (Section 5). These criteria generalize the known criteria for scalar equations.

Matrix notation is used. E and 0 denote the identity and the zero matrix of any dimension. If A is a symmetric matrix, $A > 0$ ($\geqslant 0$) means that the matrix A is positive (nonnegative) definite. $C^m(I)$ denotes the space of real-valued functions having continuous m-th derivative on an interval I. C^0 means continuity. If $A(x)$ is a matrix of real-valued functions, $A(x) \in C^m(I)$ means that each entry of $A(x)$ is of the class $C^m(I)$. A particular condition is said to hold for large x if there exists a real number c such that the condition holds on $[c, \infty)$.

2. Preliminary results. Let (U_1, V_1), (U_2, V_2) be solutions of (1.2) and let

$$W_i = \begin{pmatrix} U_i \\ V_i \end{pmatrix}, \quad i = 1, 2.$$

Then

$$W_i' = \mathcal{A}(x) W_i, \tag{2.1}$$

where

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -A^T \end{pmatrix}.$$

Write

$$\mathcal{J} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix};$$

then $\mathcal{J}\mathcal{A} + \mathcal{A}^T \mathcal{J} = 0$.

It implies $(W_1^T \mathcal{J} W_2)' = 0$, and hence $W_1^T \mathcal{J} W_2 = K$, where K is a constant $n \times n$ matrix.

A solution (U, V) of (1.2) is said to be *self-conjugate* if $U^T(x)V(x) - V^T(x)U(x) = 0$. Two solutions (U_1, V_1), (U_2, V_2) of (1.2) are said to be *linearly independent* whenever every solution (U, V) of (1.2) can be expressed in the form $(U, V) = (U_1 M + U_2 N, V_1 M + V_2 N)$, where M, N are constant $n \times n$ matrices. From the above-given considerations it follows that (U_1, V_1), (U_2, V_2) are linearly independent if and only if the $2n \times 2n$ matrix $W = (W_1, W_2)$ has rank $2n$. If (U_i, V_i), $i = 1, 2$, are self-conjugate solutions, then $W^T \mathcal{J} W = -\operatorname{diag}\{-W_1^T \mathcal{J} W_2, W_2^T \mathcal{J} W_1\}\mathcal{J}$, i.e. these solutions are linearly independent if and only if $W_1^T \mathcal{J} W_2 = U_1^T V_2 - U_2^T V_1$ is nonsingular.

Now, let $R(x)$ be a nonsingular $2n \times 2n$ matrix consisting of $n \times n$ matrices K, L, M, N,

$$R(x) = \begin{pmatrix} K(x) & M(x) \\ L(x) & N(x) \end{pmatrix}$$

for which $R^T(x)\mathcal{J} R(x) = \mathcal{J}$, i.e. $R(x)$ is $\mathcal{J}$-unitary. Then the transformation

$$W = R(x)Z \tag{2.2}$$

transforms (2.1) into the system

$$Z = \mathcal{B}(x)Z, \tag{2.3}$$

where $\mathscr{B} = R^{-1}(-R' + \mathscr{A}R)$, i.e.

$$\mathscr{B} = \begin{pmatrix} -N^T(K' - AK - BL) + M^T(L' + CK + A^T L) & -N^T(M' - AM - BN) + M^T(N' + CM + A^T N) \\ L^T(K' - AK - BL) - K^T(L' + CK + A^T L) & L^T(M' + AM + BN) - K^T(N' + CM + A^T N) \end{pmatrix}, \tag{2.4}$$

moreover,

$$\mathscr{B}^T(x)\mathscr{J} + \mathscr{J}\mathscr{B}(x) = 0, \tag{2.5}$$

see e.g. [3], i.e. system (2.3) is of the same form as system (2.1).

Two points a, b in an interval I are said to be *(mutually) conjugate* with respect to (1.1) or (1.2) if there exists a solution (u, v) of (1.1) such that $u(a) = 0 = u(b)$ and $u(x)$ is not identically zero between a and b. System (1.1) is said to be *disconjugate on I* whenever no two distinct points of I are conjugate with respect to this system and it is said to be *nonoscillatory* for large x if there exists a real number c such that this system is disconjugate on $[c, \infty)$. In the opposite case, (1.1) is said to be *oscillatory* for large x. Finally, system (1.1) is said to be *identically normal on I* whenever the only solution (u, v) of (1.1) for which $u(x) \equiv 0$ on a nondegenerate subinterval of I is the trivial solution $(u, v) = (0, 0)$.

