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Subobjects, adequacy, completeness and categories of algebras

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Introduction. In general terms, this paper is a sequel to my paper *Adequate subcategories* [5]. It develops some notions and results of the structure theory of categories, relating adequacy to ideal structure and completeness; it employs these notions in two theorems characterizing certain categories of algebras; and it continues the work of [5] on adequacy for some special functors (such as zero functors and their conjugates) and special categories (such as categories of vector spaces).

One of the theorems is much better than the others. That is the characterization of quasi-primitive categories of algebras, which is surprisingly simple, has every hypothesis nailed down by a counterexample, and is largely due to F. W. Lawvere.

THEOREM. *A category composed of some class of algebras of fixed type, closed under formation of direct products and subalgebras, with all homomorphisms between these algebras, is (up to coextension) the same as a complete category having a projectively finite projective generator.*

The language, most of which is in current use, will be explained below.

This theorem comes from my theorem (communicated to the 1962 International Congress of Mathematicians) which characterized the same categories, using roughly three hypotheses too many, and from Lawvere's theorem [8] characterizing primitive categories of algebras (which are closed also under formation of homomorphic images), using roughly one hypothesis too many. The final hypotheses are pretty nearly the intersection of the two sets, and the construction of a representation by algebras is the same in all three theorems; beyond this, the final proof is basically Lawvere's, preceded by my lemmas justifying the weaker hypotheses, and followed by my arguments establishing a conclusion not mentioned in [8]. One could reconvert this proof into a somewhat simpler proof of a somewhat simpler version of Lawvere's theorem, adding the appropriate condition [8] on congruence relations. I do not know whether the resulting set of conditions is independent.

In the present theorem, "complete" may be taken in the sense of Freyd [2], though a stronger sense is introduced in this paper for other

applications. A category is *right complete*, in the weaker sense, if every family of objects has a free sum and for every diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

there is a mapping h: $Y \to Z$ such that (1) $hf = hg$ and (2) for each t: $Y \to T$ satisfying $tf = tg$, there is exactly one k: $Z \to T$ such that $t = kh$. (Such an h is called a *coequalizer* of the pair (f, g) in [8], an *inductive limit* of the diagram in [12], a *right root* of the diagram in [2].)

Left complete is defined dually, in terms of direct products and equalizers. *Complete* means right and left complete. However, in the present theorem the existence of direct products follows from the other conditions. There are six independent conditions, three already stated (free sums, equalizers, coequalizers), three stated below.

By a *proper subobject* of an object X we mean a monomorphism m: $S \to X$ which is not an isomorphism. We call an object P *projective* if, given any diagram

$$\begin{tikzcd} & X \arrow[d, dashed, "f"] \\ P \arrow[r, dashed, "g"] & Y \end{tikzcd}$$

in which the mapping f: $X \to Y$ does not factor across any proper subobject of Y, the mapping g: $P \to Y$ factors ("lifts") across f. We call P a *generator* if for every object X, there is no proper subobject of X across which all mappings from P to X factor. (These definitions seem quite natural, though a weaker definition of "generator", too weak for the present theorem, is in common use [3]. Actually we shall use a wider notion of subobject and a correspondingly weaker definition of "projective". The corresponding stronger definition of "generator" coincides with *strict separator* in the sense of Semadeni [10]. If we use it in the present theorem, the assumption on equalizers becomes a consequence of the rest. However, there are still six conditions, for the definition of a strict separator amounts to two conditions.)

Finally, an object P is *projectively finite* if for every free sum $S = \sum_{\alpha \in A} P_\alpha$ of copies of P, every mapping $P \to S$ factors across the canonical injection of some finite subsum $\sum_{\alpha \in F} P_\alpha$ (F finite). This concept is Lawvere's, but he calls it "abstractly finite" [8]. The finiteness of such an

object may be very abstract indeed; in the category of all groups, every abelian group has this property.

It should be noted that A. I. Malcev has announced(1) a characterization [9] of quasi-primitive categories of algebras considered as concrete categories of sets and functions. The problem is not very closely related to the present one, but Malcev's work stimulated mine. I am indebted to J. P. Jans, J. Segal, and E. G. Šulgeifer for some helpful comments on some of this material.

1. Ideals. There are three ways the term "ideal" might be used in categories. We might apply it to objects which are kernels of mappings in the category (cf. [2], [7]), or to collections of mappings which are kernels of functors, or to classes of mappings which are closed under multiplication on one or both sides. The term will be used here essentially in the third way, though the difference kernels of [2] are an important special case. The present usage also has affinities with the original notion of an ideal as an "ideal number"; here it is an ideal object which one might adjoin to the category.

There are set-theoretic difficulties of two kinds in this discussion. (1) We cannot form sets of ideals, not even pairs of ideals, because an ideal is (commonly) larger than any cardinal number and in usual axiomatic set theory it cannot be a member of any set. (2) Supposing difficulty (1) somehow overcome, we might have sets of ideals (collections having a cardinal number), classes of ideals (collections of the same size as the universal class), and collections larger than the universal class. There are at least two ways around difficulty (1): one may imagine the "universal class" to be a set in some larger set theory, or one may speak of linguistic expressions defining "sets" of ideals. Cf. [6]. We shall ignore the difficulty. Difficulty (2) cannot be ignored, but we shall ignore it as far as possible.

A *set functor* on a category $\mathscr{C}$ is a functor, covariant or contravariant, from $\mathscr{C}$ to the category $\mathscr{U}$ of all sets and all functions. The contravariant *principal* set functors are the functors $\mathrm{Map}(\mathscr{C}, X)$, one for each object X of $\mathscr{C}$, which associate to each object W the set $\mathrm{Map}(W, X)$ and to each mapping $f\colon V \to W$ the function on $\mathrm{Map}(W, X)$ to $\mathrm{Map}(V, X)$ defined by right multiplication by f. Covariant principal set functors are defined dually, [5]. Set functors which are naturally equivalent to principal set functors are called *representable*, [4].

