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On generalizations of pseudo-harmonic and pseudo-Killing vectors

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Pseudo-harmonic and pseudo-Killing vectors were studied by Yano and Bochner in [2]. Srivastava in paper [1] defined and studied the properties of generalized harmonic and Killing vectors. In the present paper we have defined pseudo-harmonic and pseudo-Killing vectors of type p which, when $p = 1$, reduce to pseudo-harmonic and pseudo-Killing vectors as given in [2]. We have also obtained the properties of pseudo-harmonic and pseudo-Killing vectors of type p which correspond to those of pseudo-harmonic and pseudo-Killing vectors (of type one). In Sections 7 and 8, we have established the relations between pseudo-harmonic and pseudo-Killing vectors of type $p+1$ and of type p in a compact orientable metric manifold with torsion satisfying equations (1.4).

1. Introduction. Let V_n be a n-dimensional compact orientable manifold of class C^{p+2} with a positive definite metric

$$ds^2 = g_{ij}d\varkappa^i d\varkappa^j, \quad g_{ij} = g_{ji}, \tag{1.1}$$

and a metric connection Γ^i_{jk}, so that

$$g_{jk,l} \equiv \frac{\partial g_{jk}}{\partial \varkappa^l} - g_{sk}\Gamma^s_{jl} - g_{js}\Gamma^s_{kl} = 0, \tag{1.2}$$

where the semi-colon followed by an index denotes covariant differentiation with respect to Γ^i_{jk}.

We assume that g_{ij} and all the vectors considered in this paper are of class C^{p+1}.

We denote the torsion tensor by S^i_{jk}, where

$$S^i_{jk} = \tfrac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}), \tag{1.3}$$

and call such a manifold a *metric manifold with torsion*. We shall assume throughout that

$$S^i_{ji} = 0. \tag{1.4}$$

This condition is satisfied automatically if the covariant torsion tensor $S_{jki} = g_{il}S^l_{jk}$ is antisymmetric in all the indices.

Let R^h_{ijh} denote the curvature tensor formed with respect to the connection Γ^i_{jk} and let R_{ij} and R denote the Ricci tensor and the scalar curvature, respectively.

For the convenience of the reader, we state the following two results which are frequently used in this paper:

(a) *In a compact manifold with positive definite metric, if, for a scalar* $\varphi(x)$, *we have* [2]

$$\Delta\varphi \equiv g^{jk}\varphi_{;jk} \geqslant 0,$$

then we have $\Delta\varphi = 0$ (Bochner's lemma).

(b) *For any vector* v^i, *we have*

$$v^i{}_{;i} = \frac{\partial v^i}{\partial x^i} + v^j\Gamma^i_{ji} = v^i{}_{,i} + 2v^j S^i_{ji},$$

where a comma followed by an index denotes covariant differentiation with respect to the Christoffel symbols. Thus

$$v^i{}_{;i} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}v^i}{\partial x^i}, \quad g = |g_{ij}|, \tag{1.5}$$

by virtue of assumption (1.4). *Thus for any vector field* v^i

$$\int_{V_n} v^i{}_{;i}\,dv = 0, \tag{1.6}$$

the integral being taken over the whole manifold, dv being the volume element [2].

We shall write:

(1) $x_{i_1 i_2 \dots i_p}$ by x_{i_1/i_p},

(2) $x_{i_1 i_2 \dots i_p; i_{p+1} \dots i_{p+q}}$ by $x_{i_1/i_p; i_{p+1}/i_{p+q}}$,

(3) $g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_p j_p}$ by $g^{i_1 j_1}/g^{i_{p+1}/i_{p+q}}$,

and

(4) $g^{i_2 j_2}/g^{i_p j_p} x^{i_1}{}_{;j_2/j_p}$ by $x^{i_1; i_2/i_p}$.

