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On non-linear Volterra integral-functional equations in several variables

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Abstract. Let B be an arbitrary Banach space and $G \subset R^n_+$ be a compact set, where $R_+ = [0, +\infty)$. Assume that the functions $F \in C(G \times B^m \times B, B)$, $f_i \in C(G^2 \times B, B)$, $i = 1, \ldots, m$, $\beta \in C(G, G)$, $\alpha_i \in C(G, G)$, $i = 1, \ldots, m$, are given and $\beta(x) \leqslant x$, $\alpha_i(x) \leqslant x$ for $x \in G$, $i = 1, \ldots, m$. ($C(X, Y)$ denotes the class of continuous functions defined on X with range in Y.)

In the paper the non-linear Volterra integral-functional equation

$$\text{(V)} \quad u(x) = F\Big(x, \int_{H_1(x)} f_1\big(x, s, u(\alpha_1(s))\big)(ds)_{p_1}, \ldots$$
$$\ldots, \int_{H_m(x)} f_m\big(x, s, u(\alpha_m(s))\big)(ds)_{p_m}, u(\beta(x))\Big), \quad x \in G,$$

with $H_j(x) \subset E(x) = \{\xi\colon \xi \in G,\ \xi \leqslant x\}$ for $x \in G$, $j = 1, \ldots, m$, is considered.

In the first part of the paper equation (V) is discussed by means of a comparative method. If F and f_i satisfy the Lipschitz condition with respect to all variables except x or x, s, respectively, then, under certain additional assumptions concerning the functions β, α_j and the Lipschitz coefficients, it is proved that there exists exactly one (in a certain class of functions) continuous solution of (V). This solution is the limit of the sequence of successive approximations. It is not assumed that the Lipschitz coefficient k of the function F with respect to the last variable satisfies the condition $k < 1$.

The second part of the paper deals with equation (V) considered in a finite dimensional Banach space. A theorem on the existence of at least one solution of equation (V) is proved. Also in this case conditions milder than $k < 1$ are assumed.

Introduction. Let B be an arbitrary Banach space with norm $\|\cdot\|$. Denote by $C(X, Y)$ the set of all continuous functions defined in X taking values in Y, X, Y being arbitrary metric spaces. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in R^n$ (R^n – real Euclidean space of dimension n) we define $x \leqslant y$ as $x_i \leqslant y_i$ for $i = 1, \ldots, n$. We denote by $|\cdot|$ the Euclidean norm in R^n. Let $G \subset R^n_+$ be a compact set, where $R_+ = [0, +\infty)$. Let

$$E(x) = \{\xi\colon \xi \in G,\ \xi \leqslant x\}.$$

Assume that the functions $F \in C(G \times B^m \times B, B)$, $f_i \in C(G^2 \times B, B)$, $i = 1, \ldots, m$, $\beta \in C(G, G)$, $\alpha_i \in C(G, G)$, $i = 1, \ldots, m$, are given and $\beta(x) \leqslant x$, $\alpha_i(x) \leqslant x$ for $x \in G$, $i = 1, \ldots, m$.

We shall consider the non-linear Volterra integral-functional equation

$$(1)\quad u(x) = F\Big(x, \int_{H_1(x)} f_1\big(x, s, u(\alpha_1(s))\big)(ds)_{p_1}, \ldots \\ \ldots, \int_{H_m(x)} f_m\big(x, s, u(\alpha_m(s))\big)(ds)_{p_m}, u(\beta(x))\Big), \quad x \in G,$$

where $H_j(x) \subset E(x)$ for $x \in G$, $j = 1, \ldots, m$.

We assume further that $H_j(x)$ is contained in a p_j-dimensional hyperplane ($1 \leqslant p_j \leqslant n$), parallel to the coordinate axes, and it is Lebesgue measurable, considered as a p_j-dimensional set. Let $L_{p_j}(H_j(x))$ denotes the p_j-dimensional Lebesgue measure of $H_j(x)$. We assume that p_j does not depend on x.

If a p_j-dimensional hyperplane containing the set $H_j(x)$ and being parallel to the coordinate axes is defined by the equations

$$x_{t_1} = \mathring{x}_{t_1}, \quad x_{t_2} = \mathring{x}_{t_2}, \qquad x_{t_r} = \mathring{x}_{t_r}, \quad r = n-p_j,$$

then $\int_{H_j(x)} g(x, s)(ds)_{p_j}$, where $s = (s_1, \ldots, s_n)$, denotes the p_j-dimensional Lebesgue integral in the space $Ox_{m_1} x_{m_2} \ldots x_{m_{p_j}}$, $m_i \in \{\{1, \ldots, n\} - \{t_1, \ldots, t_r\}\}$, and $s_{t_1} = \mathring{x}_{t_1}, s_{t_2} = \mathring{x}_{t_2}, \ldots, s_{t_r} = \mathring{x}_{t_r}$.

Let $A' = \{i\colon p_i = n\}$, $B' = \{i\colon 1 \leqslant p_i < n\}$. By changing notation, if necessary, we may assume that $A' = \{1, \ldots, k_0\}$, $B' = \{k_0+1, \ldots, m\}$.

We define the sets $\sigma_j \subset \{1, \ldots, n\}$, $j = 1, \ldots, m$, in the following way: if the axis Ox_i is parallel to the p_j-dimensional hyperplane in which the set $H_j(x)$ is contained, then $i \in \sigma_j$. Put $\bar{\sigma}_j = \{1, \ldots, n\} - \sigma_j$.

For each $x \in G$ and $j = 1, \ldots, m$ we introduce the set $G_j(x)$ by

$$G_j(x) = \{s\colon s = (s_1, \ldots, s_n), s_{t_i} = \mathring{x}_{t_i} \text{ for } t_i \in \sigma_j, \\ 0 \leqslant s_{t_i} \leqslant \varphi_{t_i}^{(j)}(x) \text{ for } t_i \in \bar{\sigma}_j\},$$

where $(\varphi_{t_1}^{(j)}, \ldots, \varphi_{t_{p_j}}^{(j)}) = \varphi_j \in C(G, R^{p_j})$, $t_i \in \bar{\sigma}_j$, and $H_j(x) \subset G_j(x) \subset E(x)$. The p_j-dimensional Lebesgue-measure of $G_j(x)$ satisfies $L_{p_j}(G_j(x)) = \prod_{s \in \sigma_j} \varphi_s^{(j)}(x)$.

We adopt the following notations:

$$\int_{H(x)} f\big(x, s, z(\alpha(s))\big) ds = \Big(\int_{H_1(x)} f_1\big(x, s, z(\alpha_1(s))\big)(ds)_{p_1}, \ldots \\ \ldots, \int_{H_m(x)} f_m\big(x, s, z(\alpha_m(s))\big)(ds)_{p_m}\Big);$$

$$L(G(x)) = \big(L_{p_1}(G_1(x)), \ldots, L_{p_m}(G_m(x))\big);$$

if $K = (K_1, \ldots, K_m) \in C(G, R^m)$, then

$$K(x) \int_{H(x)} f\big(x, s, z(\alpha(s))\big) ds = \sum_{j=1}^{m} K_j(x) \int_{H_j(x)} f_j\big(x, s, z(\alpha_j(s))\big)(ds)_{p_j}, \quad x \in G,$$

and

$$K(x) \int_{H(x)} z(\alpha(s))\,ds = \sum_{j=1}^{m} K_j(x) \int_{H_j(x)} z(\alpha_j(s))(ds)_{p_j}, \qquad x \in G.$$

For $K \in C(G, R^m)$ we define

$$K(x)L(G(x)) = \sum_{j=1}^{m} K_j(x) L_{p_j}(G_j(x)), \qquad x \in G.$$

Equation (1) will be written briefly

$$u(x) = F\Big(x, \int_{H(x)} f(x, s, u(\alpha(s)))\,ds, u(\beta(x))\Big), \qquad x \in G. \tag{2}$$

There are various problems which lead to Volterra integral-functional equations of type (2). Perhaps the simplest problem in the theory of differential equations which leads to such an equation with $n = 1$ is the initial-value problem for the ordinary differential-functional equation of the neutral type

$$u'(t) = F\big(t, u(\alpha_1(t)), \ldots, u(\alpha_m(t)), u'(\beta(t))\big), \qquad t \in [0, a],\ u(0) = u_0.$$

Therefore equation (2) is a generalization of equations which have been considered in paper [3] and also of some cases of equations considered in [1], [2], [5], [7], [15].

The various initial value problems for the partial hyperbolic differential-functional equation of the neutral type

$$z_{xy}(x, y) = F\Big(x, y, z\big(\alpha_1^{(0)}(x, y), \alpha_2^{(0)}(x, y)\big), z_x\big(\alpha_1^{(1)}(x, y), \alpha_2^{(1)}(x, y)\big),$$
$$z_y\big(\alpha_1^{(2)}(x, y), \alpha_2^{(2)}(x, y)\big), z_{xy}\big(\beta_1(x, y), \beta_2(x, y)\big)\Big)$$

can be reformulated in terms of Volterra integral-functional equations. Let us consider as an example the Darboux problem, where the domain is a rectangle $\{(x, y)\colon x \in [0, a], y \in [0, b]\}$, and where initial values $u(x, 0) = \sigma(x)$, $x \in [0, a]$, $u(0, y) = \tau(y)$, $y \in [0, b]$ are prescribed. The Volterra integral-functional equation corresponding to that problem is

$$u(x, y) = F\Big(x, y, \sigma\big(\alpha_1^{(0)}(x, y)\big) + \tau\big(\alpha_2^{(0)}(x, y)\big) - \sigma(0) + \int_{H_0(x,y)} u(s, t)\,ds\,dt,$$
$$\sigma'\big(\alpha_1^{(1)}(x, y)\big) + \int_{H_1(x,y)} u(s, t)\,dt, \tau'\big(\alpha_2^{(2)}(x, y)\big) + \int_{H_2(x,y)} u(s, t)\,ds,$$
$$u\big(\beta_1(x, y), \beta_2(x, y)\big)\Big), \qquad (x, y) \in [0, a] \times [0, b],$$

where

$$H_0(x, y) = \{(s, t)\colon s \in [0, \alpha_1^{(0)}(x, y)], t \in [0, \alpha_2^{(0)}(x, y)]\},$$
$$H_1(x, y) = \{(s, t)\colon s = \alpha_1^{(1)}(x, y), t \in [0, \alpha_2^{(1)}(x, y)]\},$$
$$H_2(x, y) = \{(s, t)\colon s \in [0, \alpha_1^{(2)}(x, y)], t = \alpha_2^{(2)}(x, y)\}.$$

Therefore our equation is a generalization of the equation which was considered in paper [6] and of an adequate case of the equation disscused in [4].

The Cauchy problem and the Goursat problem for hyperbolic differential-functional equations leads to a Volterra integral-functional equation of type (2) (see [13]).

Similar initial value problems for equations in more than two variables and problems for equations of higher order can be reformulated in terms of Volterra integral-functional equations.

As a particular case of equation (2) we can obtain the system of Volterra integral equations which was considered by W. Walter in paper [12] and monograph [13]. These papers contain the extensive bibliography concerning Volterra integral equations.

In the case when u is a function of several variables equation (2) is a generalization of equations which have been considered in [8]–[11].

In this paper we give theorems concerning the existence and uniqueness of continuous solutions of (2) in a certain class of functions.

The paper is divided into two parts. In the first part we investigate equation (2) by means of the comparative method. A general formulation of this method can be found in paper [14]. If we assume that F and f_i satisfy the Lipschitz condition with respect to all variables except x or x, s, respectively, then we prove, under certain additional assumptions concerning the functions β, α_j and the Lipschitz coefficients, that there exists exactly one (in a certain class of functions) continuous solution of (2). This solution is the limit of a sequence of successive approximations. This result is obtained by means of the comparative method.