3. Main result. Consider a matrix differential system

$$S' = Q(x)C, \quad C' = -Q(x)S, \tag{3.1}$$

where $Q(x)$ is a symmetric $n \times n$ matrix. This system was studied by several authors [4], [7], [8], [15] and it was shown that its solutions have many of the properties of the sine and cosine functions (if $n = 1$, $(S(x), C(x)) = (\sin \int^x Q(s)ds, \cos \int^x Q(s)ds)$ is a solution of (3.1)). For this reason, solutions of (3.1) are said to be *trigonometric matrices*. Barrett [4] and Reid [15] showed that using solution of (3.1), the well-known Prüfer transformation for scalar equations can be extended to matrix systems (1.2) and many recently derived oscillation and disconjugacy criteria for (1.1) and (1.2) are based on this generalized Prüfer transformation. The following theorem gives another method of investigation of system (1.1) by means of trigonometric matrices, particularly, it shows that there exists a $2n \times 2n$ matrix $R(x)$ such that transformation (2.2) preserves oscillation behaviour of transformed systems and transforms system (1.2) into (3.1).

THEOREM 1. *There exist $n \times n$ matrices $H(x)$, $K(x) \in C^1(I)$, $H(x)$ being nonsingular, such that transformation* (2.2), *where*

$$R(x) = \begin{pmatrix} H(x) & 0 \\ K(x) & H^{T-1}(x) \end{pmatrix}, \tag{3.2}$$

transforms system (1.2) *into system* (3.1).

Proof. Let (U_i, V_i), $i = 1, 2$, be self-conjugate solutions of (1.2) for which

$$U_1^T(x)V_2(x) - V_1^T(x)U_2(x) = E, \tag{3.3}$$

i.e. for $W_i = \begin{pmatrix} U_i \\ V_i \end{pmatrix}$, $i = 1, 2$, and $W = (W_1, W_2)$ we have $W_1^T \mathcal{J} W_2 = E$, $W^T \mathcal{J} W = \mathcal{J}$, $W \mathcal{J} W^T = \mathcal{J}$, and hence

$$WW^T \mathcal{J} WW^T = \mathcal{J}. \tag{3.4}$$

By Cholesky factorization, see e.g. [9], Chapter 5, there exists a nonsingular $2n \times 2n$ matrix $R_0(x)$ of the form

$$R_0(x) = \begin{pmatrix} D(x) & 0 \\ F(x) & G(x) \end{pmatrix},$$

D, F, G being $n \times n$ matrices, such that $R_0 R_0^T = WW^T$. Let

$$WW^T = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix},$$

where α, β, γ are $n \times n$ matrices; then $\alpha = DD^T$, $\beta = DF^T$, $\gamma = FF^T + GG^T$ and by virtue of (3.4), $\beta\alpha = \alpha\beta^T$, $\gamma\beta = \beta^T\gamma$, $\beta^2 - \alpha\gamma = -E$, $\gamma\alpha - \beta^T\beta^T = E$. These identities imply

$$D^T F - F^T D = D^{-1}(DD^T F D^T - DF^T DD^T)D^{T-1} = D^{-1}(\alpha\beta^T - \beta\alpha)D^{T-1} = 0$$

and

$$\begin{aligned} DD^T GG^T = DD^T(\gamma - FF^T) = \alpha(\gamma - FF^T) &= \beta^2 + E - \alpha FF^T \\ &= E + DF^T DF^T - DF^T FF^T = E; \end{aligned}$$

hence $D^T G = (G^T D)^{-1}$, i.e. the matrix $D^T G$ is orthogonal. Set $R_1 = R_0 \times \operatorname{diag}\{E, G^{-1} D^{T-1}\}$; then $R_1 R_1^T = WW^T$ and

$$R_1^T \mathcal{J} R_1 = \begin{pmatrix} D^T F - F^T D & E \\ -E & 0 \end{pmatrix} = \mathcal{J}.$$