2. Let x_i be a vector field and $\varphi = x^{i_1; i_2/i_p} x_{i_1; i_2/i_p}$. The Laplacian of φ is given by $\Delta\varphi = g^{kl}\varphi_{;kl}$, i. e., by

$$\Delta\varphi = 2(x^{i_1; i_2/i_p k} x_{i_1; i_2/i_p k} + g^{kl} x_{i_1; i_2/i_p kl} x^{i_1; i_2/i_p}).$$

Now

$$x^{i_1;i_2/i_pk}x_{i_1;i_2/i_pk} = g^{i_1j_1}/g^{i_pj_p}g^{kl}x_{i_1;i_2/i_pk}x_{j_1;j_2/j_pl}$$

is a positive definite form in $x_{i_1;i_2/i_pk}$ and therefore if x_i satisfies an equation of the form

$$g^{kl}x_{i_1;i_2/i_pkl} = u_{i_1/i_pj_1/j_p}x^{j_1;j_2/j_p} + 2v_{i_1/i_pj_1/j_pr}x^{j_1;j_2/j_pr} \tag{2.1}$$

and

$$u \equiv u_{i_1/i_pj_1/j_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} + 2v_{i_1/i_pj_1/j_pr}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} \geqslant 0,$$

where $u_{i_1/i_pj_1/j_p}$ and $v_{i_1/i_pj_1/j_pr}$ are continuous tensor fields of type indicated by their indices; then $\Delta\varphi \geqslant 0$.

Consequently from Bochner's Lemma $\Delta\varphi = 0$, and in view of the above discussion

$$\begin{aligned} x_{i_1;i_2/i_pk} &= 0, \\ u_{i_1/i_pj_1/j_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} + 2v_{i_1/i_pj_1/j_pr}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} &= 0. \end{aligned} \tag{2.2}$$

Thus we have proved:

THEOREM 1. *In a compact metric manifold with torsion, if a vector x_i satisfies the relation*

$$g^{kl}x_{i_1;i_2/i_pkl} = u_{i_1/i_pj_1/j_p}x^{j_1;j_2/j_p} + 2v_{i_1/i_pj_1/j_pr}x^{j_1;j_2/j_pr},$$

then we cannot have

$$u \equiv u_{i_1/i_pj_1/j_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} + 2v_{i_1/i_pj_1/j_pr}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} \geqslant 0$$

unless we have $x_{i_1;i_2/i_pk} = 0$ and consequently $u = 0$.

Using Ricci identity, we have

$$\begin{aligned} &x_{k;i_2/i_pi_1l} - x_{i_1;i_2/i_pkl} + x_{i_1;i_2/i_pkl} - x_{k;i_2/i_pli_1} \\ &\quad = -x_{a;i_2/i_p}R^a{}_{ki_1l} - \sum_{r=2}^{p} x_{k;i_2/i_{r-1}ai_{r+1}/i_p}R^a{}_{i_ri_1l} - 2x_{k;i_2/i_pa}S^a{}_{i_1l}. \end{aligned} \tag{2.3}$$

Contraction of this equation with g^{kl} gives

$$\begin{aligned} g^{kl}x_{i_1;i_2/i_pkl} &- g^{kl}(x_{i_1;i_2/i_pk} - x_{k;i_2/i_pi_1})_{;l} - x^l{}_{;i_2/i_pli_1} \\ &= x^a{}_{;i_2/i_p}R_{ai_1} - \sum_{r=2}^{p} x^l{}_{;i_2/i_{r-1}}{}^a{}_{i_{r-1}/i_p}R_{ai_ri_1l} - 2x^l{}_{;i_2/i_pa}S^a{}_{i_1l} \\ &= \Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p} - \\ &\qquad - 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}. \end{aligned}$$

Thus if a vector field x_i satisfies

$$(2.4)\qquad g^{kl}(x_{i_1;i_2/i_pk}-x_{k;i_2/i_pi_1})_{;l}+x^l_{;i_2/i_pli_1}=0,$$

then it also satisfies

$$(2.5)\qquad g^{kl}x_{i_1;i_2/i_pkl}$$

$$=\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p}-$$

$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}$$

and consequently Theorem 1 yields

THEOREM 2. *In a compact metric manifold with torsion, there exists no vector field x_i which satisfies*

$$g^{kl}(x_{i_1;i_2/i_pk}-x_{k;i_2/i_pi_1})_{;l}+x^l_{;i_2/i_pli_1}=0$$

and

$$u'\equiv\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p}-$$

$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr}\geqslant 0$$

unless we have $x_{i_1;i_2/i_pk}=0$ and consequently $u'=0$.