A *contravariant ideal* (of $\mathscr{C}$, in X) is an arbitrary subfunctor of a prin-

(1) The characterization stated in [9] is not correct; for example, the class of all partially ordered sets and isotone functions satisfies it. Apparently what is missing is the condition on divisible homomorphisms introduced earlier in [8].

cipal set functor $\mathrm{Map}(\mathscr{C}, X)$. In particular, there are the contravariant *principal ideals* I generated by a mapping $f: W \to X$; $I(V)$ is the set of all mappings in $\mathrm{Map}(V, X)$ which are right multiples of f.

1.1. *A contravariant ideal is representable if and only if it is a principal ideal generated by a monomorphism.*

Proof. The principal ideal generated by a monomorphism $f: W \to X$ is naturally equivalent to $\mathrm{Map}(\mathscr{C}, W)$, multiples $fg: V \to X$ corresponding biuniquely to factors $g: V \to W$. Conversely, let Φ be any natural transformation from $\mathrm{Map}(\mathscr{C}, X)$ onto an ideal I of $\mathscr{C}$ in Y, and consider $i = \Phi_X(1)$. For every mapping $h: W \to X$, $\Phi_W(h) = ih$. Thus I must be the principal ideal generated by i, and if Φ is a natural equivalence then i is a monomorphism.

Grothendieck has introduced [3] equivalence classes of monomorphisms, where two monomorphisms m and m' are *equivalent* if ther arey ight multiples of each other. He calls them "sous-trucs". We shall call the representable contravariant ideals, to which they correspond biuniquely, *mono subobjects*. Representable covariant ideals in X will be called *epi quotients* of X.

A contravariant ideal I in Y will be called *polar* if for every mapping $f \in \mathrm{Map}(X, Y) - I(X)$ there is an equation $h\varphi = k\varphi$, where h, k are mappings in some $\mathrm{Map}(Y, Z)$, that is identically satisfied by all $\varphi \in I(W)$, for all W, but is not satisfied by $\varphi = f$. In short, a polar ideal is an ideal defined by equations.

Obviously any intersection of polar ideals (of fixed variance, in a fixed object) is polar. It is not hard to show that in a *concrete* category (a subcategory of $\mathscr{U}$) there are only a set of polar ideals in a given object. A stronger result is precisely formulated and proved in [6]. With the present conventions, we have

1.2. *In a concrete category the polar contravariant ideals in any object form a set, a complete lattice ordered by inclusion.*

We shall abbreviate "polar contravariant ideal" to *polar subobject* when this is convenient. One way of requiring a category to be closed under the formation of subobjects is to require that every polar subobject be a mono subobject, i.e. every (contravariant) polar ideal be principal and (it is not redundant) generated by a monomorphism. A monomorphism generating a polar subobject may be called a *polar monomorphism*.

Another way to form subobjects, suggested in [3], is to factor each mapping f in the form me, where e is an epimorphism, m is a monomorphism, and no further epimorphism can be factored out; that is to say, supposing $m = m'e'$, where e' is an epimorphism, then e' is an isomorphism. A monomorphism having this property is called an *extremal monomorphism*.

1.3. *Every polar monomorphism is extremal.*

Proof. Let the polar monomorphism m be $m'e'$, with e' epimorphic. Then the equation $hm = km$ is equivalent to $hm' = km'$. Thus m' lies in the contravariant ideal generated by m, $m' = mt$. This gives $m = mte'$; since m is monomorphic, $te' = 1$. Hence $e'te' = e'$; but e' is epimorphic, so $e't = 1$. That is, e' is an isomorphism, as was to be shown.

The converse of 1.3 is valid in most of the most familiar categories, but it is easy to find a counterexample to it. In the category of all abelian groups in which $4x = 0 \Rightarrow 2x = 0$, the monomorphism $4: Z \to Z$ taking the generator 1 to 4 is extremal but not polar. I do not know a "natural" example in which it is not true that the extremal monomorphisms are precisely the compositions of polar monomorphisms; cf. 2.1 below.

Digressing, we define a monomorphism m to be *pure*[1a] provided whenever $m = gf$, and f is an epimorphism, g is a monomorphism. Pure epimorphisms are defined dually. A monomorphism m is *copure* if whenever $m = ed$, and d is a pure epimorphism, d is an isomorphism. It is a trivial exercise to prove

1.4. *Every extremal monomorphism is both pure and copure.*

Special definitions like these can be accumulated, perhaps *ad infinitum*. The present ones have some particular interest.

1.5. *In Hausdorff spaces (also in completely regular, uniform, or proximity spaces) a pure epimorphism is exactly an onto mapping. Hence a copure monomorphism is exactly an embedding.*

Proof. In these categories, clearly, the monomorphisms are the one-to-one mappings. Hence to say that $e: X \to Y$ is a pure epimorphism is to say that it cannot be factored $X \to Z \to Y$ with $Z \to Y$ one-to-one and the image of X not dense in Z. If e is onto, this is true; if e is not onto, it is not true, for Z can be the free sum of X and a point (of $Y - e(X)$). The rest follows from the fact that an embedding is a one-to-one mapping $m: X \to Y$ which is not a mapping (continuous; uniformly continuous; etc.) with respect to any coarser structure on X.

1.5 is true also for unrestricted topological spaces, but there every epimorphism is onto and the embeddings are characterized more simply as the extremal monomorphisms.

The following result is of interest to a reader interested in uniform or proximity spaces, and for such a reader the proof is a routine exercise.

1.6. *In Hausdorff or completely regular spaces, every pure mono- or epimorphism is copure; but the converses are false. In uniform spaces, copure mono- or epimorphisms are pure, but the converses are false. In proximity spaces, pure and copure monomorphisms (also epimorphisms) coincide.*

[1a] Added in proof: Cf. Grothendieck's universal monomorphisms.