Now, let T be an orthogonal $n \times n$ matrix; then the matrix $R = R_1 \cdot \operatorname{diag}\{T, T\}$ also satisfies the identity $R^T \mathcal{J} R = \mathcal{J}$, i.e. transformation (2.2) transforms (1.2) into system (2.3), where the matrix $\mathcal{B}(x)$ satisfies (2.5). We shall show that he matrix T can be chosen in such a way that the matrix $\mathcal{B}(x)$ is of the form

$$\mathcal{B}(x) = \begin{pmatrix} 0 & Q(x) \\ -Q(x) & 0 \end{pmatrix}.$$

Indeed, let T be the solution of the differential system

$$T' = D^{-1}(AD+B\beta^T D^{T-1}-D')T, \qquad T(a) = E, \qquad a \in I.$$

As

$$\begin{aligned}
\alpha' &= (U_1 U_1^T + U_2 U_2^T)' \\
&= A(U_1 U_1^T + U_2 U_2^T) + (U_1 U_1^T + U_2 U_2^T)A^T + B(V_1 U_1^T + V_2 U_2^T) \\
&\quad + (U_1 V_1^T + U_2 V_2^T)B \\
&= A\alpha + \alpha A^T + B\beta^T + \beta B,
\end{aligned}$$

we have

$$\begin{aligned}
&D^{-1}(AD+B\beta^T D^{T-1}-D') + [D^{-1}(AD+B\beta^T D^{T-1}-D')]^T \\
&= D^{-1}(ADD^T + B\beta^T - D'D^T)D^{T-1} + D^{-1}(DD^T A^T + \beta B - DD^{T'})D^{T-1} \\
&= D^{-1}(A\alpha + \alpha A^T + B\beta^T + \beta B - \alpha')D^{T-1} = 0;
\end{aligned}$$

hence $T(x)$ is orthogonal. Further, let $H = DT$, $K = FT$. Then

$$\begin{aligned}
H' - AH - BK &= (DT)' - ADT - BFT \\
&= D'T + DT' - ADT - B\beta^T D^{T-1} T \\
&= (D' + AD + B\beta^T D^{T-1} - D' - AD - B\beta^T D^{T-1})T = 0
\end{aligned}$$

and

$$\begin{aligned}
&H^T(K' + CH + A^T K) + K^T(H' - AH - BK) \\
&= H^T[(\beta^T H^{T-1})' + CH + A^T \beta^T H^{T-1}] \\
&= H^T(\beta^{T'} H^{T-1} - \beta^T H^{T-1} H^{T'} H^{T-1} + CH + A^T \beta^T H^{T-1}) \\
&= H^T(-C\alpha - A^T\beta^T + \beta^T A^T + \gamma B)H^{T-1} - H^T \beta^T H^{T-1}(K^T B + H^T A^T)H^{T-1} \\
&\quad + H^T CH + H^T A^T \beta^T H^{T-1} \\
&= -H^T CH - H^T \gamma B H^{T-1} - H^T \beta^T H^{T-1} H^{-1} \beta B H^{T-1} + H^T CH \\
&= (H^T \gamma - H^T \beta^T H^{T-1} H^{-1} \beta) B H^{T-1} \\
&= H^{-1}(\alpha\gamma - \alpha\beta^T \alpha^{-1}\beta) B H^{T-1} \\
&= H^{-1}(\alpha\gamma - \beta^2) B H^{T-1} = HBH^{T-1},
\end{aligned}$$

where the identity $\beta' = (U_1 V_1^T + U_2 V_2^T)' = -\alpha C + B\gamma + A\beta - \beta A$ has been used. Consequently, if we write $Q = H^{-1} BH^{T-1}$, according to (2.4) we see that transformation (2.2) with the matrix $R(x)$ given by (3.2) transforms (1.2) into (3.1).

Remark 1. Let $(S(x), C(x))$ be a self-conjugate solution of (3.1) for which

$$S^T(x)S(x) + C^T(x)C(x) = E. \tag{3.5}$$

Then (S, C), $(C, -S)$ form the pair of linearly independent solutions of (3.1) and by Theorem 1 there exist $n\times n$ matrices H, K such that

$$(3.6)\quad (U_1, V_1) = (HS, KS+H^{T-1}C), \quad (U_2, V_2) = (HC, KC-H^{T-1}S)$$

form a pair of self-conjugate linearly independent solutions of (1.2). Conversely, if (U_i, V_i), $i = 1, 2$, is a pair of self-conjugate solutions of (1.2) satisfying (3.3) then there exist a self-conjugate solution (S, C) of (3.1) satisfying (3.5) and matrices H, K, H being nonsingular, such that solutions (U_i, V_i) can be expressed by (3.6). As the transformation converting system (1.2) into system (3.1) preserves oscillation behaviour of differential systems, the results concerning oscillation properties of trigonometric matrices can be used for investigation of systems (1.1) and (1.2). Some oscillation criteria for (1.1) and related systems derived by this method are given in Section 5.