Identity (2.3) can also be written as

$$-x_{i_1;i_2/i_pkl}+(x_{i_1;i_2/i_pk}+x_{k;i_2/i_pi_1})_{;l}-x_{k;i_2/i_pli_1}$$

$$=-x_{a;i_2/i_p}R^a{}_{ki_1l}-\sum_{r=2}^{p}x_{k;i_2/i_{r-1}ai_{r+1}/i_p}R^a{}_{i_ri_1l}-2x_{k;i_2/i_pa}S^a{}_{i_1l}$$

which on contraction with g^{kl} gives

$$-g^{kl}x_{i_1;i_2/i_pkl}+g^{kl}(x_{i_1;i_2/i_pk}+x_{k;i_2/i_pi_1})_{;l}-x^l_{;i_2/i_pli_1}$$

$$=\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p}-$$

$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}.$$

Thus if a vector field x_i satisfies

$$(2.6)\qquad g^{kl}(x_{i_1;i_2/i_pk}+x_{k;i_2/i_pi_1})_{;l}-x^l_{;i_2/i_pli_1}=0,$$

then it also satisfies

$$(2.7)\qquad g^{kl}x_{i_1;i_2/i_pkl}=-\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-$$

$$-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x_{j_1;j_2/j_p}+$$

$$+2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}$$

and consequently Theorem 1 yields

THEOREM 3. *In a compact metric manifold with torsion, there exists no vector field x_i which satisfies*

$$g^{kl}(x_{i_1;i_2/i_pk}+x_{k;i_2/i_pi_1})_{;l}-x^l{}_{;i_2/i_pli_1}=0$$

and

$$u'\equiv\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p}-$$
$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr}\leqslant 0$$

unless we have $x_{i_1;i_2/i_pk}=0$ and consequently $u'=0$.

3. Pseudo-harmonic and pseudo-Killing vectors of type p.

DEFINITION. We shall call a vector field x_i *pseudo-harmonic of type p*, if it satisfies the conditions

$$x_{i_1;i_2/i_pi_{p+1}}-x_{i_{p+1};i_2/i_pi_1}=0, \tag{3.1}$$

and

$$x^{i_1}{}_{;i_2/i_pi_1}=0. \tag{3.2}$$

Remark. This definition for $p=1$ reduces to the classical case [2].

Now, if x_i is a pseudo-harmonic vector field of type p, then it satisfies (2.4) and consequently we have (2.5). Thus, as a special case of Theorem 2, we can state:

THEOREM 4. *In a compact metric manifold with torsion, there exists no pseudo-harmonic vector field x_i of type p which satisfies*

$$u'\equiv\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p}-$$
$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr}\geqslant 0$$

unless we have $x_{i_1;i_2/i_{p+1}}=0$ and consequently $u'=0$.

DEFINITION. We shall call a vector field x_i *pseudo-Killing of type p*, if it satisfies the conditions

$$x_{i_1;i_2/i_pi_{p+1}}+x_{i_{p+1};i_2/i_pi_1}=0 \tag{3.3}$$

and then automatically

$$x^{i_1}{}_{;i_2/i_pi_1}=0. \tag{3.4}$$

Remark. This definition for $p=1$ reduces to the classical case [2].

Now, if x_i is a pseudo-Killing vector field of type p, then it satisfies (2.6) and consequently we have (2.7). Thus, as a special case of Theorem 3 we have:

THEOREM 5. *In a compact metric manifold with torsion, there exists no pseudo-Killing vector field x_i of type p which satisfies*

$$u' \equiv \Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \\ - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} - \\ - 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} \leqslant 0$$

unless we have $x_{i_1;i_2/i_{p+1}} = 0$ *and consequently* $u' = 0$.

4. In this section we establish an integral formula for a compact orientable metric manifold with torsion satisfying equations (1.4) and use it to obtain alternative proofs of Theorems 4 and 5.

Let us consider an arbitraty vector field x_i and form the vector field

$$x^{i_1}{}_{;i_2/i_pi_{p+1}}x^{i_{p+1};i_2/i_p} - x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}$$

whose divergence is

$$D \equiv x_{i_1;i_2/i_pi_{p+1}}x^{i_{p+1};i_2/i_pi_1} - \\ - (x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}i_1} - x^{i_{p+1}}{}_{;i_2/i_pi_1i_{p+1}})x^{i_1;i_2/i_p} - x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}{}_{i_1}.$$

From the Ricci identity we have

$$x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}i_1} - x^{i_{p+1}}{}_{;i_2/i_pi_1i_{p+1}} \\ = -x^{a}{}_{i_2/i_p}R_{ai_1} + \sum_{r=2}^{p} x^{i_{p+1}}{}_{;i_2/i_{r-1}ai_{r+1}/i_p}R^{a}{}_{i_ri_1i_{p+1}} - 2x^{i_{p+1}}{}_{;i_2/i_pa}S^{a}{}_{i_1i_{p+1}}.$$