The essential fact in our considerations is that we do not assume that the Lipschitz coefficient k of the function F with respect to the last variable satisfies the condition $k < 1$ (see Lemmas 4–9). If $k < 1$, then we have a theorem on the existence and uniqueness of solutions of (2), which can be obtained by means of the Banach fixed-point theorem.

The second part of the paper concerns equation (2) considered in a finite dimensional Banach space. We prove here a theorem on the existence of at least one solution of equation (2). In this case it is an important fact that we also do not assume that the Lipschitz coefficient k of the function F with respect to the last variable satisfies the condition $k < 1$ (see Lemma 14). This part of the paper is an extension of the result contained in paper [3], where an equation of type (2) with the unknown function of one variable was considered.

Remark 1. Let

$$G^*(x) = \{\xi\colon \xi \leqslant x\}, \quad \tilde{G} = \bigcup_{s \in G} G^*(s).$$

(We do not assume that $G^*(x) \subset G$.) Suppose that the functions $F \in C(G \times B^m \times B, B)$, $f_i \in C(G^2 \times B, B)$, $\alpha_i \in C(\tilde{G}, \tilde{G})$, $i = 1, \dots, m$, $\beta \in C(G, \tilde{G})$, $\varphi \in C(\tilde{G} - G, B)$ are given and $\beta(x) \leqslant x$, $\alpha_i(x) \leqslant x$ for $x \in G$, $i = 1, \dots, m$.

Let us consider the equation

$$\text{(1')} \qquad \begin{aligned} u(x) &= F\Big(x, \int_{H_1(x)} f_1\big(x, s, u(\alpha_1(s))\big)(ds)_{p_1}, \dots \\ &\qquad \dots, \int_{H_m(x)} f_m\big(x, s, u(\alpha_m(s))\big)(ds)_{p_m}, u\big(\beta(x)\big)\Big), \quad x \in G, \\ u(x) &= \varphi(x) \quad \text{for } x \in \tilde{G} - G, \end{aligned}$$

where $H_j(x) \subset \tilde{G}$. (We do not assume that the sets $H_j(x)$ satisfy the condi tion $H_j(x) \subset G$.)

We want to point out that equation (1′) is equivalent to some equation of type (1). We shall prove this only for the case $m = 1$, i.e., for the equation

$$\text{(1'')} \qquad \begin{aligned} u(x) &= F\Big(x, \int_{H(x)} f\big(x, s, u(\alpha(s))\big)(ds)_p, u\big(\beta(x)\big)\Big), \quad x \in G, \\ u(x) &= \varphi(x) \quad \text{for } x \in \tilde{G} - G. \end{aligned}$$

We define for $x \in G$

$$\tilde{H}(x) = \{s \colon s \in H(x) \cap G \text{ and } \alpha(s) \in G\}, \quad \tilde{\tilde{H}}(x) = H(x) - \tilde{H}(x).$$

Then we have

$$\int_{H(x)} f\big(x, s, u(\alpha(s))\big)(ds)_p = \int_{\tilde{H}(x)} f\big(x, s, u(\alpha(s))\big)(ds)_p + \int_{\tilde{\tilde{H}}(x)} f\big(x, s, \varphi(\alpha(s))\big)(ds)_p.$$

Let

$$\tilde{\Delta} = \{x \in G \colon \beta(x) \in G\}, \quad \tilde{\tilde{\Delta}} = \{x \in G \colon \beta(x) \in \tilde{G} - G\}.$$

Let $\tilde{\beta}$ be a function satisfying the following conditions:

(a) $\tilde{\beta} \in C(G, G)$,

(b) $\tilde{\beta}(x) = \beta(x)$ for $x \in \tilde{\Delta}$, $\tilde{\beta}(x) \leqslant x$ for $x \in G$.

Put

$$\tilde{F}(x, u, v) = \begin{cases} F(x, u, v) & \text{for } x \in \tilde{\Delta}, \\ F\big(x, u, \varphi(\beta(x))\big) & \text{for } x \in \tilde{\tilde{\Delta}}. \end{cases}$$

Now equation (1″) is equivalent to the equation

$$u(x) = \tilde{F}\Big(x, \int_{\tilde{H}(x)} f\big(x, s, u(\alpha(s))\big)(ds)_p + \int_{\tilde{\tilde{H}}(x)} f\big(x, s, \varphi(\alpha(s))\big)(ds)_p, u\big(\tilde{\beta}(x)\big)\Big), \quad x \in G,$$

which is of type (1).

PART I

1. Assumptions. Let $x \in G$, $h \in R^n$, $x+h \in G$, $i \in B'$. Suppose that the set $H_i(x)$ is contained in a p_i-dimensional hyperplane ($1 \leqslant p_i < n$) parallel to the $n-p_i$ coordinate axes. We denote this hyperplane by $S_i(x)$. Let the set $H_i(x+h)$ be contained in a p_i-dimensional hyperplane $S_i(x+h)$ parallel to the hyperplane $S_i(x)$. There exists a vector $t_i(x,h) \in R^n$ such that the set $-t_i(x,h)+H_i(x+h)$ is contained in $S_i(x)$.

We introduce

ASSUMPTION H_1 (see [12], p. 970; [13], p. 134). Suppose that:

1° for $i \in A'$ we have $\lim_{h \to 0} L_n[H_i(x) \dot{-} H_i(x+h)] = 0$ uniformly with respect to $x \in G$ (the sign $\dot{-}$ denotes the symmetric difference of two sets),

2° for $i \in B'$ we have, uniformly with respect to $x \in G$,

(a) $\lim_{h \to 0} t_i(x,h) = 0$,

(b) $\lim_{h \to 0} L_{p_i}[H_i(x) \dot{-} (-t_i(x,h)+H_i(x+h))] = 0$.

Remark 2. If $x \in S_i(x)$ for $x \in G$ and $i \in B'$, we may assume that $t_i(x,h) = h$. Condition (a) of Assumption H_1 is satisfied in this case.

ASSUMPTION H_2. Suppose that:

1° the functions $k, \tilde{h} \in C(G, R_+)$, $K = (K_1, \ldots, K_m) \in C(G, R_+^m)$, $\beta \in C(G, G)$ are given and $\beta(x) \leqslant x$ for $x \in G$,

2° we have

$$\tilde{m}(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\tilde{h}(\beta^{(i)}(x)) < +\infty \quad \text{for } x \in G, \tag{3}$$

where

$$k^{(0)}(x) = 1 \quad \text{for } x \in G, \quad k^{(i+1)}(x) = k(x)k^{(i)}(\beta(x)) \quad \text{for } x \in G,\ i = 0,1,2,\ldots,$$

$$\beta^{(0)}(x) = x \quad \text{for } x \in G, \quad \beta^{(i+1)}(x) = \beta(\beta^{(i)}(x)) \quad \text{for } x \in G,\ i = 0,1,2,\ldots,$$

3° we have

$$M(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\left[K(\beta^{(i)}(x))L\big(G(\beta^{(i)}(x))\big)\right] < +\infty \quad \text{for } x \in G, \tag{4}$$

4° $M, \tilde{m} \in C(G, R_+)$, the function

$$\tilde{M}(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\Big[\sum_{j=1}^{m} K_j(\beta^{(i)}(x))L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\Big(\prod_{s \in \ddot{\sigma}_j} x_s\Big)^{-1}\Big]$$

is bounded for $x \in G$.

We adopt the following notation:

$$\bar{m}(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\bar{h}(\beta^{(i)}(x)),$$

$$(Vz)(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\Big[K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z(\alpha(s))\,ds\Big].$$

Remark 3. If

(a) conditions 1° H_2 – 3° H_2 are satisfied,

(b) $\bar{h} \in C(G, R_+)$, $\bar{h}(x) \leqslant \tilde{h}(x)$ for $x \in G$,

(c) z is a non-negative and upper-semicontinuous function,

then $\bar{m}$ and Vz are functions defined in G.

2. The main lemma.

LEMMA 1. *If Assumptions* H_1, H_2 *are satisfied and* $\tilde{h} \in C(G, R_+)$, $\bar{h}(x) \leqslant \tilde{h}(x)$ *for* $x \in G$, *then*:

1° *There exist solutions* $\bar{z}, \tilde{z} \in C(G, R_+)$ *of the equations*

(5) $$z(x) = \bar{m}(x) + (Vz)(x), \quad x \in G$$

and

(6) $$z(x) = \tilde{m}(x) + (Vz)(x), \quad x \in G,$$

respectively. The solutions $\bar{z}$ *and* $\tilde{z}$ *of* (5) *and* (6), *respectively, are unique in the set* $M(G, R_+)$ *of non-negative upper-semicontinuous functions.*

2° *The functions* $\bar{z}$ *and* $\tilde{z}$ *are solutions of the equations*

(7) $$z(x) = K(x) \int_{H(x)} z(\alpha(s))\,ds + k(x) z(\beta(x)) + \bar{h}(x), \quad x \in G$$

and

(8) $$z(x) = K(x) \int_{H(x)} z(\alpha(s))\,ds + k(x) z(\beta(x)) + \tilde{h}(x), \quad x \in G,$$

respectively. Moreover, these solutions are unique in the class $\tilde{M}(G, R_+, \tilde{z})$, *where*

$$\tilde{M}(G, R_+, \tilde{z}) = \{z\colon\ z \in M(G, R_+) \text{ and } \inf[c\colon\ z(x) \leqslant c\tilde{z}(x)] < +\infty\}.$$

The function $\tilde{z}$ *satisfies the condition*

(9) $$\lim_{r\to\infty} k^{(r)}(x)\tilde{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G.$$

3° *The function* $z(x) = 0$ *for* $x \in G$ *is the unique solution of the inequality*

(10) $$z(x) \leqslant K(x) \int_{H(x)} z(\alpha(s))\,ds + k(x) z(\beta(x)), \quad x \in G,$$

in the class $\tilde{M}(G, R_+, \tilde{z})$.

4° *If* $k, \tilde{h}, \bar{h}, K, \beta$ *are non-decreasing in* G *and* $H_j(x) \subset H_j(\bar{x})$ *for* $x < \bar{x}$, $x, \bar{x} \in G$, $j = 1, 2, \ldots, m$, *then* $\bar{z}$ *and* $\tilde{z}$ *are non-decreasing in* G.

Proof. We shall show that equation (5) has exactly one solution in the set $M(G, R_+)$. Let T be the operator defined by the right-hand side of equation (5). We prove that T: $M(G, R_+) \to M(G, R_+)$. Let $z \in M(G, R_+)$, $v_{ij}(x) = \int_{H_j(\beta^{(i)}(x))} z(\alpha_j(s))(ds)_{p_j}$. Then there exists a sequence $\{z_r\}$ such that $z_r \in C(G, R_+)$ and

$$(11) \qquad z_{r+1}(x) \leqslant z_r(x), \quad x \in G, \; r = 1, 2, \ldots, \qquad \text{and} \quad z(x) = \lim_{r \to \infty} z_r(x), \quad x \in G.$$

Let $v_{ij}^{(r)}(x) = \int_{H_j(\beta^{(i)}(x))} z_r(\alpha_j(s))(ds)_{pj}$, $x \in G$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $r = 1, 2, \ldots$ The functions $v_{ij}^{(r)}$ are continuous in G (cf. [12], p. 972), and $v_{ij}^{(r+1)}(x) \leqslant v_{ij}^{(r)}(x)$. From (11) and by the Lebesgue theorem on integration of non-increasing sequences we have $v_{ij}(x) = \lim_{r \to \infty} v_{ij}^{(r)}(x)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $x \in G$. Since v_{ij} is the limit of the non-increasing sequence of continuous functions, we see that $v_{ij} \in M(G, R_+)$. It follows from Dini's theorem and from assumptions 2° of H_1, 3° of H_1 that series (3) and (4) are uniformly convergent in G. From this fact and by the conditions

$$k^{(i)}(x)\bar{h}(\beta^{(i)}(x)) \leqslant k^{(i)}(x)\tilde{h}(\beta^{(i)}(x)), \qquad i = 0, 1, 2, \ldots, \; x \in G,$$

$$k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z(\alpha(s))\, ds\big]$$
$$\leqslant [\sup_{x \in G} z(x)]\, k^{(i)}(x)\big[K(\beta^{(i)}(x)) \cdot L\big(G(\beta^{(i)}(x))\big)\big], \qquad i = 0, 1, 2, \ldots, x \in G,$$

we infer the uniform convergence in G of the following series:

$$\sum_{i=0}^{\infty} k^{(i)}(x)\bar{h}(\beta^{(i)}(x)), \qquad \sum_{i=0}^{\infty} k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z(\alpha(s))\, ds\big].$$

Hence we get $\bar{m} \in C(G, R_+)$, $Vz \in M(G, R_+)$ and consequently, T: $M(G, R_+) \to M(G, R_+)$.