2. Completeness. We call a category *left complete* if every polar contravariant ideal is representable and every family (i.e. indexed set) of objects has a direct product.

Subject to various mild set-theoretic restrictions, left completeness is equivalent to (a) the dual property of *right completeness* (b) a self-dual completeness property indicated in 2.2 and 2.3 and (c) the weaker property originally called left completeness by Freyd [2]. We shall not prove (c) here. It should be noted that (1) there is an oversight in the definition appearing in [2]; (2) allowing for (1), a sufficient condition for equivalence of the definitions is that the category is isomorphic with a concrete category.

2.1. Theorem. *In a left complete category in which each object has only a set of mono subobjects,*

(i) *every intersection of mono subobjects of a given object X is a mono subobject of X;*

(ii) *every intersection of extremal subobjects is an extremal subobject;*

(iii) *every composition of extremal monomorphisms is extremal.*

Specifically, the extremal monomorphisms are the smallest class of monomorphisms containing the polar ones and closed under composition and intersection.

Proof. Given any set of mono subobjects of X, let $\{m_a\}$ be a set of monomorphisms $m_a\colon S_a \to X$ representing them. Adjoin, if necessary, $S_0 = X$, $m_0 = 1$. Form the product P of all S_a, with coordinate projections $\pi_a\colon P \to S_a$. Let $n\colon I \to P$ be a polar monomorphism generating the polar ideal J composed of all $f_\beta\colon W_\beta \to X$ such that $m_a \pi_a f_\beta = \pi_0 f_\beta$ for all coordinate indices a. Then $\pi_0 n$ is a monomorphism. To see this, consider any two distinct mappings $h\colon W \to I$, $k\colon W \to I$. We have $nh \neq nk$; so in some coordinate, $\pi_a nh \neq \pi_a nk$. As m_a is monomorphic, $m_a \pi_a nh \neq m_a \pi_a nk$. Since n belongs to $J(I)$, $\pi_0 nh \neq \pi_0 nk$. Finally, the subobject generated by $\pi_0 n = m_a \pi_a n$ is in each subobject generated by m_a, and every mapping into X which factors across all m_a factors across $\pi_0 n$; so (i) is proved.

Next we establish the form of an extremal monomorphism $m\colon X \to Y$, more specifically than the theorem states it. Let $n_1\colon S_1 \to Y$ be a polar monomorphism generating the smallest polar subobject of Y which includes m in its values; $m = n_1 p_1$. Having $n_a\colon S_a \to Y$ (for any ordinal number a) with m factored as $n_a p_a$, let $s_{a+1}\colon S_{a+1} \to S_a$ be a monomorphic generator of the smallest polar subobject including p_a; $p_a = s_{a+1} p_{a+1}$, $n_{a+1} = n_a s_{a+1}$. At a limit ordinal β, let n_β represent the intersection of the subobjects of Y represented by n_a for $a < \beta$. By (i), there is $p_\beta\colon X \to S_\beta$ such that $n_\beta p_\beta = m$. As there are only a set of subobjects of Y, the n_a must be mutually equivalent from some point on. This can hap-

pen only when p_α is epimorphic. But an epimorphic right factor of m is isomorphic; so we may identify X with S_α, and m is n_α, formed by composition and intersection from the polar monomorphisms $s_{\alpha+1}$.

Conversely, suppose a monomorphism n_α: $S_\alpha \to Y$ has this form, and also has the form gf, where f: $S_\alpha \to T$ is epimorphic. Then for any h: $Y \to Z$, k: $Y \to Z$, if $hgf = kgf$ then $hg = kg$. Hence g factors across S_1, $g = n_1 g_1$. By an evident induction one finds factorizations $g = n_\beta g_\beta$ for all $\beta \leqslant \alpha$. Finally $g = n_\alpha g_\alpha$, $n_\alpha = gf = n_\alpha g_\alpha f$. As n_α is monomorphic, $g_\alpha f = 1$. Thus $fg_\alpha f = f$, with f epimorphic; $fg_\alpha = 1$ too, and f is an isomorphism.

From this, (iii) is immediate; and (ii) reduces to an exercise, which we omit.

As a corollary, a category satisfying 2.1 may be made into a *bicategory*. This means [10] that one can define two classes of mappings, the class $\mathscr{P}$ of *projections* (or *surjections*) and the class $\mathscr{I}$ of *injections* so that $\mathscr{P}$ is a subcategory of epimorphisms (i.e. every identity is in $\mathscr{P}$, $\mathscr{P}$ is closed under composition, and every mapping in $\mathscr{P}$ is an epimorphism), $\mathscr{I}$ is a subcategory of monomorphisms, the intersection of $\mathscr{P}$ and $\mathscr{I}$ is exactly the class of all isomorphisms, and every mapping f can be represented as a composition ip, with $i \in \mathscr{I}$ and $p \in \mathscr{P}$, this representation being unique except for the isomorphic variants $(i\lambda)(\lambda^{-1}p)$. In the present application we define $\mathscr{P}$ as the class of all epimorphisms, $\mathscr{I}$ as the class of all extremal monomorphisms. All the required properties are immediate, using 2.1 where needed; for the factorization $f = ip$, intersect all the extremal subobjects which include f.

2.2. COROLLARY. *Every left complete category in which each object has only a set of mono subobjects can be made into a bicategory in which every epimorphism is a projection.*

I do not know whether there is always another bicategorical structure in which every monomorphism is an injection. The only difficulty is that compositions of extremal epimorphisms might fail to be extremal. A counterexample of course could not satisfy the hypotheses dual to the present hypotheses, and therefore (in view of 2.5 below) must presumably be "unnatural".