Substituting this into the divergence D we obtain

$$\text{(4.1)}\quad D \equiv x_{i_1;i_2/i_pi_{p+1}}x^{i_{p+1};i_2/i_pi_1} + \Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \\ - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} - \\ - 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} - x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}{}_{i_1}.$$

Integrating both members of (4.1) over the whole manifold and applying (1.6) we obtain the formula

$$\text{(4.2)}\quad \int_{v_n}\Big[\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \\ - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{i_1;i_2/i_p}x^{j_1;j_2/j_p} - \\ - 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p} - x^{i_1;i_2/i_p}x^{j_1;j_2/j_pr} + \\ + x_{i_1;i_2/i_pi_{p+1}}x^{i_{p+1};i_2/i_pi_1} - x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}_{i_1}{}_{i_1}\Big]dv = 0,$$

which is valid for any vector field x_i.

Using (4.2) one obtains alternative proofs of Theorems 4 and 5.

5. A necessary and sufficient condition for a vector field to be pseudo-harmonic of type p or pseudo-Killing of type p.

From Section 3 a pseudo-harmonic vector field x_i of type p satisfies

$$
\begin{aligned}
(5.1)\qquad & g^{kl}x_{i_1;i_2/i_pkl}-\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\\
&-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)\,x^{j_1;j_2/j_p}+\\
&+2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}=0\,.
\end{aligned}
$$

In this section we prove the converse.

For arbitrary vector field x_i we put

$$\varphi = x^{i_1;i_2/i_p}x_{i_1;i_2/i_p}$$

and form

$$\Delta\varphi = 2\,(g^{kl}x_{i_1;i_2/i_pkl}x^{i_1;i_2/i_p}+x^{i_1;i_2/i_pi_{p+1}}x_{i_1;i_2/i_pi_{p+1}})\,.$$

Integrating both members over the whole manifold and applying (1.6), we get

$$(5.2)\qquad \int_{v_n}(g^{kl}x_{i_1;i_2/i_pkl}x^{i_1;i_2/i_p}+x^{i_1;i_2/i_pi_{p+1}}x_{i_1;i_2/i_pi_{p+1}})\,dv=0\,.$$

Using the integral formulae (4.2) and (5.2) we obtain

$$
\begin{aligned}
\int_{v_n}\Big[\Big\{&g^{kl}x_{i_1;i_2/i_pkl}-\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\\
&-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)\,x^{j_1;j_2/j_p}+\\
&+2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}\,x^{j_1;j_2/j_pr}\Big\}x^{i_1;i_2/i_p}+\\
&+x^{i_1;i_2/i_pi_{p+1}}(x_{i_1;i_2/i_pi_{p+1}}-x_{i_{p+1};i_2/i_pi_1})+\\
&+x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}{}_{i_1}\Big]\,dv=0
\end{aligned}
$$

which can be written in the form

$$
\begin{aligned}
(5.3)\qquad \int_{v_n}\Big[\Big\{&g^{kl}x_{i_1;i_2/i_pkl}-\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\\
&-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)\,x^{j_1;j_2/j_p}+\\
&+2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}\,x^{j_1;j_2/j_pr}\Big\}x^{i_1;i_2/i_p}+\\
&+\tfrac{1}{2}(x^{i_1;i_2/i_pi_{p+1}}-x^{i_{p+1};i_2/i_pi_1})(x_{i_1;i_2/i_pi_{p+1}}-x_{i_{p+1};i_2/i_pi_1})+\\
&+x^{i_{p+1}}{}_{;i_2/i_pi_{p+1}}x^{i_1;i_2/i_p}{}_{i_1}\Big]\,dv=0\,.
\end{aligned}
$$

This equation shows that if x_i satisfies (5.1), then we must have

$$x_{i_1;i_2/i_pi_{p+1}} - x_{i_{p+1};i_2/i_pi_1} = 0$$

and

$$x^{i_1}{}_{;i_2/i_pi_1} = 0,$$

i. e., the vector must be pseudo-harmonic of type p. Thus we have proved

THEOREM 6. *In a compact orientable metric manifold with torsion satisfying equation* (1.4), *a necessary and sufficient condition that a vector field x_i be pseudo-harmonic of type p is that it satisfies*

$$g^{kl}x_{i_1;i_2/i_pkl} - \Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \\ - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p} + \\ + 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr} = 0.$$