Now we prove that the operator T is a contraction. Let

$$\|z\|_0 = \max_{x \in G} [e^{-\lambda(x_1 + \ldots + x_n)} |z(x)|],$$

where $z \in M(G, R_+)$, and $\lambda > \Lambda = \max[1, \sup_{x \in G} \tilde{M}(x)]$. For $z, w \in M(G, R_+)$

we get

$$|(Tz)(x)-(Tw)(x)| \leqslant \sum_{i=0}^{\infty} k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} |z(\alpha(s))-w(\alpha(s))|\,ds\big]$$

$$\leqslant \|z-w\|_0 \sum_{i=0}^{\infty} k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} e^{\lambda(s_1+\dots+s_n)}\,ds\big].$$

We have the following estimates:

$$\int_{H_j(\beta^{(i)}(x))} e^{\lambda(s_1+\dots+s_n)}(ds)_{p_j} \leqslant \exp(\lambda \sum_{p\in\sigma_j} x_p) \int_{G_j(\beta^{(i)}(x))} \exp(\lambda \sum_{p\in\bar{\sigma}_j} s_p)(ds)_{p_j}$$

$$= \exp(\lambda \sum_{p\in\sigma_j} x_p) \prod_{p\in\bar{\sigma}_j} \left\{\frac{1}{\lambda}\big(\exp[\lambda\varphi_p^{(j)}(\beta^{(i)}(x))]-1\big)\right\}$$

$$\leqslant \frac{1}{\lambda}\exp(\lambda \sum_{p\in\sigma_j} x_p) \prod_{p\in\bar{\sigma}_j} \left\{\exp\left[\lambda x_p \frac{\varphi_p^{(j)}(\beta^{(i)}(x))}{x_p}\right]-1\right\}$$

$$\leqslant \frac{1}{\lambda} e^{\lambda(x_1+\dots+x_n)} L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\big(\prod_{p\in\bar{\sigma}_j} x_p\big)^{-1}.$$

The last inequality is a consequence of the obvious inequality

$$e^{\gamma t}-1 \leqslant \gamma e^t \quad \text{for } \gamma\in[0,1],\ t \geqslant 0.$$

Finally, we obtain

$$|(Tz)(x)-(Tw)(x)|$$

$$\leqslant \frac{1}{\lambda}\|z-w\|_0 \sum_{i=0}^{\infty} k^{(i)}(x)\big[\sum_{j=1}^{m} K_j(\beta^{(i)}(x)) L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\times$$

$$\times\big(\prod_{p\in\bar{\sigma}_j} x_p\big)^{-1}\big] e^{\lambda(x_1+\dots+x_n)}$$

$$\leqslant \frac{1}{\lambda}\Lambda\|z-w\|_0 e^{\lambda(x_1+\dots+x_n)},$$

and consequently

$$\|Tz-Tw\|_0 \leqslant \frac{\Lambda}{\lambda}\|z-w\|_0.$$

Since $\Lambda < \lambda$, then by the Banach fixed point theorem we infer that equation (5) has a unique solution $\bar{z}$ being an upper-semicontinuous function.

We prove that $\bar{z}\in C(G, R_+)$. The solution $\bar{z}$ of equation (5) is the limit of the sequence $\{z_r\}$ which is defined in the following way:

$$z_0\in M(G, R_+), \qquad z_0 \text{ — arbitrarily fixed},$$

$$z_{r+1}(x) = \bar{m}(x)+(Vz_r)(x), \qquad x\in G,\ r = 0, 1, 2, \dots$$

For $z \in M(G, R_+)$ we define $(V^0 z)(x) = z(x)$, $(V^{i+1} z)(x) = \big(V(V^i z)\big)(x)$, $x \in G$, $i = 0, 1, \ldots$ We easily see that

$$z_{r+1}(x) = \sum_{i=0}^{r} (V^i \bar{m})(x) + (V^{r+1} z_0)(x). \tag{12}$$

$\{V^r z_0\}$ is the sequence of successive approximations for the equation $z(x) = (Vz)(x)$. Since this equation has a solution $z(x) = 0$, $x \in G$, which is unique in the set $M(G, R_+)$, it follows that

$$\lim_{r\to\infty} (V^r z_0)(x) = 0 \quad \text{uniformly with respect to } x \in G.$$

Since the functions $V^i \bar{m}$ are continuous in G and the sequence $\{z_r\}$ is uniformly convergent, it follows from (12) that

$$\bar{z}(x) = \sum_{i=0}^{\infty} (V^i \bar{m})(x)$$

is a continuous function in G. This completes the proof of assertion 1° of Lemma 1.

Now we shall prove assertion 2°. At first we prove that equality (9) holds true. It is easy to check that functions $k^{(r)}$ and $\beta^{(r)}$ satisfy the conditions

$$k^{(r)}(x)\, k^{(i)}\big(\beta^{(r)}(x)\big) = k^{(r+i)}(x), \qquad \beta^{(i)}\big(\beta^{(r)}(x)\big) = \beta^{(r+i)}(x), \qquad x \in G,\ r, i = 0, 1, \ldots \tag{13}$$

Formulas (13) and (6) imply

$$k^{(r)}(x)\,\bar{z}\big(\beta^{(r)}(x)\big) = \sum_{i=0}^{\infty} k^{(r+i)}(x)\,\bar{h}\big(\beta^{(r+i)}(x)\big) + \sum_{i=0}^{\infty} k^{(r+i)}(x)\Big[K\big(\beta^{(r+i)}(x)\big) \int_{H(\beta^{(r+i)}(x))} \bar{z}\big(\alpha(s)\big)\,ds\Big].$$

This last equality and (3), (4) imply (9). The uniform convergence of $\{k^{(r)}(x)\,\bar{z}(\beta^{(r)}(x))\}$ follows from the uniform convergence of series (3) and (4).

We observe that any solution of equation (5) is a solution of (7). Indeed, if $\bar{z}$ is a solution of equation (5), we have

$$\begin{aligned}
&\bar{z}(x) - K(x) \int_{H(x)} \bar{z}\big(\alpha(s)\big)\,ds - k(x)\,\bar{z}\big(\beta(x)\big) \\
&= \sum_{i=0}^{\infty} k^{(i)}(x)\,\bar{h}\big(\beta^{(i)}(x)\big) + \sum_{i=0}^{\infty} k^{(i)}(x)\Big[K\big(\beta^{(i)}(x)\big) \int_{H(\beta^{(i)}(x))} \bar{z}\big(\alpha(s)\big)\,ds\Big] - \\
&\quad - K(x) \int_{H(x)} \bar{z}\big(\alpha(s)\big)\,ds - k(x)\Big\{ \sum_{i=0}^{\infty} k^{(i)}\big(\beta(x)\big)\,\bar{h}\big(\beta^{(i)}(\beta(x))\big) + \\
&\quad + \sum_{i=0}^{\infty} k^{(i)}\big(\beta(x)\big)\Big[K\big(\beta^{(i)}(\beta(x))\big) \int_{H(\beta^{(i)}(\beta(x)))} \bar{z}\big(\alpha(s)\big)\,ds\Big]\Big\} \equiv \bar{h}(x),
\end{aligned}$$

which means that $\bar{z}$ is a solution of (7).

Now we prove that $\bar{z}$ is a unique solution of (7) in the set $\tilde{M}(G, R_+, \tilde{z})$. In fact, if $\bar{\bar{z}} \in \tilde{M}(G, R_+, \tilde{z})$ is a solution of (7), then for $r = 1, 2, \ldots$ and any $x \in G$ the equality

$$(14) \quad \bar{\bar{z}}(x) = \sum_{i=0}^{r-1} k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int\limits_{H(\beta^{(i)}(x))} \bar{\bar{z}}(\alpha(s))\,ds\big] + \\ + \sum_{i=0}^{r-1} k^{(i)}(x)\bar{h}(\beta^{(i)}(x)) + k^{(r)}(x)\bar{\bar{z}}(\beta^{(r)}(x))$$

holds.

Since $\bar{\bar{z}} \in \tilde{M}(G, R_+, \tilde{z})$, we have for some $c \in R_+$: $0 \leqslant \bar{\bar{z}}(x) \leqslant c\tilde{z}(x)$ for $x \in G$. Now, according to (9), we obtain

$$(15) \quad \lim_{r\to\infty} k^{(r)}(x)\bar{\bar{z}}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G.$$

If we let $r \to \infty$ in relation (14), we obtain

$$\bar{\bar{z}}(x) = \bar{m}(x) + (V\bar{\bar{z}})(x), \qquad x \in G,$$

i.e. $\bar{\bar{z}}$ is the solution of equation (5). This equation has only the solution $\bar{z}$; hence it results that $\bar{z} = \bar{\bar{z}}$. Thus the proof of 2° is completed.

Now we are going to prove 3°. Let us suppose that $z^* \in \tilde{M}(G, R_+, \tilde{z})$ and z^* is a solution of inequality (10). We obtain easily for $r = 1, 2, \ldots$ and $x \in G$

$$(16) \quad z^*(x) \leqslant \sum_{i=0}^{r-1} k^{(i)}(x)\big[K(\beta^{(i)}(x)) \int\limits_{H(\beta^{(i)}(x))} z^*(\alpha(s))\,ds\big] + k^{(r)}(x)z^*(\beta^{(r)}(x)).$$

Since $z^* \in \tilde{M}(G, R_+, \tilde{z})$, we have for some $c \in R_+$: $0 \leqslant z^*(x) \leqslant c\tilde{z}(x)$ for $x \in G$. By (9) the last inequalities implies

$$\lim_{r\to\infty} k^{(r)}(x)z^*(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G.$$

Letting r in (16) tend to ∞ we get

$$(17) \qquad z^*(x) \leqslant (Vz^*)(x), \qquad x \in G.$$

Let $\{z_r\}$ be the sequence defined in the following way:

$$(18) \quad z_0(x) = \tilde{z}(x), \quad x \in G, \quad z_{r+1}(x) = (Vz_r)(x), \quad x \in G,\ r = 0, 1, 2, \ldots$$

From assertions 1° and 2° of this Lemma and from (18) it follows that

$$(19) \qquad 0 \leqslant z_{r+1}(x) \leqslant z_r(x), \qquad x \in G,\ r = 0, 1, 2, \ldots$$

and

$$(20) \qquad \lim_{r\to\infty} z_r(x) = 0 \quad \text{uniformly in } G.$$

Further, by (17), we obtain

$$z^*(x) \leqslant c z_r(x), \qquad x \in G,\ r = 0, 1, \ldots$$

The last formula together with (20) gives $z^*(x) = 0$ for $x \in G$, which completes the proof of assertion 3°.

The simple proof of assertion 4° is omitted.