4. Reciprocal systems. In this section, we shall suppose that $B(x)\geqslant 0$, $C(x)\geqslant 0$ on an interval I. Simultaneously with system (1.2) consider the reciprocal system

$$(4.1)\qquad Y' = -A^T(x)Y+C(x)Z, \qquad Z' = -B(x)Y+A(x)Z.$$

Note that if (U, V) is a solution of (1.2) then $(Y, Z) = (V, -U)$ is a solution of (4.1).

There exists considerable duality between disconjugacy criteria for (1.2) and its reciprocal system (4.1). This duality has been studied by Ahlbrandt [1], [2] and by Jakubovič [10], in particular, it is known that under certain assumptions the system (1.2) and the reciprocal system (4.1) have the same oscillation behaviour. In this section we shall use Theorem 1 and some properties of trigonometric matrices in order to give an alternative proof of this result which, in the form given here, is due to Jakubovič [10].

First, recall the following properties of trigonometric matrices.

LEMMA 1 [7]. *Let $Q_i(x)$, $i = 1, 2$, be symmetric continuous $n\times n$ matrices and let (S_i, C_i) be self-conjugate solutions of differential systems*

$$(4.2)_i\qquad S_i' = Q_i(x)C_i, \qquad C_i' = -Q_i(x)S_i,$$

for which $S_i^T(x)S_i(x)+C_i^T(x)C_i(x) = E$. If the matrix $S_1(x)C_2^T(x)-C_1(x)S_2^T(x)$ is nonsingular on I then for every $a\in I$ there exist a real constant $c\in[0, \pi/n)$ and an integer k such that

$$(4.3)\qquad c+k\pi < \frac{1}{n}\int_a^x \operatorname{tr}(Q_1(s)-Q_2(s))ds < c+(k+1)\pi$$

for every $x\in I$.

Proof. See [7], Theorem 4.

LEMMA 2 [15]. *Let $Q(x) \geqslant 0$ on I and suppose that neither the matrix $S(x)$ nor the matrix $C(x)$ can be identically singular on any nondegenerate subinterval of I for every self-conjugate solution (S, C) of* (3.1). *Then* (3.1) *is oscillatory for large x if and only if $\int^{\infty} \operatorname{tr} Q(x)dx = \infty$.*

Proof. See Reid [14], p. 352.

THEOREM 2. *Let $B(x) \geqslant 0$, $C(x) \geqslant 0$ on $I = [a, \infty)$ and suppose that neither the matrix $U(x)$ nor the matrix $V(x)$ can be identically singular on any nondegenerate subinterval of I for every self-conjugate solution (U, V) of* (1.2). *Then system* (1.2) *is oscillatory for large x if and only if the reciprocal system* (4.1) *is oscillatory for large x.*

Proof. Let (U_i, V_i), $i = 1, 2$, be self-conjugate solutions of (1.2) for which (3.3) holds. Set

$$R_0 = \begin{pmatrix} U_1 & U_2 \\ V_1 & V_2 \end{pmatrix}, \quad \tilde{R}_0 = \begin{pmatrix} V_1 & V_2 \\ -U_1 & -U_2 \end{pmatrix}.$$

By the same method as in the proof of Theorem 1 we can show that these matrices generate transformations

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = R \cdot \begin{pmatrix} S_1 \\ C_1 \end{pmatrix}, \quad \begin{pmatrix} Y_2 \\ Z_2 \end{pmatrix} = \tilde{R} \cdot \begin{pmatrix} S_2 \\ C_2 \end{pmatrix},$$

where .

$$R = \begin{pmatrix} H_1 & 0 \\ K_1 & H_1^{T-1} \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} H_2 & 0 \\ K_2 & H_2^{T-1} \end{pmatrix},$$

and these transformations transform (1.2) and (4.1) into systems $(4.2)_1$, and $(4.2)_2$, respectively. According to Remark 1 the pair of solutions (U_i, V_i) can be expressed in the form

$$(U_1, V_1) = (H_1 S_1, K_1 S_1 + H_1^{T-1} C_1),$$
$$(U_2, V_2) = (H_1 C_1, K_1 C_1 - H_1^{T-1} S_1)$$

and

$$(V_1, -U_1) = (H_2 S_2, K_2 S_2 + H_2^{T-1} C_2),$$
$$(V_2, -U_2) = (H_2 C_2, K_2 C_2 - H_2^{T-1} S_2).$$

Comparing equalities for V_1 and V_2, we have

$$(4.4) \qquad H_2 S_2 = K_1 S_1 + H_1^{T-1} C_1, \quad H_2 C_2 = K_1 C_1 - H_1^{T-1} S_1.$$