A similar approach proves

THEOREM 7. *In a compact orientable metric manifold with torsion, satisfying* (1.4), *a necessary and sufficient condition that a vector field x_i be pseudo-Killing of type p is that it satisfies*

$$g^{kl}x_{i_1;i_2/i_pkl} + \Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p} - \\ - \sum_{r=2}^{p} R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p} - \\ - 2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr} = 0$$

and

$$x^{i_1}{}_{;i_2/i_pi_1} = 0.$$

7. In this section we prove that a pseudo-Killing vector of type $p+1$ is a pseudo-harmonic vector of type p.

For an arbitrary vector field x_i the Ricci identity may be written in the form

$$(x_{i_1;i_2/i_pi_{p+1}l} + x_{l;i_2/i_pi_{p+1}i_1}) - x_{l;i_2/i_pi_{p+1}i_1} - x_{i_1;i_2/i_pli_{p+1}} \\ = -x^a{}_{;i_2/i_p}R_{i_1ali_{p+1}} + \sum_{r=2}^{p} x_{i_1;i_2/i_{r-1}}{}^a{}_{i_{r+1}/i_p}R_{ai_rli_{p+1}} - 2x_{i_1;i_2/i_p}{}^aS_{i_{p+1}la}.$$

Contraction of this equation with $g^{i_1i_{p+1}}$ gives

$$g^{i_1i_{p+1}}(x_{i_1;i_2/i_pi_{p+1}l} + x_{l;i_2/i_pi_{p+1}i_1}) - g^{i_1i_{p+1}}x_{l;i_2/i_pi_{p+1}i_1} - x^{i_{p+1}}{}_{;i_2/i_pli_{p+1}} \\ = -x^a{}_{;i_2/i_p}R_{al} + \sum_{r=2}^{p} x^{i_{p+1}}{}_{;i_2/i_{r-1}}{}^a{}_{i_{r+1}/i_p}R_{ai_rli_{p+1}} - 2x^{i_{p+1}}{}_{;i_2/i_p}{}^aS_{i_{p+1}la}.$$

Thus if x_i is a pseudo-Killing vector of type $p+1$, then from above it satisfies

$$-g^{i_1 i_{p+1}} x_{l;i_2/i_p i_{p+1} i_1} = -x^a{}_{;i_2/i_p} R_{al} + \\ + \sum_{r=2}^{p} x^{i_{p+1}}{}_{;i_2/i_{r-1}}{}^{a}{}_{i_{r+1}/i_p} R_{a i_r l i_{p+1}} - 2x^{i_{p+1}}{}_{;i_2/i_p}{}^{a}\, S_{i_{p+1} l a},$$

i. e., it satisfies

$$g^{kl} x_{i_1;i_2/i_p kl} - \\ - \Big(R_{i_1 j_1} g_{i_2 j_2}/g_{i_p j_p} - \sum_{r=2}^{p} R_{i_r j_r j_1 i_1} g_{i_2 j_2}/g_{i_{r-1} j_{r-1}} g_{i_{r+1} j_{r+1}}/g_{i_p j_p}\Big) x^{j_1;j_2/j_p} + \\ + 2S_{i_1 j_1 r} g_{i_2 j_2}/g_{i_p j_p} x^{j_1;j_2/j_p r} = 0,$$

and by Theorem 6, the vector is pseudo-harmonic of type p.

Hence we have proved

THEOREM 8. *In a compact orientable metric manifold with torsion, satisfying* (1.4) *a pseudo-Killing vector field of type* $p+1$ *is pseudo-harmonic of type* p.

Now if a vector field x_i is a pseudo-Killing vector of type p as well as of type $p+1$, we have

$$x_{i_1;i_2/i_p i_{p+1}} + x_{i_{p+1};i_2/i_p i_1} = 0 \tag{7.1}$$

and from previous theorem it is also a pseudo-harmonic vector of type p so that

$$x_{i_1;i_2/i_p i_{p+1}} - x_{i_{p+1};i_2/i_p i_1} = 0. \tag{7.2}$$

Equations (7.1) and (7.2) give

$$x_{i_1;i_2/i_{p+1}} = 0.$$

Conversely, if a vector field x_i satisfies

$$x_{i_1;i_2/i_{p+1}} = 0,$$

it is both a pseudo-Killing vector of type p as well as of type $p+1$. Hence we have proved

THEOREM 9. *In a compact orientable metric manifold with torsion, satisfying* (1.4) *a necessary and sufficient condition for a vector field* x_i *to be pseudo-Killing of type* p *as well as of type* $p+1$ *is that it satisfies*

$$x_{i_1;i_2/i_{p+1}} = 0.$$

8. In this section we prove that a pseudo-harmonic vector field of type $p+1$ satisfying a certain condition is pseudo-Killing of type p.