LEMMA 2. *If Assumptions* H_1 *and* H_2 *are satisfied and the sequence* $\{w_r\}$ *is defined by the formulas*

$$(21)\qquad w_0(x) = \tilde{z}(x), \qquad w_{r+1}(x) = K(x) \int_{H(x)} w_r(\alpha(s))\,ds + k(x)\,w_r(\beta(x)),$$
$$x \in G,\ r = 0, 1, \ldots$$

then

$$(22)\qquad 0 \leqslant w_{r+1}(x) \leqslant w_r(x) \leqslant w_0(x), \quad x \in G,\ r = 0, 1, 2, \ldots,$$

$$(23)\qquad \lim_{r \to \infty} w_r(x) = 0 \quad \textit{uniformly with respect to } x \in G.$$

Proof. Relations (22) follow by induction. The convergence of the sequence $\{w_r\}$ is implied by (22). Since $w_r \in C(G, R_+)$, it follows that the limit $\bar{w}$ of the sequence $\{w_r\}$ is an upper-semicontinuous function. From (21) it follows that $\bar{w}$ satisfies inequality (10). According to assertion 3° of Lemma 1 we have $\bar{w}(x) = 0$ for $x \in G$. The uniform convergence of the sequence $\{w_r\}$ follows from Dini's theorem.

3. The existence and uniqueness of solutions of equation (2). We introduce the following

ASSUMPTION H_3. Suppose that:

1° There exist functions $l_i \in C(G, R_+)$, $i = 1, \ldots, m$, $k \in C(G, R_+)$ such that

$$\|F(x, u, v) - F(x, \bar{u}, \bar{v})\| \leqslant \sum_{i=1}^{m} l_i(x)\,\|u_i - \bar{u}_i\| + k(x)\,\|v - \bar{v}\|,$$

where $u = (u_1, \ldots, u_m)$, $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, $x \in G$, $u, \bar{u} \in B^m$, $v, \bar{v} \in B$.

2° There exist functions $\bar{l}_i \in C(G, R_+)$, $i = 1, \ldots, m$, such that

$$\|f_i(x, s, z) - f_i(x, s, \bar{z})\| \leqslant \bar{l}_i(x)\,\|z - \bar{z}\|, \quad x, s \in G,\ z, \bar{z} \in B.$$

3° There exists a function $u_0 \in C(G, B)$ such that Assumption H_2 is fulfilled for $\tilde{h}, K$ defined by relations

$$\tilde{h}(x) = \Big\|F\Big(x, \int_{H(x)} f\big(x, s, u_0(\alpha(s))\big)\,ds,\ u_0(\beta(x))\Big) - u_0(x)\Big\|,$$
$$K(x) = \big(l_1(x)\bar{l}_1(x), \ldots, l_m(x)\bar{l}_m(x)\big),$$

and for k defined by condition 1° of Assumption H_3.

LEMMA 3. *If Assumptions* H_1 *and* H_3 *are satisfied and the sequence* $\{u_r\}$ *is defined by the relations*

$$(24)\quad u_{r+1}(x) = F\Big(x, \int_{H(x)} f\big(x, s, u_r(\alpha(s))\big)ds, u_r(\beta(x))\Big), \qquad x\in G,\ r = 0, 1, 2, \ldots,$$

where u_0 *is given by condition* 3° *of Assumption* H_3, *then*

$$(25)\qquad \|u_r(x)-u_0(x)\| \leqslant \tilde{z}(x), \qquad x\in G,\ r = 0, 1, 2, \ldots,$$

$$(26)\qquad \|u_{r+p}(x)-u_r(x)\| \leqslant w_r(x), \qquad x\in G,\ r = 0, 1, 2, \ldots,$$

where $\tilde{z}$ *is defined in Lemma* 1, *and the sequence* $\{w_r\}$ *is defined by relations* (21).

Proof. We prove that (25) is fulfilled. For $r = 0$ this inequality is evidently satisfied. If we assume that $\|u_r(x)-u_0(x)\| \leqslant \tilde{z}(x)$ for $x\in G$, then

$$\begin{aligned}\|u_{r+1}(x)-u_0(x)\| &\leqslant \Big\|F\Big(x, \int_{H(x)} f\big(x, s, u_r(\alpha(s))\big)ds, u_r(\beta(x))\Big)-\\ &\qquad -F\Big(x, \int_{H(x)} f\big(x, s, u_0(\alpha(s))\big)ds, u_0(\beta(x))\Big)\Big\| +\tilde{h}(x)\\ &\leqslant K(x)\int_{H(x)} \tilde{z}(\alpha(s))ds+k(x)\tilde{z}(\beta(x))+\tilde{h}(x) = \tilde{z}(x), \quad x\in G.\end{aligned}$$

Now we obtain (25) by induction.

Next we prove (26). From (21) and (25) it follows that (26) is satisfied for $r = 0$, $p = 0, 1, 2, \ldots$, $x\in G$. If we assume that (26) holds for arbitrarily fixed r and any $p = 0, 1, 2, \ldots$, $x\in G$, then

$$\begin{aligned}\|u_{r+1+p}(x)-u_{r+1}(x)\| &\leqslant \Big\|F\Big(x, \int_{H(x)} f\big(x, s, u_{r+p}(\alpha(s))\big)ds, u_{r+p}(\beta(x))\Big)-\\ &\qquad -F\Big(x, \int_{H(x)} f\big(x, s, u_r(\alpha(s))\big)ds, u_r(\beta(x))\Big)\Big\|\\ &\leqslant K(x)\int_{H(x)} w_r(\alpha(s))ds+k(x)w_r(\beta(x)) = w_{r+1}(x).\end{aligned}$$

Now (26) follows by induction with respect to r.

We have the following

THEOREM 1. *If Assumptions* H_1 *and* H_3 *are satisfied, then there exists a solution* $\bar{u}\in C(G, B)$ *of equation* (2) *such that*

$$(27)\qquad \|\bar{u}(x)-u_0(x)\| \leqslant \tilde{z}(x), \qquad x\in G,$$

$$(28)\qquad \|\bar{u}(x)-u_r(x)\| \leqslant w_r(x), \qquad x\in G,\ r = 0, 1, 2, \ldots,$$

where u_r *and* w_r *are defined by formulas* (24) *and* (21), *respectively. The solution* $\bar{u}$ *of* (2) *is unique in the class*

$$X(G, B) \stackrel{\text{df}}{=} \bigcup_{c\geqslant 0} \{u\colon\ u\in C(G, B),\ \|u(x)-u_0(x)\| \leqslant c\tilde{z}(x),\ x\in G\}.$$

Proof. It follows from (23) and (26) that the sequence $\{u_r\}$ is uniformly convergent in G to a certain function $\bar{u} \in C(G, B)$. Obviously $\bar{u}$ is a solution of (2). The estimates (27) and (28) are implied by (25) and (26), respectively. To prove that the solution $\bar{u}$ of (2) is unique in the class considered, let us suppose that there exists another solution $\bar{\bar{u}}$ of equation (2) and $\bar{\bar{u}} \in X(G, B)$. It is easy to check that the function $z(x) = \|\bar{\bar{u}}(x) - \bar{u}(x)\|$ is an element of the set $\tilde{M}(G, R_+, \tilde{z})$ and

$$z(x) \leqslant K(x) \int_{H(x)} z(\alpha(s))\, ds + k(x) z(\beta(x)), \qquad x \in G.$$

By assertion 3° of Lemma 1 we get $z(x) = 0$ for $x \in G$, and hence $\bar{u}(x) = \bar{\bar{u}}(x)$ for $x \in G$. Thus the proof of Theorem 1 is complete.

4. Continuous dependence of solutions on the right-hand side of equation (2). Let us consider another equation:

$$v(x) = \tilde{F}\Big(x, \int_{\tilde{H}(x)} \tilde{f}\big(x, s, v(\tilde{\alpha}(s))\big)\, ds, v\big(\tilde{\beta}(x)\big)\Big), \qquad x \in G, \tag{29}$$

where the functions $\tilde{F}$, $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m)$, $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m)$, $\tilde{\beta}$ and the sets $\tilde{H}(x) = (\tilde{H}_1(x), \ldots, \tilde{H}_m(x))$ have the same properties as $F, f, \alpha, \beta, H(x)$, which are formulated Assumptions H_1, H_3. Suppose that $\bar{u}$ and $\bar{v}$ are solutions of equations (2) and (29), respectively. Let $\tilde{r} \in C(G, R_+)$ be a function such that $\|\bar{u}(x) - \bar{v}(x)\| \leqslant \tilde{r}(x)$ for $x \in G$. Let

$$q(x) \overset{\text{def}}{=} \Big\| F\Big(x, \int_{H(x)} f\big(x, s, \bar{v}(\alpha(s))\big)\, ds, \bar{v}\big(\beta(x)\big)\Big) -$$

$$- \tilde{F}\Big(x, \int_{\tilde{H}(x)} \tilde{f}\big(x, s, \bar{v}(\tilde{\alpha}(s))\big)\, ds, \bar{v}\big(\tilde{\beta}(x)\big)\Big)\Big\|,$$

$$\tilde{h}_1(x) = \max[\tilde{r}(x), q(x), \tilde{h}(x)], \qquad x \in G.$$

Now we have the following

THEOREM 2. *If the functions F, f, α, β and $\tilde{F}, \tilde{f}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{h}_1$ satisfy Assumption* H_3 *and the sets $H(x)$, $\tilde{H}(x)$, $x \in G$, satisfy Assumption* H_1, *then there exists a solution $\bar{w} \in C(G, R_+)$ of the equation*

$$z(x) = K(x) \int_{H(x)} z(\alpha(s))\, ds + k(x) z(\beta(x)) + q(x), \qquad x \in G, \tag{30}$$

such that

$$\|\bar{u}(x) - \bar{v}(x)\| \leqslant \bar{w}(x) \quad \textit{for } x \in G. \tag{31}$$

Proof. Let $\tilde{z}$ be a solution of the equation

$$z(x) = K(x) \int_{H(x)} z(\alpha(s))\, ds + k(x) z(\beta(x)) + \tilde{h}_1(x), \qquad x \in G.$$

Put

$$w_0(x) = \bar{z}(x), \quad x \in G,$$

$$w_{r+1}(x) = K(x) \int_{H(x)} w_r(\alpha(s))\, ds + k(x) w_r(\beta(x)) + q(x), \quad x \in G,\ r = 0, 1, 2, \ldots$$

By induction we get $0 \leqslant w_{r+1}(x) \leqslant w_r(x) \leqslant \bar{z}(x)$, $x \in G$, $r = 0, 1, 2, \ldots$ From these inequalities we see that the sequence $\{w_r\}$ is convergent to the solution $\bar{w}$ of equation (30) and $0 \leqslant \bar{w}(x) \leqslant \bar{z}(x)$ for $x \in G$. However, in view of Lemma 1, there exists only one solution of this equation in the class of upper-semicontinuous functions satisfying the condition $0 \leqslant \bar{w}(x) \leqslant \bar{z}(x)$ for $x \in G$.