There are two (or more) types of partial converse to Theorem 2.1, going from conditions on mono subobjects and auxiliary constructions to representability of polar subobjects. One was given already by Freyd [2]: if every family of objects has a direct product, and every set of mono subobjects of a given object has representable intersection, then every polar subobject defined by a set of equations is representable. Left completeness (in the present sense) follows if we assume either that there are only a set of mono subobjects in any object or that the category is concrete.

It may be well to pause to consider these "mild" set-theoretic restrictions. In the opinion of the author, it is not worth much effort to remove the assumption that a category is concrete from a theorem, but the assumption that there are only a set of mono subobjects is more serious. I have not thought of a natural example in which it is not trivial that there are only a set of mono subobjects, but I have spent time on the problem whether every semigroup has only a set of epi quotients without solving it. From this point of view, Freyd's result just stated is not marred by set-theoretic restrictions. Neither is 2.1 (i); for in any left complete category one can at least intersect sets of mono subobjects. However, for the rest of 2.1—for an intersection of two extremal subobjects or a composition of two extremal monomorphisms—the set-theoretic part of the hypothesis should be removed, or weakened to something unobjectionable, or shown to be necessary by satisfactory examples. Of course, these remarks are intended to apply to all theorems in general categories involving set-theoretic restrictions; I do not mean to suggest that the sharpening of Theorem 2.1 is an especially important problem.

Returning to the second partial converse: the digression explains why I take time to weaken a hypothesis, defining an *injected subobject* in a bicategory as a mono subobject generated by an injection.

2.3. *In a bicategory in which every family of objects has a free sum, and each object has only a set of injected subobjects, every polar subobject is representable.*

The proof is a routine exercise.

2.3, unlike the preceding and following results, seems to be isolated and to lead nowhere. Of course it has a dual, and if both 2.3 and its dual apply, the category is left and right complete. But the presence of sums and subobjects does not lead to products or quotients, even in the category of free abelian groups.

We define the *sum functor* of a family of objects X_a as the direct product of all the covariant principal set functors $\mathrm{Map}(X_a, \mathscr{C})$. (The sum functor is a product because $\mathrm{Map}(X_a, \mathscr{C})$ depends contravariantly on X_a; the product functor is also a product.) Evidently the existence of a sum object is equivalent to the representability of the sum functor.

A set functor F, let us say covariant, is *dominated* by a set S of objects if for every object Y, for every member p of $F(Y)$, there exist $X \in S$ and $f\colon X \to Y$ such that $p \in F(f)(f(X))$ [5]; a set functor dominated by a set of objects is called *proper*.

2.4. Theorem. *In a left complete category, every family of objects whose sum functor is proper has a free sum and every proper covariant polar ideal is representable.*

Proof. The argument is essentially the same for the two cases. Let $\{X_a\colon a \in A\}$ be a family of objects, perhaps reducing to a single object;

let F be the set functor to be represented, that is, the sum functor or a covariant polar ideal. Let $\{Y_\beta\}$ be a set of objects dominating F. Let M be the set of all families of mappings of the following form: the family $\mu = \{m_a\colon a \in A\}$ has a common range Y_β, $\beta = \beta(\mu)$, and each m_a has domain X_a — and for the second case, $m_a = m \in F(Y_\beta)$.

Let P be the direct product of the indexed family $\{Y_{\beta(\mu)}\colon \mu \in M\}$. For each a in A, let $p_a\colon X_a \to P$ be the map whose μth coordinate ($\mu \in M$) is the ath member m_a of μ. Then the smallest polar contravariant ideal in P containing all p_a in its values is representable, so generated by some monomorphism $n\colon S \to P$. We shall show (i) that S is the direct sum of $\{X_a\}$, with coordinate injections $i_a\colon X_a \to S$ defined as the unique solutions of $ni_a = p_a$ (respectively, (ii) that S is the required polar quotient object).

(i) Given any family $\{g_a\colon a \in A\}$ of maps $g_a\colon X_a \to Z$, there is at least one $\mu = \{m_a\}$ in M such that for some $h\colon Y_{\beta(\mu)} \to Z$, $g_a = hm_a$ for all a. Let π_μ denote the μth coordinate projection, $\pi_\mu\colon P \to Y_\beta$; then $g = h\pi_\mu n$ has coordinates $gi_a = g_a$.

It remains to show that g is unique, i.e. that no proper polar contravariant ideal in S includes all i_a in its values. Consider the family $\{i_a\} \in F(S)$. There is a map q (of the form $h\pi_\mu$) from P to S such that for all a, $qp_a = i_a$. Then $nqp_a = ni_a = p_a$. If 1_P denotes the identity of P, we have $nqp_a = 1_P p_a$; since n determines the same polar ideal as the p_a, $nqn = 1_P n$ also. Here $nqn = n$, with n a monomorphism; hence qn is the identity 1_S. That is, S is a retract of P. Now if we had $g'\colon S \to Z$ with $g'i_a = gi_a$ for all a, we should have $g'qni_a = g'qp_a = gqp_a$; so $g'qn = gqn$ and $g' = g$.

(ii) The generating epimorphism $e\colon X \to S$ will be the solution of $ne = p$ (p replaces $\{p_a\}$ in this case). There are three things to prove: $e \in F(S)$, e generates F, and e is epimorphic. First $p \in F(P)$; for any equations $\varphi f = \varphi g$ satisfied by all the coordinates of p are satisfied by p. Since n is a monomorphism, it follows that e satisfies the same equations as ne, which is p. Thus $e \in F(S)$. The rest is just as in case (i).

As every representable set functor is proper, 2.4 gives necessary and sufficient conditions. More useful sufficient conditions can be given. Let us say that an object Y is an *amalgamation*[2] of a family of objects X_a if there are mappings $f_a\colon X_a \to Y$ such that no proper subobject of Y, either mono or polar, includes all f_a in its values. If sums exist, this means Y is an extremal quotient of a sum (with repetitions) of the X_a.