By multiplication of these equations from the right by C_1^T and $-S_1^T$, respectively, and by addition of these equations, we obtain $S_2 C_1^T - C_2 S_1^T = H_2^{-1} H_1^{T-1}$, i.e. the matrix $S_1 C_2^T - C_1 S_2^T$ is nonsingular. Hence, by Lemma 1, there exist a real constant $c \in [0, \pi/n)$ and an integer k such that (4.3) holds. Consequently, the integrals $\int^{\infty} \operatorname{tr} Q_1(x)dx$ and $\int^{\infty} \operatorname{tr} Q_2(x)dx$ simultaneously converge or diverge and hence systems $(4.2)_{1,2}$ have the same oscillation behaviour for large x. As the transformation converting (1.2) and (4.1) into $(4.2)_1$ and $(4.2)_2$, respectively, preserves oscillation properties of differential systems, system (1.2) and the reciprocal system (4.1) are simultaneously oscillatory or nonoscillatory for large x.

5. Applications. The aim of this section is to give some oscillation criteria for (1.1) and related systems which are based on the transformation described in Theorem 1. For general treatment of oscillation of (1.1) and related systems, the reader is referred to the lectures notes of Coppel [5] and Kreith [11], the books of Reid [14], [16], and the references contained therein.

In the sequel we shall need the following modification of Lemma 2.

LEMMA 3. *Let $Q(x) \geqslant 0$ on $I = [a, \infty)$ and suppose that system* (3.1) *is identically normal for large x. If $\int^{\infty} \operatorname{tr} Q(x)dx = \infty$ then* (3.1) *is oscillatory for large x.*

Proof. See Reid [14], p. 352.

LEMMA 4. *Let $B \geqslant 0$ be a symmetric $n \times n$ matrix and let R be a nonsingular $n \times n$ matrix for which $\|R^{-1}\| \leqslant k$, where $\| \ \|$ denotes the spectral matrix norm and k is a positive real constant. Then $\operatorname{tr} RBR^T \geqslant k^{-2} \operatorname{tr} B$.*

Proof. First observe that if A, D are square matrices, then

$$\|AD\|_E^2 \leqslant \|A\|^2 \|D\|_E^2,$$

where $\|C\|_E^2 = \operatorname{tr}(C^T C)$. Indeed, denote by c_j and d_j the j-th column of the matrix $C = AD$ and D, respectively. We have

$$\|AD\|_E^2 = \operatorname{tr}(C^T C) = \sum_{j=1}^{n} \|c_j\|^2 = \sum_{j=1}^{n} \|Ad_j\|^2 \leqslant \sum_{j=1}^{n} \|A\|^2 \|d_j\|^2 = \|A\|^2 \|D\|_E^2.$$

Now, passing to the proof of Lemma 4, denote by K the symmetric matrix such that $B = K^2$. Then using the above proved inequality, we have

$$\begin{aligned} \operatorname{tr} B = \operatorname{tr} K^2 &= \|K\|_E^2 = \|R^{-1} RK\|_E^2 \\ &\leqslant \|R^{-1}\|^2 \|RK\|_E^2 = \|R^{-1}\|^2 \|KR\|_E^2 = \|R^{-1}\|^2 \operatorname{tr}(R^T BR), \end{aligned}$$

which completes the proof.

THEOREM 3. *Let $B(x) \geqslant 0$ on $I = [a, \infty)$, $\int^{\infty} \operatorname{tr} B(x)dx = \infty$ and suppose that* (1.2) *is identically normal for large x. If for every solution $(U(x), V(x))$ of* (1.2) *the function $\|U(x)\|$ is bounded for large x then system* (1.2) *is oscillatory for large x.*