Now, we show that

$$\|\bar{u}(x) - \bar{v}(x)\| \leqslant w_r(x) \quad \text{for } x \in G,\ r = 0, 1, 2, \ldots \tag{32}$$

Since $\tilde{r}(x) \leqslant \tilde{h}_1(x) \leqslant \bar{z}(x) = w_0(x)$, $x \in G$, it follows that (32) is satisfied for $r = 0$ and $x \in G$. If we assume that $\|\bar{u}(x) - \bar{v}(x)\| \leqslant w_r(x)$ for $x \in G$ and for some r, then

$$\begin{aligned}
\|\bar{u}(x) - \bar{v}(x)\| \leqslant & \Big\| F\Big(x, \int_{H(x)} f\big(x, s, \bar{u}(\alpha(s))\big) ds, \bar{u}(\beta(x))\Big) - \\
& \quad - F\Big(x, \int_{H(x)} f\big(x, s, \bar{v}(\alpha(s))\big) ds, \bar{v}(\beta(x))\Big) \Big\| + \\
& + \Big\| F\Big(x, \int_{H(x)} f\big(x, s, \bar{v}(\alpha(s))\big) ds, \bar{v}(\beta(x))\Big) - \\
& \quad - \tilde{F}\Big(x, \int_{\tilde{H}(x)} \tilde{f}\big(x, s, \bar{v}(\tilde{\alpha}(s))\big) ds, \bar{v}(\tilde{\beta}(x))\Big) \Big\| \\
\leqslant & K(x) \int_{H(x)} \|\bar{u}(\alpha(s)) - \bar{v}(\alpha(s))\|\, ds + k(x) \|\bar{u}(\beta(x)) - \bar{v}(\beta(x))\| + q(x) \\
\leqslant & K(x) \int_{H(x)} w_r(\alpha(s))\, ds + k(x) w_r(\beta(x)) + q(x) = w_{r+1}(x), \quad x \in G.
\end{aligned}$$

Now (32) follows by induction. Letting $r \to \infty$ in (32), we get estimation (31).

5. Some effective conditions. We give here simple sufficient conditions for assumptions 2°–4° in H_2 to be satisfied.

LEMMA 4. *Assume that*

1° $k(x) \leqslant \bar{k}$, $K(x) = (K_1(x), \ldots, K_m(x)) \leqslant (\bar{K}_1, \ldots, \bar{K}_m)$, $\bar{k}, \bar{K}_i \in R_+$,

2° $\varphi_i(x) = (\varphi_{t_1}^{(i)}(x), \ldots, \varphi_{t_{p_i}}^{(i)}(x)) \leqslant (\bar{\alpha}_{t_1}^{(i)} x_{t_1}, \ldots, \bar{\alpha}_{t_{p_i}}^{(i)} x_{t_{p_i}})$, *where* $\bar{\alpha}_{t_j}^{(i)} \in R_+$, $t_j \in \sigma_i$, $\bar{\alpha}_{t_j}^{(i)} \leqslant 1$,

3° $\beta(x) = (\beta_1(x), \ldots, \beta_n(x)) \leqslant (\bar{\beta}_1 x_1, \ldots, \bar{\beta}_n x_n)$, $\bar{\beta}_i \in R_+$, $\bar{\beta}_i \leqslant 1$,

4° $\sum_{i=0}^{\infty} \bar{k}^i \tilde{h}(\bar{\beta}^i x_1, \ldots, \bar{\beta}^i x_n) < +\infty$,

5° $\bar{k} \prod_{s \in \sigma_j} \bar{\beta}_s < 1$ *for* $j = 1, 2, \ldots, m$.

Under these assumptions conditions 2°–4° *of Assumption* H_2 *are satisfied.*

Proof. By induction we easily obtain the estimates $k^{(i)}(x) \leqslant \bar{k}^i$, $x \in G$, $i = 0, 1, 2, \ldots$ and $\beta^{(i)}(x) \leqslant (\bar{\beta}_1^i x_1, \ldots, \bar{\beta}_n^i x_n)$, $x \in G$, $i = 0, 1, 2, \ldots$ From these inequalities we get the following estimation for series (3):

$$\sum_{i=0}^{\infty} k^{(i)}(x)\bar{h}(\beta^{(i)}(x)) \leqslant \sum_{i=0}^{\infty} \bar{k}^i \bar{h}(\bar{\beta}_1^i x_1, \ldots, \bar{\beta}_n^i x_n), \qquad x \in G.$$

Now 4° implies condition 2° of Assumption H_2. Since

$$\sum_{i=0}^{\infty} k^{(i)}(x)\left[K(\beta^{(i)}(x))L\big(G(\beta^{(i)}(x))\big)\right]$$
$$= \sum_{j=1}^{m} \Big[\sum_{i=0}^{\infty} k^{(i)}(x)K_j(\beta^{(i)}(x))L_{p_j}\big(G(\beta^{(i)}(x))\big)\Big],$$

then for a fixed index j we have

$$\sum_{i=0}^{\infty} k^{(i)}(x)K_j(\beta^{(i)}(x))L_{p_j}\big(G(\beta^{(i)}(x))\big) \leqslant \sum_{i=0}^{\infty} \bar{k}^i \bar{K}_j \prod_{s \in \sigma_j} \varphi_s^{(j)}(\beta^{(i)}(x))$$
$$\leqslant \bar{K}_j \sum_{i=0}^{\infty} \bar{k}^i \prod_{s \in \bar{\sigma}_j} \bar{\alpha}_s^j \cdot \bar{\beta}_s^{(i)} \cdot x_s \leqslant \bar{K}_j\Big(\prod_{s \in \bar{\sigma}_j} \bar{\alpha}_s^j\Big)\Big(\prod_{s \in \bar{\sigma}_j} x_s\Big) \sum_{i=0}^{\infty} \Big(\bar{k} \prod_{s \in \bar{\sigma}_j} \bar{\beta}_s\Big)^i.$$

Hence and from assumption 5° of this Lemma it follows that condition 3° of Assumption H_2 is satisfied. From Dini's theorem and from the last inequalities it follows that condition 4° of Assumption H_2 is satisfied, too.

Remark 4. Suppose that conditions 1°–4° of Lemma 4 are satisfied. A sufficient condition for the existence of a solution of equations of the type (2) given in Lemma 11 in [11] is of the form

$$\bar{k} \max_{1 \leqslant s \leqslant n} \bar{\beta}_s < 1. \tag{33}$$

We see easily that condition 5° of Lemma 4 is more general than condition (33).

By a similar argument we can prove the following lemmas:

Lemma 5. *If*

1° $k(x) \leqslant \bar{k}$, $K_j(x) \leqslant \bar{K}_1 x_1 + \ldots + \bar{K}_n x_n$, $j = 1, 2, \ldots, m$, $\bar{k}, \bar{K}_i \in R_+$,

2° *assumptions* 2°–4° *of Lemma* 4 *are satisfied, then conditions* 2°–4° *of Assumption* H_2 *are fulfilled.*

Lemma 6. *If*

1° $G = [0, a]$, $a = (a_1, \ldots, a_n)$, $a_i > 0$, $i = 1, \ldots, n$,

2° $k(x) \leqslant \bar{k}_1 x_1 + \ldots + \bar{k}_n x_n$, $K(x) = \big(K_1(x), \ldots, K_n(x)\big) \leqslant (\bar{K}_1, \ldots, \bar{K}_n)$, $\bar{k}_i, \bar{K}_i \in R_+$,

3° *assumptions* 2°, 3° *of Lemma* 4 *are satisfied,* 4° $\bar{k}_1 \bar{\beta}_1 a_1 + \ldots + \bar{k}_n \bar{\beta}_n a_n < 1$,

then conditions 2°–4° *of Assumption* H_2 *are fulfilled.*

Lemma 7. *If*

1° $k(x) \leqslant \bar{k}$, $K_j(x) \leqslant \bar{K}_1 x_1 + \ldots + \bar{K}_n x_n$, $j = 1, \ldots, m$, $\bar{k}, \bar{K}_i \in R_+$,

2° $\varphi_j(x) \leqslant (\bar{\alpha}^{(j)}_{t_1} x^2_{t_1}, \ldots, \bar{\alpha}^{(j)}_{t_{p_j}} x^2_{t_{p_j}})$, $t_i \in \bar{\sigma}_j$, $\bar{\alpha}^{(j)}_{t_s} \in R_+$,

3° *assumptions* 3°, 4° *of Lemma* 4 *are satisfied,*

4° $\bar{k}\bar{\beta}_i (\prod_{s \in \bar{\sigma}_j} \bar{\beta}_s)^2 < 1$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$,

then conditions 2°–4° *of Assumption* H_2 *are satisfied.*

Lemma 8. *If*

1° $G = [0, a]$, $a = (a_1, \ldots, a_n)$, $0 < a_i \leqslant 1$, $i = 1, \ldots, n$, $\prod_{s \in \bar{\sigma}_j} a_s^2 < 1$, $j = 1, \ldots, m$,

2° $\beta(x) \leqslant (x_1^2, \ldots, x_n^2)$,

3° *assumptions* 1°, 2° *of Lemma* 4 *are satisfied,*

4° $\sum_{i=0}^{\infty} \bar{k}^i \bar{h}(x_1^{2^i}, \ldots, x_n^{2^i}) < \infty$,

then conditions 2°–4° *of Assumption* H_2 *are fulfilled.*

Lemma 9. *If*

1° *assumptions* 1°, 2°, 3° *of Lemma* 4 *are satisfied,*

2° $\bar{h}(x) \leqslant h x^p$, $h = \text{const}$, $x^p = x_1^p \cdot x_2^p \cdot \ldots \cdot x_n^p$,

3° $\bar{k}(\prod_{s \in \bar{\sigma}_j} \bar{\beta}_s)^{\nu} < 1$, *where* $\nu = \min[1, p]$,

then conditions 2°–4° *of Assumption* H_2 *are fulfilled.*

PART II

In this part of the paper we give sufficient conditions for the existence of at least one continuous solution of equation (2) considered in a finite dimensional Banach space B. Now we do not assume the Lipschitz condition for the function F with respect to u_i for $i \in A'$ and for the functions f_i with respect to z for $i \in A'$ (see Assumption H_3 in Part I).

1. Assumptions. We introduce the following

Assumption H_4. Suppose that:

1° There exist functions $h_i \in C(G, R_+)$, $i = 0, 1, \ldots, m$, $g, \bar{h}_i, \bar{g}_i \in C(G, R_+)$, $i = 1, 2, \ldots, m$, such that

$$\|F(x, u, v)\| \leqslant \sum_{i=1}^{m} h_i(x) \|u_i\| + h_0(x) \|v\| + g(x), \qquad x \in G,$$

where $u = (u_1, \ldots, u_m) \in B^m$, $v \in B$, B is a finite dimensional Banach space and

$$\|f_i(x, s, z)\| \leqslant \bar{h}_i(x)\|z\| + g_i(x), \quad i = 1, 2, \ldots, m, \ x, s \in G.$$

2° There exist functions $l_i \in C(G, R_+)$, $i = k_0+1, \ldots, m$, $k \in C(G, R_+)$ and $\bar{l}_i \in C(G, R_+)$, $i = k_0+1, \ldots, m$, such that

$$\|F(x, u_A, u_B, v) - F(x, u_A, \bar{u}_B, \bar{v})\| \leqslant \sum_{i=k_0+1}^{m} l_i(x)\|u_i - \bar{u}_i\| + k(x)\|v - \bar{v}\|,$$

where $x \in G$, $u_A = (u_1, \ldots, u_{k_0})$, $u_B = (u_{k_0+1}, \ldots, u_m)$, $u_i, \bar{u}_i, v, \bar{v} \in B$, and

$$\|f_i(x, s, z) - f_i(x, s, \bar{z})\| \leqslant \bar{l}_i(x)\|z - \bar{z}\| \quad \text{for } i = k_0+1, \ldots, m, \ x, s \in G, \ z, \bar{z} \in B.$$

3° Assumption H_2 is fulfilled for $\tilde{h}, K$ defined by the relations

$$K(x) = \big(h_1(x)\bar{h}_1(x), \ldots, h_m(x)\bar{h}_m(x)\big),$$

$$\tilde{h}(x) = \sum_{i=1}^{m} \big[h_i(x)\bar{g}_i(x) \sup_{x \in G} L_{p_i}(H_i(x))\big] + g(x)$$

and for k defined in condition 2° of Assumption H_4.