For an arbitrary vector field x_i the Ricci identity may also be written in the form

$$(x_{i_1;i_2/i_pi_{p+1}l}-x_{l;i_2/i_pi_{p+1}i_1})+x_{l;i_2/i_pi_{p+1}i_1}-x_{i_1;i_2/i_pli_{p+1}}$$
$$=-x^a{}_{;i_2/i_p}R_{i_1ali_{p+1}}+\sum_{r=2}^{p}x_{i_1;i_2/i_{r-1}}{}^a{}_{i_{r+1}/i_p}R_{ai_rli_{p+1}}-2x_{i_1;i_2/i_p}{}^aS_{i_{p+1}la}.$$

This by contraction with $g^{i_1i_{p+1}}$ gives

$$g^{i_1i_{p+1}}(x_{i_1;i_2/i_pi_{p+1}l}-x_{l;i_2/i_pi_{p+1}i_1})+g^{i_1i_{p+1}}x_{l;i_2/i_pi_{p+1}i_1}-x^{i_{p+1}}{}_{;i_2/i_pli_{p+1}}$$
$$=-x^a{}_{;i_2/i_p}R_{al}+\sum_{r=2}^{p}x^{i_{p+1}}{}_{;i_2/i_{r-1}}{}^a{}_{i_{r+1}/i_p}R_{ai_rli_{p+1}}-2x^{i_{p+1}}{}_{;i_2/i_p}{}^aS_{i_{p+1}la}.$$

Thus if x_i is a pseudo-harmonic vector field of type $p+1$, then it satisfies

$$g^{i_1i_{p+1}}x_{l;i_2/i_pi_{p+1}i_1}$$
$$=-x^a{}_{;i_2/i_p}R_{al}+\sum_{r=2}^{p}x^{i_{p+1}}{}_{;i_2/i_{r-1}}{}^a{}_{i_{r+1}/i_p}R_{ai_rli_{p+1}}-2x^{i_{p+1}}{}_{;i_2/ip}{}^aS_{i_{p+1}la},$$

i. e., it satisfies

$$g^{kl}x_{i_1;i_2/i_pkl}+$$
$$+\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_pj_p}-\sum_{r=2}^{p}R_{i_rj_rj_1i_1}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_pj_p}\Big)x^{j_1;j_2/j_p}-$$
$$-2S_{i_1j_1r}g_{i_2j_2}/g_{i_pj_p}x^{j_1;j_2/j_pr}=0,$$

and by Theorem 7, if it satisfies

$$x^{i_1}{}_{;i_2/i_pi_1}=0,$$

it is a pseudo-Killing vector of type p. Hence we have

THEOREM 10. *In a compact orientable metric manifold with torsion, satisfying* (1.4) *a pseudo-harmonic vector field x_i of type $p+1$ satisfying an additional condition*

$$x^{i_1}{}_{;i_2/i_pi_1}=0,$$

is a pseudo-Killing vector of type p.

Now if a vector field x_i is a pseudo-harmonic vector of type p as well as of type $p+1$, we have

$$x_{i_1;i_2/i_pi_{p+1}}-x_{i_{p+1};i_2/i_pi_1}=0 \tag{8.1}$$

and from previous theorem it is also a pseudo-Killing vector of type p so that

$$x_{i_1;i_2/i_pi_{p+1}}+x_{i_{p+1};i_2/i_pi_1}=0. \tag{8.2}$$

The two equations (8.1) and (8.2) give

$$x_{i_1;i_2/i_{p+1}} = 0.$$

Conversely, if a vector field x_i satisfies

$$x_{i_1;i_2/i_{p+1}} = 0,$$

it is both a pseudo-harmonic vector of type p as well as of type $p+1$. Thus we have proved the following

THEOREM 11. *In a compact orientable metric manifold with torsion, satisfying* (1.4) *a necessary and sufficient condition for a vector field x_i to be pseudo-harmonic of type p as well as of type $p+1$ is that it satisfies*

$$x_{i_1;i_2/i_{p+1}} = 0.$$

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