2.5. Corollary. *Let $\mathscr{C}$ be a left complete category in which every intersection of mono subobjects is a mono subobject. If each object has only a set*

[2] In groups, an amalgamation of a family of objects means an amalgamated product of any quotients of the objects.

of extremal quotients, then every covariant polar ideal is representable. If each family of objects has only a set of amalgamations, then $\mathscr{C}$ is right complete.

Proof. The appropriate sets of extremal quotients or of amalgamations dominate the functors to be represented, because mappings factor across a smallest subobject of the range.

2.6. *Examples of left complete concrete categories which are not right complete.*

It is convenient to describe the dual examples instead; the dual of any concrete category is isomorphic with a concrete category [6].

The category $\mathscr{W}$ of all ordinal numbers (each ordinal α being the set of all smaller ordinals) and inclusion functions is right complete; it fails to be left complete only because the empty set of objects has no product.

Let $\mathscr{T}$ be the category of all non-empty topological spaces (or semigroups, if the reader prefers; or sets). Adjoin to each object T of $\mathscr{T}$ a "base point" 0 (not an ordinal number; $\{T\}$ will do), and extend each mapping f by defining $f(0) = 0$. Call this category T'. Let $\mathscr{A}$ consist of the objects and mappings of $\mathscr{W}$ and of $\mathscr{T}'$ and for each $\alpha \in \mathscr{W}$, $X \in \mathscr{T}'$, a mapping $g_{\alpha X}$: $\alpha \to X$ defined $g_{\alpha X}(x) \equiv 0$. One verifies easily that $\mathscr{A}$ is right complete. Also every set of objects has a product; but the smallest contravariant polar ideal in any $X \in \mathscr{T}'$ (corresponding to the missing empty space) is not representable. For the same reason, no mapping $g_{\alpha X}$ is me, where m is an extremal monomorphism and e an epimorphism.

If the reader prefers an example with a pair of objects having no product, one can be constructed by amalgamating two copies of $\mathscr{A}$ along $\mathscr{W}$ and the product of the empty set of objects.

3. Adequate and reflexive. There are two notions of *conjugate* and reflexive functors in the literature [5], [12], having quite different purposes. The theory of Fuks and Švarc concerns functors on $\mathscr{C}$ to $\mathscr{C}$, and the conjugate is an "opposite"; it is a duality theory. My theory concerns set functors and is a theory of extension. For example, an ideal Φ may stand for an "ideal subobject" F of some object, each set $\Phi(X)$ standing for $\mathrm{Map}(X, F)$; then the conjugate Φ^* can be used to define $\mathrm{Map}(F, Y) = \Phi^*(Y)$.

The (left) *regular representation* of a category $\mathscr{C}$ [5] represents objects of $\mathscr{C}$ by set functors on $\mathscr{C}$ and represents mappings of $\mathscr{C}$ by natural transformations; specifically, the object X goes to the contravariant set functor $\mathrm{Map}(\mathscr{C}, X)$, and the mapping f: $X \to Y$ goes to the natural transformation from $\mathrm{Map}(\mathscr{C}, X)$ to $\mathrm{Map}(\mathscr{C}, Y)$ defined by left multiplication by f. This is an isomorphic full representation ("full" means that every natural transformation between principal set functors is multiplication by a mapping). For any subcategory $\mathscr{A}$ of $\mathscr{C}$, there is a left

subregular representation, $X \to \mathrm{Map}\,(\mathscr{A}, X) = \mathrm{Map}\,(\mathscr{C}, X) \mid \mathscr{A}$. $\mathscr{A}$ is called *left adequate* in $\mathscr{C}$ if $\mathscr{A}$ is a full subcategory and the subregular representation on $\mathscr{A}$ is isomorphic and full.

The *right regular* (dually isomorphic) *representation* and *right adequacy* are defined dually.

For any proper covariant set functor F on $\mathscr{C}$, the *conjugate* F^* is a contravariant set functor on $\mathscr{C}$, defined only up to natural equivalence. For each object X, the set $F^*(X)$ is in one-to-one correspondence with the "collection" of all natural transformations from F to the principal set functor $\mathrm{Map}\,(X, \mathscr{C})$. (Such a set exists, since F is proper.) For a mapping $f\colon X \to Y$, $F^*(f)$ takes $F^*(Y)$ to $F^*(X)$ by composition with the natural transformation f induces from $\mathrm{Map}\,(Y, \mathscr{C})$ to $\mathrm{Map}\,(X, \mathscr{C})$. Then F^* is well defined, but not necessarily proper. If F^* is proper, there is a well defined covariant set functor F^{**}, and there is an evaluation transformation $\varepsilon\colon F \to F^{**}$. F is *reflexive* if F and F^* are proper and ε is a natural equivalence.

The relation between reflexivity and adequacy is essentially that $\mathscr{A}$ is a two-sided adequate subcategory of $\mathscr{C}$ if and only if $\mathscr{C}$ is representable (subregularly) as a full category of reflexive contravariant set functors on $\mathscr{A}$.

There are set-theoretic complications, treated not quite correctly in [5] and correctly in [6].

3.1. *The conjugate functor of a principal ideal is a polar ideal; specifically, the conjugate of the contravariant principal ideal generated by $f\colon X \to Y$ is the smallest covariant polar ideal in X having f among its values. Hence in case both of the principal ideals generated by f are polar, they are mutually conjugate and reflexive.*

Proof. Let I be the contravariant ideal in Y generated by f; let J be the smallest covariant polar ideal in X including f. We define a natural equivalence $\Phi\colon J \to I^*$. For each object Z, Φ_Z will take maps $g\colon X \to Z$ in $J(Z)$ to natural transformations Γ in $I^*(Z)$. In turn $\Gamma\colon I \to \mathrm{Map}\,(\mathscr{C}, Z)$ is made up of functions $\Gamma_W\colon I(W) \to \mathrm{Map}\,(W, Z)$. For each $h \in I(W)$, we can write h as ft (since f generates I); we define $\Gamma_W(h)$ as gt. The definition does not depend on the choice of t, because $g \in J(Z)$. Naturality of Γ is immediate. $\Gamma_V(fts) = gts$; naturality of Φ is clear, from a diagram. Each function Φ_Z is one-to-one, because each g in $J(Z)$ goes to a transformation Γ satisfying $\Gamma_X(f) = g$. Moreover, Φ_Z is onto; to get any Γ in $I^*(Z)$, consider $g = \Gamma_X(f)$. Since Γ is natural, $\Gamma(ft)$ is always gt, which proves $g \in J(Z)$ and $\Phi_Z(g) = \Gamma$. Thus Φ is a natural equivalence, $I^* = J$. The concluding assertion follows at once.