Assumption H_5. We assume that the functions $D_j, d_i, \bar{d}_i, \bar{\bar{d}}_i, \tilde{d}_i, \bar{\omega}_s, \bar{\bar{\omega}}_s \in C(R_+, R_+)$, $i = 1, \ldots, m$, $j = 0, 1, \ldots, m+1$, $s = k_0+1, \ldots, m$ are subadditive, non-decreasing and such that $D_j(0) = 0$, $d_i(0) = \bar{d}_i(0) = \bar{\bar{d}}_i(0) = \tilde{d}_i(0) = 0$, $\bar{\omega}_s(0) = \bar{\bar{\omega}}_s(0) = 0$, and, moreover:

1° $\|F(x, u, v) - F(\bar{x}, \bar{u}, \bar{v})\| \leqslant D_0(|x - \bar{x}|) + \sum_{i=1}^{m} D_i(\|u_i - \bar{u}_i\|) + D_{m+1}(\|v - \bar{v}\|)$ for $x, \bar{x} \in G$, $\|u_i\|$, $\|\bar{u}_i\| \leqslant R_i$, $\|v\|$, $\|\bar{v}\| \leqslant \tilde{r}$, where $\tilde{r} \overset{\text{def}}{=} \sup_{x \in G} \tilde{z}(x)$, $R_i \overset{\text{def}}{=} \sup_{x \in G} L_{p_i}(H_i(x)) \sup_{x \in G} \bar{h}_i(x) \sup_{x \in G} \tilde{z}(x) + \sup_{x \in G} \bar{g}_i(x)$, and $\tilde{z}$ defined in Lemma 1.

2° $\|f_i(x, s, z) - f_i(\bar{x}, \bar{s}, \bar{z})\| \leqslant d_i(|x - \bar{x}|) + \bar{d}_i(|s - \bar{s}|) + \bar{\bar{d}}_i(\|z - \bar{z}\|)$, $i = 1, \ldots, m$ for $x, \bar{x}, s, \bar{s} \in G$, $\|z\|, \|\bar{z}\| \leqslant \tilde{r}$.

3° $L_n(H_i(x+h) - H_i(x)) \leqslant \tilde{d}_i(|h|)$ for $i \in A'$, $x, x+h \in G$,

$$L_{p_i}\big[H_i(x) - \big(-t_i(x, h) + H_i(x+h)\big)\big] \leqslant \tilde{d}_i(|h|) \quad \text{for } i \in B', \ x, x+h \in G.$$

4° $|t_i(x, h)| \leqslant \bar{\omega}_i(|h|)$ for $i \in B'$, $|h| \in [0, r_0]$, where r_0 is the diameter of the set G,

$$|\alpha_i(x+h) - \alpha_i(x)| \leqslant \bar{\bar{\omega}}_i(|h|) \quad \text{for } i \in B', \ x \in G, \ |h| \in [0, r_0].$$

Let $\tilde{\omega}_i(t) = \bar{\bar{\omega}}_i(\bar{\omega}_i(t))$, $i \in B'$, $t \in [0, r_0]$.

2. A certain functional equation.

Lemma 10. *If*

1° *Assumption* H_5 *and conditions* 1°, 3° *from Assumption* H_4 *are satisfied,* 2° *the Lipschitz condition*

$$\|F(x,u,v)-F(x,u,\bar v)\| \leqslant k(x)\|v-\bar v\|, \quad x\in G,\ v,\bar v\in B \tag{35}$$

holds,

3° $W \overset{\text{def}}{=} \{y\colon\ y\in C(G,B),\ \|y(x)\| \leqslant \tilde z(x)\ \textit{for}\ x\in G\}$,

then for any $y\in W$ *there exists the unique* $u(\cdot, y)\in W$ *being a solution of the equation*

$$u(x) = F\Big(x, \int_{H(x)} f\big(x,s,y(\alpha(s))\big)ds, u\big(\beta(x)\big)\Big), \quad x\in G. \tag{36}$$

Proof. Put

$$u_0(x) = 0, \quad u_{r+1}(x) = F\Big(x, \int_{H(x)} f\big(x,s,y(\alpha(s))\big)ds, u_r\big(\beta(x)\big)\Big), \qquad x\in G\ r = 0,1,\ldots \tag{37}$$

We prove that

$$\|u_r(x)\| \leqslant \tilde z(x), \quad x\in G,\ r = 0,1,2,\ldots \tag{38}$$

For $r=0$ this inequality is evidently satisfied. If we assume that $\|u_r(x)\| \leqslant \tilde z(x)$ for $x\in G$, then

$$\begin{aligned}\|u_{r+1}(x)\| &\leqslant \Big\|F\Big(x, \int_{H(x)} f\big(x,s,y(\alpha(s))\big)ds, u_r\big(\beta(x)\big)\Big)-\\ &\quad -F\Big(x, \int_{H(x)} f\big(x,s,y(\alpha(s))\big)ds, 0\Big)\Big\| + \Big\|F\Big(x, \int_{H(x)} f\big(x,s,y(\alpha(s))\big)ds, 0\Big)\Big\|\\ &\leqslant k(x)\tilde z\big(\beta(x)\big)+K(x)\int_{H(x)} \tilde z\big(\alpha(s)\big)ds+\tilde{\bar h}(x) = \tilde z(x)\end{aligned}$$

for $x\in G$. Now we obtain (38) by induction.

Next we prove that u_r are continuous in G. Since u_0 is continuous in G, it is sufficient to prove that the continuity of u_r implies the continuity of u_{r+1}. Let

$$\bar R_1 = \max_j\,[\sup_{x\in G} \bar h_j(x)\sup_{x\in G}\|y(x)\| + \sup_{x\in G}\bar g_j(x)].$$

By Assumption H_5 we have

$$\begin{aligned}&\|u_{r+1}(x+h)-u_{r+1}(x)\|\\ &\leqslant D_0(|h|)+\sum_{j=1}^{m} D_j\Big(\Big\|\int_{H_j(x+h)} f_j\big(x+h,s,y(\alpha_j(s))\big)(ds)_{p_j}-\\ &\quad -\int_{H_j(x)} f\big(x,s,y(\alpha_j(s))\big)(ds)_{p_j}\Big\|\Big)+D_{m+1}\big(\|u_r(\beta(x+h))-u_r(\beta(x))\|\big).\end{aligned} \tag{39}$$

Now for $j \in A'$ we get, writing ds instead of $(ds)_n$

(40) $$\Big\| \int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)\,ds - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)\,ds \Big\|$$
$$\leqslant \int_{H_j(x+h)\cap H_j(x)} \big\| f_j\big(x+h, s, y(\alpha_j(s))\big) - f_j\big(x, s, y(\alpha_j(s))\big)\big\|\,ds + \int_{H_j(x+h)-H_j(x)} \tilde{R}_1\,ds$$
$$\leqslant L_n(G)\,d_j(|h|) + \tilde{R}_1\,\bar{d}_j(|h|).$$

If for $j \in B'$ we define the sets $H_j^0(x,h)$, $H_j^1(x,h)$, $H_j^2(x,h)$, $H_j^3(x,h)$ by

(41) $$\begin{aligned} H_j^0(x,h) &= H_j(x) - \big(-t_j(x,h) + H_j(x+h)\big),\\ H_j^1(x,h) &= \big(-t_j(x,h) + H_j(x+h)\big) \cap H_j(x),\\ H_j^2(x,h) &= \big(-t_j(x,h) + H_j(x+h)\big) - H_j(x),\\ H_j^3(x,h) &= H_j(x) - \big(-t_j(x,h) + H_j(x+h)\big),\end{aligned}$$

then

(42) $$\Big\| \int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)(ds)_{p_j} - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j} \Big\|$$
$$\leqslant \int_{H_j^1(x,h)} \Big\| f_j\Big(x+h, s+t_j(x,h), y\big(\alpha_j(s+t_j(x,h))\big)\Big) - f_j\big(x, s, y(\alpha_j(s))\big)\Big\|\,(ds)_{p_j} +$$
$$+ \int_{H_j^2(x,h)} \Big\| f_j\Big(x+h, s+t_j(x,h), y\big(\alpha_j(s+t_j(x,h))\big)\Big)\Big\|\,(ds)_{p_j} +$$
$$+ \int_{H_j^3(x,h)} \big\| f_j\big(x, s, y(\alpha_j(s))\big)\big\|\,(ds)_{p_j}$$
$$\leqslant L_{p_j}(H_j(x))\Big[d_j(|h|) + \bar{d}_j(\tilde{\omega}_j(|h|)) +$$
$$+ \bar{d}_j\Big(\sup_{s\in G}\big\| y\big(\alpha_j(s+t_j(x,h))\big) - y(\alpha_j(s))\big\|\Big)\Big] + \int_{H_j^0(x,h)} \tilde{R}_1\,(ds)_{p_j}$$
$$\leqslant L_{p_j}(H_j(x))\big[d_j(|h|) + \bar{d}_j(\tilde{\omega}_j(|h|)) + \bar{d}_j\big(\bar{\bar{d}}(\tilde{\omega}_j(|h|))\big) + \tilde{R}_1\,\bar{d}_j(|h|)\big],$$

where $\bar{\bar{d}}$ is a modulus of continuity for the function y. It follows from (39), (40), (42) and from the continuity of y, u_r, α_j, β that u_{r+1} is continuous.

We put

$$z_0(x) = \tilde{z}(x) \quad \text{for } x \in G, \qquad z_r(x) = k(x)\,z_{r-1}(\beta(x)), \quad r = 1, 2, \ldots,\ x \in G.$$

By induction we get

(43) $$z_r(x) = k^{(r)}(x)\,\tilde{z}(\beta^{(r)}(x)), \quad r = 1, 2, \ldots,\ x \in G.$$

In virtue of condition (9) of Lemma 1 it follows from (43) that

(44) $$\lim_{r\to\infty} z_r(x) = 0$$

and the convergence is uniform with respect to $x \in G$. Further, we get easily

$$(45) \qquad \|u_{r+p}(x) - u_r(x)\| \leqslant z_r(x), \quad x \in G,\ r, p = 0, 1, 2, \ldots$$

Indeed, from (37) and (38) it follows that (45) is satisfied for $r = 0$, $p = 0, 1, 2, \ldots$, $x \in G$. If we assume that (45) holds for a fixed r and $p = 0, 1, 2, \ldots$, $x \in G$, then

$$\begin{aligned} \|u_{r+1+p}(x) - u_{r+1}(x)\| &\leqslant k(x) \left\|u_{r+p}(\beta(x)) - u_r(\beta(x))\right\| \\ &\leqslant k(x) z_r(\beta(x)) = z_{r+1}(x). \end{aligned}$$

Now we obtain (45) by induction.

By (37), (44), (45) we infer that the sequence $\{u_r\}$ is uniformly convergent in G to the solution $\bar{u}$ of equation (36). Since the sequence $\{u_r\}$ is uniformly convergent in G and $u_r \in C(G, B)$, we conclude by (38) that $\bar{u} \in W$.

To prove that the solution $\bar{u}$ of (36) is unique in W, let us suppose that there exists another solution $\bar{\bar{u}} \neq \bar{u}$ and $\bar{\bar{u}} \in W$. Now, from (35) we have

$$\|\bar{u}(x) - \bar{\bar{u}}(x)\| \leqslant k(x) \left\|\bar{u}(\beta(x)) - \bar{\bar{u}}(\beta(x))\right\|, \quad x \in G,$$

and by induction we get

$$(46) \qquad \|\bar{u}(x) - \bar{\bar{u}}(x)\| \leqslant k^{(r)}(x) \left\|\bar{u}(\beta^{(r)}(x)) - \bar{\bar{u}}(\beta^{(r)}(x))\right\|, \quad r = 0, 1, 2, \ldots$$

Since

$$\begin{aligned} k^{(r)}(x) \left\|\bar{u}(\beta^{(r)}(x))\right\| &\leqslant k^{(r)}(x) \bar{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, \ldots, \ x \in G, \\ k^{(r)}(x) \left\|\bar{\bar{u}}(\beta^{(r)}(x))\right\| &\leqslant k^{(r)}(x) \bar{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, \ldots, \ x \in G, \end{aligned}$$

and

$$\lim_{r \to \infty} k^{(r)}(x) \bar{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G,$$

we infer by (46) that $\bar{u} = \bar{\bar{u}}$. This contradiction proves the uniqueness of $\bar{u}$ in W.