Proof. Let (U_i, V_i), $i = 1, 2$, be self-conjugate solutions of (1.2) for which (3.3) holds. As these solutions are linearly independent, every solution (U, V) of (1.2) can be expressed in the form $(U, V) = (U_1 M + U_2 N,\ V_1 M + V_2 N)$, where M, N are constant $n \times n$ matrices. By Theorem 1 there exist $n \times n$ matrices $H, K \in C^1(I)$, H being nonsingular, such that transformation (2.2) with the matrix R given by (3.2) transforms (1.2) into (3.1), i.e. $U_1 = HS$, $U_2 = HC$, where (S, C) is a self-conjugate solution of (3.1) for which (3.5) holds. This identity implies that $\|S(x)\| \leqslant 1$, $\|C(x)\| \leqslant 1$, hence $\|U(x)\|$ is bounded if and only if $\|H(x)\|$ is bounded, say $\|H(x)\| \leqslant k$, k being a positive real constant. As the matrix Q in (3.1) is given by the relation $Q = H^{-1} B H^{T-1}$, it follows from Lemma 4 that

$$\int^{\infty} \operatorname{tr} Q(x)dx = \int^{\infty} \operatorname{tr}\big(H^{-1}(x)B(x)H^{T-1}(x)\big)dx \geqslant k^{-2} \cdot \int^{\infty} \operatorname{tr} B(x)dx = \infty.$$

Now, by Lemma 3 system (3.1) is oscillatory for large x and hence system (1.2) is also oscillatory for large x.

Remark 2. If $n = 1$ and $B(x) = 1$, Theorem 3 states the well-known fact that a scalar differential equation $y'' + C(x)y = 0$ is oscillatory if all its solutions are bounded, see e.g. [17], p. 64.

Remark 3. Similarly we can prove the following more general version of Theorem 3:

Let $B(x) \geqslant 0$ and suppose that (1.2) *is identically normal for large x. If there exists a nonsingular $n \times n$ matrix $H(x) \in C^1(I)$ such that*

$$\int^{\infty} \operatorname{tr}\big(H^{-1}(x) B(x) H^{T-1}(x)\big)dx = \infty$$

and for every solution (U, V) of (1.2) *the matrix $\|H^{-1}(x) U(x)\|$ is bounded for large x, then* (1.2) *is oscillatory for large x.*

Indeed, the transformation

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & H^{T-1} \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}$$

transforms (1.2) into the system of the same form whose solution is $(H^{-1} U, H^T V)$, with the matrix $H^{-1} BH^{T-1}$ instead of B. The remaining part is the same as in the proof of Theorem 3.

Now, consider a scalar linear differential equation of the even order

$$\sum_{j=0}^{n} \big(p_j(x)\, y^{(j)}\big)^{(j)} = 0, \tag{5.1}$$

where $p_j(x)$, $j = 0, \ldots, n$, are real-valued functions and $p_n(x) > 0$. Setting $u_j = y^{(j-1)}$, $j = 1, \ldots, n$, $v_n = p_n y^{(n)}$, $v_{n-j} = -v_{n-j+1} + (-1)^j p_{n-j} y^{(n-j)}$,

$j = 1, \ldots, n-1$, we can rewrite equation (5.1) in the form (1.1), where $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T$ and

$$B = \operatorname{diag}\{0, \ldots, 0, p_n^{-1}\},$$

$$C = \operatorname{diag}\{(-1)^{n-j-1} p_j\}, \quad j = 0, \ldots, n-1, \tag{5.2}$$

$$A_{ij} = \begin{cases} 0 & \text{for } j \neq i+1,\ 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant n-1 \\ 1 & \text{for } j = i+1,\ i = 1, \ldots, n-1. \end{cases}$$

Recall that real points a, b are said to be conjugate with respect to equation (5.1) if there exists a nontrivial solution y of this equation such that $y^{(j)}(a) = 0 = y^{(j)}(b)$, $j = 0, \ldots, n-1$. Thus these points are conjugate with respect to equation (5.1) if and only if they are conjugate with respect to system (1.1) with the above given matrices A, B, C, i.e. the definitions of oscillation, disconjugacy, etc. for system (1.1) comply with definitions of these concepts for equation (5.1).

THEOREM 4. *Let* $k \in \{1, \ldots, n\}$ *and suppose*

(i) $\int^{\infty} p_n^{-1}(x) \cdot x^{2(k-1)} dx = \infty$,

(ii) *for every solution* y *of* (5.1) *the functions* $y, \ldots, y^{(n-k)}$ *are bounded on* $I = [a, \infty)$ *and* $y^{(n-k+j)} = O(x^{-j})$ *for* $x \to \infty$, $1 \leqslant j \leqslant k-1$.