I do not know any significant example of a mapping f generating polar principal ideals on both sides, beyond the example of any idempotent mapping $f\colon X \to X$. That example was already treated in [5]

One might guess from 3.1 that for a principal ideal to be reflexive, it must be polar. This conjecture is quickly refuted when one recalls that reflexivity is invariant under natural equivalence and polarity is not; but it is unknown whether (or how generally) every reflexive principal ideal is naturally equivalent to a polar ideal. There seems to be no hope for any comparably simple condition either necessary or sufficient for reflexivity of a non-principal ideal, cf. 6.1.

3.2. THEOREM. *Let $\mathscr{C}$ be a category in which every family of objects has a free sum and a direct product, and every mapping has a factorization me where m is a monomorphism and e an epimorphism. Let F be a reflexive set functor on a full subcategory $\mathscr{A}$ of $\mathscr{C}$. Then F is representable in $\mathscr{C}$; in fact, for some object X which is both an epiquotient of a sum, and a mono subobject of a product, of objects of $\mathscr{A}$, F is naturally equivalent to* Map $(\mathscr{A}, X)$ *and F^* is naturally equivalent to* Map $(X, \mathscr{A})$.

Proof. Let D be a set of objects of $\mathscr{A}$ dominating both F and F^*. Let H be the union of all $F^*(Z)$, $Z \in D$; let I be the union of all $F(W)$, $W \in D$. (We suppose for notational convenience that all the sets $F(W)$ are disjoint; and similarly for $F^*(Z)$. We shall use the usual convention to speak of the natural transformations h: $F \to \mathrm{Map}(\mathscr{A}, Z)$ as members of $F^*(Z)$.) Form the sum S of the family $\{W_i: i \in I\}$, where W_i is defined by $i \in F(W_i)$, and the product P of $\{Z_h: h \in H\}$, $h \in F^*(Z_h)$. We define a (natural) mapping λ: $S \to P$ by its coordinate projections λ_h: $S \to Z_h$; λ_h in turn is defined by its coordinate injections λ_h^i. Let us write W for W_i. Then h_W is a function from $F(W)$ to $\mathrm{Map}(W, Z_h)$; put $\lambda_h^i = \lambda_W(i)$. Finally we factor λ as me, where e: $S \to X$ is epimorphic and m: $X \to P$ is monomorphic. Thus X is a mono subobject of P and an epi quotient of S, as required. Taking account of duality, it will be sufficient to show that F is naturally equivalent to $\mathrm{Map}(\mathscr{A}, X)$.

For $i \in F(W)$, let $\bar{i}$ denote the ith coordinate injection $\bar{i}$: $W \to S$. We define a natural equivalence Φ from F to $\mathrm{Map}(\mathscr{A}, X)$ by $\Phi_W(i) = e\bar{i}$, for $W \in D$. For V in $\mathscr{A}$, not in D, each j in $F(V)$ is $F(f)(i)$ for some f: $V \to W$, $W \in D$, $i \in F(W)$. Then $\Phi_V(j)$ is defined as $\Phi_W(i)f = e\bar{i}f$ for any such representation. It is independent of the choice of f and i because the hth coordinate of $me\bar{i}f$ is just $h_W(i)f = h_V(j)$, for each $h \in H$. The same analysis shows that Φ is a natural transformation. Each Φ_V is one-to-one because [F is reflexive and D dominates F^*. On the other hand, let g be any mapping from V to X in $\mathscr{C}$. Define a natural transformation ν in $F^{**}(V)$ as follows. For $h \in F^*(Z)$, $Z \in D$, $\nu_Z(h)$ is the hth coordinate $\pi_h mg$ of mg: $V \to P$. For Z not in D, $\nu_Z(h)$ is defined by means of representations $h = F^*(l)(k)$, $K \in F^*(Y)$, $Y \in D$. Again we must check independence of the representation and naturality. The necessary implication is $l\pi_k mg = l'\pi_{k'} mg$ when $F^*(l)(k) = F^*(l')(k')$. Since e is an epimorphism, it suffices to prove $l\pi_k\lambda = l'\pi_{k'}\lambda$. Passing to coordinate

injections λ^i, this reduces to the given formula $F^*(l)(k) = F^*(l')(k')$. Thus ν is natural, $\nu \in F^{**}(V)$. Since F is reflexive, there is $c \in F(V)$ such that $\nu_Z(h) \equiv h_V(c)$. That is, for all h, $h_V(c)$, which must be $\pi_h m \Phi_V(c)$, is $\pi_h mg$; thus $\Phi_V(c) = g$. Therefore Φ is a natural equivalence.

Remarks. The conclusion of this theorem is not as strong as one might want, and I do not know what are the strongest possible results. Nine sharpened forms of 3.2 seem to deserve mention. First, from the proof, the factorization of λ: $S \to X \to P$ is arbitrary. Thus (1) if $\mathscr{C}$ is a bicategory, we may suppose e: $S \to X$ is surjective and m: $X \to P$ is injective. In particular, (2) if $\mathscr{C}$ satisfies 2.1, we may take m to be an extremal monomorphism, e some epimorphism. (3) Alternatively, on the same hypothesis, we may take e to be extremal. By a separate proof, (4) we may choose X to be a polar subobject of a product of objects of $\mathscr{A}$, with no additional conclusion (no epimorphism); the hypothesis on $\mathscr{C}$ can be correspondingly weakened. (5) If we add to the hypotheses of 3.2 that $\mathscr{A}$ is right adequate in $\mathscr{C}$, we can conclude (by further proof) that X is unique, that m is a polar monomorphism, and that e is an extremal epimorphism. Four more propositions (6)-(9) are the exact duals of (2)-(5).