3. Further assumptions. We introduce

ASSUMPTION H_6. Suppose that

1° $m_0(x, \delta_1, \delta_2, \ldots, \delta_m) = \sum_{i=0}^{\infty} k^{(i)}(x) \sum_{j=1}^{m} D_j\big(\delta_j L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\big) < +\infty$ for $x \in G$, $\delta_j \in R_+$,

2° the function m_0 is continuous with respect to $(x, \delta_1, \ldots, \delta_m) \in G \times R_+^m$.

Remark 5. It is obvious that Assumption H_6 is fulfilled if, for instance, $k(x) \leqslant \bar{k} < 1$ for $x \in G$. If condition 1° of H_6 is satisfied and the functions k, β, α_j are non-decreasing in G, then condition 2° of H_6 is fulfilled.

Remark 6. If the functions φ_i and β satisfy conditions 2° and 3° of Lemma 4, respectively, and

$$k(x) \leqslant \bar{k} = \text{const}, \quad D_j(t) \leqslant Dt, \quad D = \text{const}, \quad j = 1, \ldots, m,$$

$$\bar{k} \prod_{s \in \bar{\sigma}_j} \bar{\beta}_s < 1, \quad j = 1, 2, \ldots, m,$$

then Assumption H_6 is fulfilled (see the proof of Lemma 4).

We adopt the following notation:

$$\bar{K} = (K_{k_0+1}, \ldots, K_m), \quad \bar{L}(G(x)) = \big(L_{p_{k_0+1}}(G_{k_0+1}(x)), \ldots, L_{p_m}(G_m(x))\big),$$

$$\bar{K}(x) \cdot \bar{L}(G(x)) = \sum_{j=k_0+1}^{m} K_j(x) \cdot L_{p_j}(G_j(x)),$$

$$\int_{\bar{H}(x)} d(\alpha(s), \tilde{\omega}(t))\, ds = \Big(\int_{H_{k_0+1}(x)} d(\alpha_{k_0+1}(s), \tilde{\omega}_{k_0+1}(t))(ds)_{p_{k_0+1}}, \ldots$$

$$\ldots, \int_{H_m(x)} d(\alpha_m(s), \tilde{\omega}_m(t))(ds)_{p_m}\Big),$$

$$\bar{K}(x) \int_{\bar{H}(x)} d(\alpha(s), \tilde{\omega}(t))\, ds = \sum_{j=k_0+1}^{m} K_j(x) \int_{H_j(x)} d(\alpha_j(s), \tilde{\omega}_j(t))(ds)_{p_j},$$

where $\tilde{\omega}_i$ are real-valued functions of one variable.

We introduce

Assumption H_7. Suppose that

1° $|\beta(x+h) - \beta(x)| \leqslant \omega(|h|)$, for $x, x+h \in G$, where $\omega \in C(R_+, R_+)$ is subadditive and non-decreasing and

$$\omega(0) = 0, \quad \tilde{\omega} = (\tilde{\omega}_{k_0+1}, \ldots, \tilde{\omega}_m) \in C(R_+, R_+^{m-k_0}),$$

$\tilde{\omega}_i$ are subadditive and non-decreasing, and $\tilde{\omega}_i(0) = 0$,

2° there is given a function p such that

(a) $p \in C(G \times [0, r_0], R_+)$, where $r_0 \in R_+$ is defined in 4° of H_5,

(b) p is non-decreasing and subadditive with respect to the last variable,

(c) $p(x, 0) = 0$ for $x \in G$,

3° $\bar{m}(x, t) = \sum_{i=0}^{\infty} k^{(i)}(x)\, p(\beta^{(i)}(x), \omega^{(i)}(t)) < +\infty$ for $(x, t) \in G \times [0, r_0]$,

where $\omega^{(0)}(t) = t$, $\omega^{(i+1)}(t) = \omega(\omega^{(i)}(t))$, $i = 0, 1, 2, \ldots$, $t \in [0, r_0]$,

4° $\bar{M}(x) = \sum_{i=0}^{\infty} k^{(i)}(x) \big[\bar{K}(\beta^{(i)}(x)) \cdot \bar{L}\big(G(\beta^{(i)}(x))\big)\big] < +\infty$, $x \in G$,

5° $\bar{m} \in C(G \times [0, r_0], R_+)$, $\bar{M} \in C(G, R_+)$, 6° the function

$$\bar{M}(x) = \sum_{i=0}^{\infty} k^{(i)}(x)\Big[\sum_{j=k_0+1}^{m} K_j(\beta^{(i)}(x))L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\cdot\big(\prod_{s\in\bar{\sigma}_j} x_s\big)^{-1}\Big]$$

is bounded in G.

We have the following

LEMMA 11. *If Assumption* H_7 *and condition* 2° *of Assumption* H_1 *are satisfied, then*:

1° *There exists a solution* $\tilde{d}\in C(G\times[0,r_0],R_+)$ *of the equation*

$$(47)\qquad d(x,t) = \sum_{i=0}^{\infty} k^{(i)}(x)p(\beta^{(i)}(x),\omega^{(i)}(t))+ \\ +\sum_{i=0}^{\infty} k^{(i)}(x)\Big[\bar{K}(\beta^{(i)}(x))\cdot\int_{\bar{H}(\beta^{(i)}(x))} d\big(\alpha(s),\bar{\omega}(\omega^{(i)}(t))\big)ds\Big], \\ (x,t)\in G\times[0,r_0].$$

The solution $\tilde{d}$ *of* (47) *is unique in the class* $M(G\times[0,r_0],R_+)$ *of non-negative upper-semicontinuous functions defined on* $G\times[0,r_0]$. *The function* $\tilde{d}$ *is non-decreasing and subadditive with respect to the last variable and* $\tilde{d}(x,0)=0$ *for* $x\in G$.

2° *The function* $\tilde{d}$ *is a solution of the equation*

$$(48)\qquad d(x,t) = \bar{K}(x)\int_{\bar{H}(x)} d\big(\alpha(s),\bar{\omega}(t)\big)ds+k(x)d\big(\beta(x),\omega(t)\big)+p(x,t), \\ (x,t)\in G\times[0,r_0].$$

Moreover, this solution is unique in the class $\tilde{M}(G\times[0,r_0],R_+,\tilde{d})$, *where* $\tilde{M}(G\times[0,r_0],R_+,\tilde{d}) = \{z\colon z\in M(G\times[0,r_0],R_+),\ \inf\,[c\colon z(x,t)\leqslant c\tilde{d}(x,t)] < +\infty\}$.

The proof of this Lemma is similar to the proof of assertions 1°, 2° of Lemma 1.

4. Properties of the operator U. Let $\tilde{W}=\{y\colon y\in C(G,B),\ \|y(x)\|\leqslant\tilde{z}(x),\ \|y(x+h)-y(x)\|\leqslant\tilde{d}(x,|h|)\}$, where the functions $\tilde{z}$ and $\tilde{d}$ are defined by Lemma 1 and Lemma 11, respectively. We consider the operator U defined by the formula $Uy=u(\cdot,y)$, where $u(\cdot,y)$ is the solution of functional equation (36).

We have

LEMMA 12. *If Assumptions* H_5, H_6, *conditions* 1° *and* 3° *from* H_4 *and the Lipschitz condition* (35) *are satisfied, then the operator* U *is continuous in the set* W.

Proof. Let $y_1, y_2 \in W$, $u_1 = u(\cdot, y_1)$, $u_2(\cdot, y_2)$, $v(x) = \|u_1(x) - u_2(x)\|$. Then we have for $x \in G$

$$\begin{aligned}
v(x) = &\Big\|F\Big(x, \int_{H(x)} f\big(x, s, y_1(\alpha(s))\big)ds, u_1(\beta(x))\Big) - \\
&-F\Big(x, \int_{H(x)} f\big(x, s, y_2(\alpha(s))\big)ds, u_2(\beta(x))\Big)\Big\| \\
\leqslant &\sum_{j=1}^{m} D_j\Big(\Big\|\int_{H_j(x)} \big[f_j\big(x, s, y_1(\alpha_j(s))\big) - f_j\big(x, s, y_2(\alpha_j(s))\big)\big](ds)_{p_j}\Big\|\Big) + \\
&+k(x)\|u_1(\beta(x)) - u_2(\beta(x))\| \\
\leqslant &\sum_{j=1}^{m} D_j\Big(\int_{H_j(x)} \bar{d}_j\big(\|y_1(\alpha_j(s)) - y_2(\alpha_j(s))\|\big)(ds)_{p_j}\Big) + k(x)v(\beta(x)).
\end{aligned}$$

Let $\delta_j = \bar{d}_j\big(\sup_{s \in G} \|y_1(s) - y_2(s)\|\big)$. Then we have the inequality

$$v(x) \leqslant \sum_{j=1}^{m} D_j\big(\delta_j L_{p_j}(G_j(x))\big) + k(x)v(\beta(x)), \quad x \in G,$$

and we get by induction

$$v(x) \leqslant \sum_{i=0}^{r-1} k^{(i)}(x)\Big[\sum_{j=1}^{m} D_j\Big(\delta_j L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\Big)\Big] + k^{(r)}(x)v(\beta^{(r)}(x)), \tag{49}$$

$$x \in G,\ r = 1, 2, \ldots$$

Since

$$0 \leqslant k^{(r)}(x)v(\beta^{(r)}(x)) \leqslant 2k^{(r)}(x)\tilde{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, \ldots, x \in G$$

and

$$\lim_{r \to \infty} k^{(r)}(x)\tilde{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G,$$

we get, making $r \to \infty$ in (49), that

$$v(x) \leqslant \sum_{i=0}^{\infty} k^{(i)}(x)\Big[\sum_{j=1}^{m} D_j\Big(\delta_j L_{p_j}\big(G_j(\beta^{(i)}(x))\big)\Big)\Big] = m_0(x, \delta_1, \ldots, \delta_m).$$

In view of the continuity of the function m_0 we conclude the assertion of Lemma 12.

LEMMA 13. *Suppose that:*

1° *Assumptions* H_4, H_5, H_6 *are satisfied and Assumption* H_7 *is fulfilled for* p, $\bar{K}$ *defined by the relations*

$$(50)\qquad p(x,t) = D_0(t) + \sum_{j=1}^{k_0} D_j\big(L_n(G)d_j(t) + P_j\tilde{d}_j(t)\big) + \\ + \sum_{j=k_0+1}^{m} l_j(x)\big[L_{p_j}(H_j(x))\big(d_j(t)+\bar{d}_j(\bar{\omega}_j(t))\big) + P_j\tilde{d}_j(t)\big],$$

where $P_j = \sup_{x\in G}\bar{h}_j(x)\sup_{x\in G}\tilde{z}(x) + \sup_{x\in G}\bar{g}_j(x)$,

$$(51)\qquad \bar{K}(x) = \big(l_{k_0+1}(x)\bar{l}_{k_0+1}(x), \ldots, l_m(x)\bar{l}_m(x)\big), \quad x\in G.$$

2° *For* $x, x+h\in G$ *we have*

$$(52)\qquad \lim_{r\to\infty} k^{(r)}(x)\tilde{z}\big(\beta^{(r)}(x+h)\big) = 0 \quad \textit{uniformly with respect to } x, x+h\in G.$$

Under these assumptions the operator U *maps* $\tilde{W}$ *into itself.*

Proof. In virtue of Lemma 10 it follows that for each $y\in\tilde{W}$ the function Uy satisfies the condition $\|(Uy)(x)\| \leqslant \tilde{z}(x)$ for $x\in G$. To prove that $Uy\in\tilde{W}$ for $y\subset\tilde{W}$, it is sufficient to show that $\|(Uy)(x+h)-(Uy)(x)\| \leqslant \tilde{d}(x,|h|)$ for $x, x+h\in G$.