Then equation (5.1) *is oscillatory for large* x.

Proof. First, let $k = 1$. Then for every solution of (5.1) the functions $y, \ldots, y^{(n-1)}$ are bounded and hence for every solution (u, v) of (1.1) with the matrices A, B, C given by (5.2) the function $\|u(x)\|$ is also bounded. The statement then follows from Theorem 3, since system (1.1) with A, B, C given by (5.2) is identically normal, see e.g. [2]. If $k > 1$, let $H(x)$ be the solution of the system $H' = \bar{A}H$, $H(0) = E$, where $\bar{A}$ is the $n \times n$ matrix with entries

$$\bar{A}_{ij} = \begin{cases} 1 & \text{for } j = i+1,\ n-k+1 \leqslant i \leqslant n, \\ 0 & \text{elsewhere},\ 1 \leqslant i,\ j \leqslant n. \end{cases}$$

Then $H = \operatorname{diag}\{E_{n-k}, \bar{H}\}$, where E_{n-k} is the $(n-k)$-dimensional identity matrix and $\bar{H}_{ij} = 0$ for $i > j$, $\bar{H}_{ij} = ((j-i)!)^{-1} x^{j-i}$ for $j \geqslant i$, $1 \leqslant i$, $j \leqslant k$. Denote $G = H^{-1}$, then $G = \operatorname{diag}\{E_{n-k}, \bar{G}\}$, where $\bar{G}_{ij} = (-1)^{j-i}\bar{H}_{ij}$, $1 \leqslant i, j \leqslant k$. The transformation $u = Hy$, $v = H^{T-1} z$ transforms system (1.1) into the system

$$y' = A_1(x)y + B_1(x)z, \qquad z' = -C_1(x)y - A_1^T(x)z, \tag{5.3}$$

where $B_1 = GBG^T$. If $B(x)$ is given by (5.2), $\operatorname{tr} B_1 = \sum_{j=0}^{k-1} p_n^{-1}(j!)^{-2} \cdot x^{2j}$, hence $\int^{\infty} \operatorname{tr} B_1(x) dx = \infty$. Let $u = (u_1, \ldots, u_n)^T$. Then

$$u_j = y_j, \quad j = 1, \ldots, n-k,$$

$$y_{n-k+j} = \sum_{i=j}^{k} (-1)^{i-j} [(i-j)!]^{-1} \cdot x^{i-j} u_{n-k+i}, \quad j = 1, \ldots, k,$$

i.e. the functions $y_1, \ldots, y_n$ are bounded and the same argument as in the case $k = 1$ implies that (5.3) is oscillatory. As the matrix $H(x)$ is nonsingular system (1.1) corresponding to equation (5.1) is also oscillatory.

Remark 4. Oscillation criteria of a similar kind for two term equation

$$(p_n(x) y^{(n)})^{(n)} + p_0(x) y = 0 \tag{5.4}$$

have been recently derived by Müller-Pfeiffer [12], [13], but using variation principle. Particularly, the theorem of Müller-Pfeiffer states that (5.4) is oscillatory if assumption (i) of Theorem 4 is satisfied and there exists a real-valued polynomial $Q_{n-k}(x) = a_0 + \ldots + a_{n-k} x^{n-k}$ such that

$$\int^{\infty} (-1)^n p_0(x) Q_{n-k}^2(x) dx = -\infty.$$

Remark 5. Consider a self-adjoint linear differential system of the even order

$$\sum_{k=0}^{n} (P_k(x) y^{(k)})^{(k)} = 0, \tag{5.5}$$

where $P_k(x)$, $k = 0, \ldots, n$, are symmetric $n \times n$ matrices and $P_n(x) > 0$. Oscillation and disconjugacy of this system is defined similarly as in the case of scalar equation (5.1). By the same method as in Theorem 4 we can prove the following statement:

Let $k \in \{1, \ldots, n\}$ and suppose that (i) $\int^{\infty} \operatorname{tr}(P_n^{-1}(x)) dx = \infty$, (ii) for every solution y of (5.5) the functions $\|y(x)\|, \ldots, \|y^{(n-k)}(x)\|$ are bounded on $I = [a, \infty)$ and $\|y^{(n-k+j)}(x)\| = O(x^{-j})$ for $x \to \infty$, $1 \leqslant j \leqslant k-1$. Then system (5.5) is oscillatory for large x.

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