Of these nine sharpened forms, (2) generalizes Theorem 2.1 of [5].

4. Full categories of algebras. A *category of algebras* means, here, not an abstract category but a collection of algebras and homomorphisms forming a category. The category of algebras is called *full* if it includes all homomorphisms between its objects.

In [5] it is shown that if the full category of algebras A has operations at most n-ary, where $n \geqslant 1$, and if A contains a free algebra F on n generators, then F with its endomorphisms forms a left adequate subcategory of $\mathscr{A}$. The same proof shows that in any full category of algebras, either a single free algebra on infinitely many generators or a set of free algebras on unbounded finite numbers of generators will form the objects of a left adequate subcategory, if they exist. Now it is well known that every full category of algebras is contained in a full category of algebras having free algebras on arbitrary sets of generators. Hence

4.1. *Every full category of algebras is a full subcategory of a category having a left adequate set of objects.*

4.2. Theorem. *The categories which are isomorphic with full categories of algebras are precisely those which can be embedded as full subcategories of categories having left adequate sets of objects; moreover, they are all isomorphic with full categories of algebras with unary operations.*

Proof. It suffices to show that a category $\mathscr{C}$ having a left adequate set S is isomorphic with a full unary category. First embed $\mathscr{C}$ in a larger category, coextensive with $\mathscr{C}$, but having at least two objects in each

isomorphism class; and enlarge S accordingly. In particular, each object in S will now certainly be the range of a mapping whose domain is a different, perhaps isomorphic, object in S. We associate to each object X of $\mathscr{C}$ the algebra X^*, whose elements are all the mappings $g\colon W \to X$ for all W in S, and whose operations are defined as follows. There is one operation Q_f for each $f\colon V \to W$, with both V and W in S; the element $Q_f(g)$ of X^* is gf if gf is defined, g otherwise. To a mapping $h\colon X \to Y$ we associate $h^*\colon X^* \to Y^*$ defined by $h^*(g) = hg$. Obviously h^* is a homomorphism, and (since S is left adequate) we have an isomorphism of the category $\mathscr{C}$ into a full category of unary algebras. Finally consider any homomorphism $k\colon X^* \to Y^*$. For any mapping $g\colon U \to X$, U in S, consider $k(g)\colon W \to Y$. By the preliminary construction, there is $f\colon V \to W$ with $V \neq W$ in S. $k(Q_f(g)) = Q_f(k(g)) = k(g)f \neq k(g)$; so $Q_f(g) \neq g$. This implies $Q_f(g) = gf$, and $W = U$; that is, k preserves domains. Since k preserves the operations Q_f, it defines a natural transformation from $\mathrm{Map}(\mathscr{S}, X)$ to $\mathrm{Map}(\mathscr{S}, Y)$, where $\mathscr{S}$ is the full subcategory on the set S of objects. Consequently k is h^* for some mapping $h\colon X \to Y$, and the proof is complete.

The representation with unary operations is helpful in the next proof. The reader is supposed to have some acquaintance with the notion of a *measurable cardinal* (see [6] or [11]). In particular, the assumption that no measurable cardinals exist is known to be consistent with all usual systems of set theory.

4.3. *Assuming no measurable cardinals exist, the dual of a full category of algebras is a full category of algebras.*

Proof. It was shown in [5] that on this hypothesis, a countable set N is right adequate in the category $\mathscr{U}$. Then given a full category of algebras $X, Y, \ldots$, with unary operations Q_α, we represent the dual by associating to each algebra X the set X^* of all functions $\beta\colon X \to N$. Unary operations on X^* are defined as follows: for each function $e\colon N \to N$, $e^*(\beta) = e\beta$; for each operation Q_α, $Q^*_\alpha(\beta) = \beta Q_\alpha$. The verification is immediate.

4.3 answers (essentially) a question I raised at the 1961 Prague Symposium on Topology; the category of compact Hausdorff spaces is isomorphic with a full category of algebras, unless measurable cardinals exist. Several different representations can be described, not that there is much chance that any will ever be useful.

The problem of exhibiting a concrete category that is not isomorphic with a full category of algebras remains open. The last example in 5.3 below is isomorphic with a concrete category, perhaps not with a full category of algebras.

5. Quasi-primitive categories of algebras. The main theorem of this section has been rather fully stated in the Introduction, but there

are some questions of alternative definitions and alternative hypotheses to be cleared up. First, a quasi-primitive category of algebras is both left and right complete in the sense of Freyd ([2] and Introduction to this paper) and in the stronger sense of Section 2 of this paper. One can see at once that algebraic direct products are categorical direct products, and that every polar subobject is representable by a subalgebra. Thus the category is left complete. 2.5 applies (and is well known in this context) because all the amalgamations of a given family of algebras are algebras whose size is bounded by a cardinal number determined by the family. This establishes completeness in the stronger sense, and the definition given in the Introduction obviously follows. It should be noted that the latter is a simplification of Freyd's definition, shown in [8] to be equivalent to it. As we want to use the weakest possible conditions in the sufficiency proof, we note, from [8]:

5.1 (Lawvere). *If every family of objects has a free sum and every pair of mappings $X \to Y$ has a coequalizer, then every polar quotient defined by a set of equations is representable.*

The proof goes as follows. For the polar quotient of the object U made up of all mappings h which satisfy $hf_a = hg_a$, $a \in A$, f_a and g_a being mappings $T_a \to U$, define X as the free sum of two copies of T_a, for each a in A, and one copy of U; define Y as the free sum of one copy of each T_a and one copy of U. Let d: $X \to Y$ map each summand T_a of X (each copy) to the summand T_a of Y by the identity mapping, and U to U by the identity. Let e: $X \to Y$ map every summand of X to U, the two copies of T_a being mapped one by f_a and one by g_a, U by the identity. Then one readily verifies that the coequalizer of (d, e) induces on U a polar epimorphism generating the given ideal.