Let us suppose that $y\in\tilde{W}$ and $u(x) = (Uy)(x)$. We show that

$$(53)\qquad \|u(x+h)-u(x)\| \leqslant \tilde{d}(x,|h|), \quad x, x+h\in G.$$

For $j\in A'$ we get, writing ds instead of $(ds)_n$ for simplicity,

$$\Big\|\int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)ds - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)ds\Big\| \\ \leqslant \int_{H_j(x+h)\cap H_j(x)} \big\|f_j\big(x+h, s, y(\alpha_j(s))\big) - f_j\big(x, s, y(\alpha_j(s))\big)\big\|\,ds + \\ + \int_{H_j(x+h)-H_j(x)} \big[\bar{h}_j(x+h)\,\|y(\alpha_j(s))\| + \bar{g}_j(x+h)\big]\,ds + \\ + \int_{H_j(x)-H_j(x+h)} \big[\bar{h}_j(x)\,\|y(\alpha_j(s))\| + \bar{g}_j(x)\big]\,ds \\ \leqslant L_n(G)d_j(|h|) + P_j\tilde{d}_j(|h|).$$

In the case $j\in B'$ we define the sets $H_j^k(x,h)$, $k = 0,1,2,3$, by (41) and note that integration over the set $H_j(x+h)$ is equivalent to integration over $-t_j(x,h)+H_j(x+h)$ if one replaces the variable s by $s+t_j(x,h)$.

In this way we arrive at

$$
\begin{aligned}
&\Big\| \int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)(ds)_{p_j} - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j}\Big\| \\
&= \Big\| \int_{H_j^1(x,h)} f_j\Big(x+h, s+t_j(x,h), y\big(\alpha_j(s+t_j(x,h))\big)\Big)(ds)_{p_j} - \\
&\quad - \int_{H_j^1(x,h)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j} + \\
&\quad + \int_{H_j^2(x,h)} f_j\Big(x+h, s+t_j(x,h), y\big(\alpha_j(s+t_j(x,h))\big)\Big)(ds)_{p_j} - \\
&\quad - \int_{H_j^3(x,h)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j}\Big\| \\
&\leqslant \int_{H_j^1(x,h)} \Big\| f_j\Big(x+h, s+t_j(x,h), y\big(\alpha_j(s+t_j(x,h))\big)\Big) - f_j\big(x, s, y(\alpha_j(s))\big)\Big\| (ds)_{p_j} + \\
&\quad + \int_{H_j^2(x,h)} P_j(ds)_{p_j} + \int_{H_j^3(x,h)} P_j(ds)_{p_j} \\
&\leqslant \int_{H_j^1(x,h)} \Big[d_j(|h|) + \bar{d}_j(\bar{\omega}_j(|h|)) + \bar{l}_j(x)\, \Big\| y\big(\alpha_j(s+t_j(x,h))\big) - y(\alpha_j(s))\Big\|\Big](ds)_{p_j} + \\
&\quad + \int_{H_j^0(x,h)} P_j(ds)_{p_j} \\
&\leqslant L_{p_j}(H_j(x))\big[d_j(|h|) + \bar{d}_j(\bar{\omega}_j(|h|))\big] + \\
&\quad + \bar{l}_j(x) \int_{H_j(x)} \Big\| y\big(\alpha_j(s+t_j(x,h))\big) - y(\alpha_j(s))\Big\| (ds)_{p_j} + P_j \bar{d}_j(|h|).
\end{aligned}
$$

It follows from Assumption H_5 and from the above estimates that

$$
\begin{aligned}
\|u(x+h) - u(x)\| &= \Big\| F\Big(x+h, \int_{H(x+h)} f\big(x+h, s, y(\alpha(s))\big)ds, u(\beta(x+h))\Big) - \\
&\quad - F\Big(x, \int_{H(x)} f\big(x, s, y(\alpha(s))\big)ds, u(\beta(x))\Big)\Big\| \\
&\leqslant D_0(|h|) + \sum_{j=1}^{k_0} D_j \Big[\Big\| \int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)(ds)_{p_j} - \\
&\quad - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j}\Big\|\Big] + \\
&\quad + \sum_{j=k_0+1}^{m} l_j(x) \Big\| \int_{H_j(x+h)} f_j\big(x+h, s, y(\alpha_j(s))\big)(ds)_{p_j} - \\
&\quad - \int_{H_j(x)} f_j\big(x, s, y(\alpha_j(s))\big)(ds)_{p_j}\Big\| + k(x)\, \|u(\beta(x+h)) - u(\beta(x))\|
\end{aligned}
$$

$$\leqslant D_0(|h|)+\sum_{j=1}^{k_0} D_j[L_n(G)d_j(|h|)+P_j\tilde{d}_j(|h|)]+$$

$$+\sum_{j=k_0+1}^{m} l_j(x)\{L_{p_j}(H_j(x))[d_j(|h|)+\bar{d}_j(\tilde{\omega}_j(|h|))]+P_j\tilde{d}_j(|h|)\}+$$

$$+\sum_{j=k_0+1}^{m} l_j(x)\cdot\bar{l}_j(x)\int_{H_j(x)}\left\|y\big(\alpha_j(s+t_j(x,h))\big)-y(\alpha_j(s))\right\|(ds)_{p_j}+$$

$$+k(x)\left\|u(\beta(x+h))-u(\beta(x))\right\|$$

$$\leqslant p(x,|h|)+\sum_{j=k_0+1}^{m} K_j(x)\int_{H_j(x)}\left\|y\big(\alpha_j(s+t_j(x,h))\big)-y(\alpha_j(s))\right\|(ds)_{p_j}+$$

$$+k(x)\left\|u(\beta(x+h))-u(\beta(x))\right\|.$$

Since $\|y(x+h)-y(x)\| \leqslant \tilde{d}(x,|h|)$, we have

$$\left\|y\big(\alpha_j(s+t_j(x,h))\big)-y(\alpha_j(s))\right\| \leqslant \tilde{d}(\alpha_j(s),\tilde{\omega}_j(|h|))$$

and consequently

$$\|u(x+h)-u(x)\| \leqslant p(x,|h|)+\bar{K}(x)\int_{\bar{H}(x)}\tilde{d}(\alpha(s),\tilde{\omega}(|h|))\,ds+$$

$$+k(x)\left\|u(\beta(x+h))-u(\beta(x))\right\|.$$

The last inequality implies the following:

$$\|u(x+h)-u(x)\| \leqslant \sum_{i=0}^{r-1} k^{(i)}(x)\,p\big(\beta^{(i)}(x),|\beta^{(i)}(x+h)-\beta^{(i)}(x)|\big)+ \tag{54}$$

$$+\sum_{i=0}^{r-1} k^{(i)}(x)\Big[\bar{K}(\beta^{(i)}(x))\int_{\bar{H}(\beta^{(i)}(x))}\tilde{d}\big(\alpha(s),\tilde{\omega}(|\beta^{(i)}(x+h)-\beta^{(i)}(x)|)\big)\,ds\Big]+$$

$$+k^{(r)}(x)\left\|u(\beta^{(r)}(x+h))-u(\beta^{(r)}(x))\right\|,\qquad x,x+h\in G,\ r=1,2,\ldots$$

It follows from the inequalities

$$k^{(r)}(x)\left\|u(\beta^{(r)}(x+h))-u(\beta^{(r)}(x))\right\| \leqslant k^{(r)}(x)\left\|u(\beta^{(r)}(x))\right\|+k^{(r)}(x)\left\|u(\beta^{(r)}(x+h))\right\|$$

$$\leqslant k^{(r)}(x)\tilde{z}(\beta^{(r)}(x))+k^{(r)}(x)\tilde{z}(\beta^{(r)}(x+h)),\qquad x,x+h\in G,\ r=0,1,2,\ldots,$$

and from conditions (9) and (52) that

$$\lim_{r\to\infty} k^{(r)}(x)\left\|u(\beta^{(r)}(x+h))-u(\beta^{(r)}(x))\right\| = 0 \quad \text{uniformly in } G. \tag{55}$$

By induction we easily obtain

$$|\beta^{(i)}(x+h)-\beta^{(i)}(x)| \leqslant \omega^{(i)}(|h|),\qquad x,x+h\in G,\ i=0,1,2,\ldots \tag{56}$$

Now, from (55), (56) and by the definition of $\tilde{d}$, we have, letting $r \to \infty$ in (54),

$$\|u(x+h)-u(x)\| \leqslant \sum_{i=0}^{\infty} k^{(i)}(x)\,p\big(\beta^{(i)}(x), \omega^{(i)}(|h|)\big)+$$

$$+\sum_{i=0}^{\infty} k^{(i)}(x)\Big[\bar{K}\big(\beta^{(i)}(x)\big) \int_{H(\beta^{(i)}(x))} \tilde{d}\Big(\alpha(s), \bar{\omega}\big(\omega^{(i)}(|h|)\big)\Big)\,ds\Big] = \tilde{d}(x, |h|),$$

which completes the proof of (53).

Remark 7. If the functions $k, \tilde{h}, K, \beta$ are non-decreasing in G and $H_j(x) \subset H_j(\bar{x})$ for $x < \bar{x}$, $x, \bar{x} \in G$, $j = 1, 2, \ldots, m$, then assumption 2° of Lemma 13 is satisfied. This fact follows from assertion 4° of Lemma 1 and from (9).

Now, we have the following

THEOREM 3. *Suppose that:*

1° *Assumptions* H_4, H_5, H_6 *are satisfied,*

2° *Assumption* H_7 *is fulfilled for* $p, \bar{K}$ *defined by relations* (50), (51),

3° *condition* (52) *of Lemma* 13 *holds.*

Under these assumptions equation (2) *has at least one solution* $\tilde{u} \in \tilde{W}$.

Proof. It follows from Lemmas 10, 12, 13 that the continuous operator U maps the compact and convex set $\tilde{W} \subset C(G, B)$ into itself. By the Schauder fixed point theorem there exists at least one solution $\tilde{u} \in \tilde{W}$ of equation (2).

LEMMA 14. *If*

1° $k(x) \leqslant \bar{k} = \text{const}$, $\bar{K}(x) = \big(K_{k_0+1}(x), \ldots, K_m(x)\big) \leqslant (\bar{K}_{k_0+1}, \ldots, \bar{K}_m) = \text{const}$,

2° *the functions* φ_i *and* β *satisfy conditions* 2° *and* 3°, *respectively, of Lemma* 4,

3° *there exist constants* ω_0 *and* D *such that* $D_i(t)$, $d_j(t)$, $\tilde{d}_j(t)$, $\bar{d}_r(t)$, $\bar{\omega}_r(t) \leqslant Dt$, $i = 1, \ldots, k_0$, $j = 1, \ldots, m$, $r = k_0+1, \ldots, m$, *and* $\omega(t) \leqslant \omega_0 \cdot t$,

4° $\bar{k} \prod_{s \in \sigma_j} \bar{\beta}_s < 1$ *for* $j = k_0+1, \ldots, m$,

5° $\bar{k} \cdot \omega_0 < 1$,

then conditions 3°–6° *of Assumption* H_7 *are fulfilled.*

The proof of this lemma is similar to the proof of Lemma 4. Using this Lemma we can easily formulate a theorem which is more effective than Theorem 3.

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