Remark. All that is needed to complete the proof that the two definitions of right completeness agree for concrete categories is to verify that there every polar quotient is defined by a set of equations. This is a special case of a result in [6].

It is rather well known (see e.g. [9]) that in a quasi-primitive category of algebras there are free algebras on any set of generators; and it is easy to verify that an algebra freely generated by a non-empty finite set is projectively finite, is projective in the strong sense defined in the Introduction, and is not only a generator but a strict separator in the following sense (from [10]). A *strict separator* is a object P such that for any object X, no proper mono subobject of X and no proper polar subobject of X has the value Map (P, X) on P. In fact, in any category having equalizers, any generator is a strict separator.

As was indicated in the Introduction, it seems better to define a *projective* object P in the following weaker way: for any extremal epimorphism e: $X \to Y$ and any mapping f: $P \to Y$, there exists g: $P \to X$

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such that $eg = f$. A further weakening, trivial in the present application and probably undesirable in general, would require e to be a polar epimorphism. Because of the dominating influence of projective and injective objects in many applications, one might be interested in allowing e to be an arbitrary epimorphism; but in general (e.g. in the category of all commutative rings), epimorphisms need not be onto, and not even the free algebras need be projective in this sense.

5.2. THEOREM. *Quasi-primitive categories of algebras are characterized (up to coextension) by the following properties. Every family of objects has a free sum. Every pair of mappings $X \to Y$ has a coequalizer. Some object P is at once projectively finite, projective, and a generator; and finally, P is a strict separator (or we may assume that every pair of mappings $X \to Y$ has an equalizer).*

Proof. Necessity is established, once we observe that these properties are invariant under coextension. For the converse, the parenthesis is justified by preceding remarks. Next, every polar quotient Λ of any object X is defined by a set of equations in Map (P, X). For this let Λ^* be the class of all pairs (f, g) of mappings $A \to X$ such that $hf = hg$ for every h in every value of Λ. Suppose k: $X \to Z$ not in $\Lambda(Z)$, and select (f, g) in Λ^*, mapping A to X, such that $kf \neq kg$. Since P is a strict separator, there is e: $P \to A$ such that $kfe \neq kge$. Evidently $(fe, ge) \in \Lambda^* \cap$ $\cap$Map (P, X); so the assertion is proved.

Consequently a composition of polar epimorphisms q: $X \to Y$, r: $Y \to Z$ is polar. For let Λ be the smallest polar quotient of X having rq in $\Lambda(Z)$. If t: $X \to T$ is in $\Lambda(T)$, it is certainly in the polar quotient generated by q; hence $t = uq$. In turn u must be a multiple vr (showing that rq generates Λ) if we can show $ua = ub$ for every pair (a, b) in Map (P, Y) such that $ra = rb$. But since P is projective, there are f and g: $P \to X$ such that $qf = a$, $qg = b$. Hence $rqf = rqg$. Since $t \in \Lambda(T)$, $tf = tg$; that is, $uqf = uqg$, $ua = ub$.

Every mapping f: $X \to Y$ has the form me, where m is a monomorphism and e a polar epimorphism; for if we take e to be an epimorphic generator of the smallest polar quotient of X including f in its values (using 5.1), we have $f = me$, and if m: $M \to Y$ were not monomorphic we could factor out a non-trivial polar epimorphism q: $M \to N$, making qe a factor of f generating a smaller polar quotient.

Every object X is a polar quotient of a sum of copies of P; the summands may be indexed by Map (P, X) and the mapping defined so that its fth coordinate is f. The mapping is a polar epimorphism since P is a generator.

With these preliminaries, we construct a representation by algebras. The object X is represented by the set Map (P, X). The algebraic operations are defined by the mappings a: $P \to nP$, nP being any finite sum

of copies of P; an n-tuple $(f_1, \ldots, f_n)$ in Map (P, X) forms the set of coordinates of a unique mapping f: $nP \to X$, and we define $a(f_1, \ldots, f_n) = fa$. For a mapping g: $X \to Y$, we represent it by the homomorphism $f \to gf$. Clearly this is a homomorphism, and the representation is functorial. Since P is a strict separator, the representation is isomorphic.

We must show that every homomorphism F: Map $(P, X) \to$ Map (P, Y) is induced by a mapping. Represent X as a polar quotient of a sum of copies of P, by p: $\sum P_i \to X$. F applies to the coordinates p_i of P, giving $F(p_i) = q_i$: $P \to Y$. These mappings are coordinates of a mapping q: $\sum P_i \to Y$. We wish to factor q as fp, which can be done if $qa = qb$ whenever $pa = pb$ for a and b: $P \to \sum P_i$. Since P is projectively finite, both a and b factor through a finite subsum of $\sum P_i$. Hence the desired relation follows since F is homomorphic. Thus $q = fp$. F is precisely multiplication by f since $fp_i = q_i = F(p_i)$ for all p_i, which means all of Map (P, X).

Thus the given category $\mathscr{C}$ is represented isomorphically by a full category of algebras $\mathscr{D}$. Evidently P and all free sums of copies of P are represented by free algebras, freely generated by the coordinate injections.

Let $\{X_a\}$ be any family of algebras in $\mathscr{D}$. Their direct product is a well-defined algebra Y, a homomorphic image of a free algebra K in $\mathscr{D}$. The homomorphism e: $K \to Y$ may not be in $\mathscr{D}$, but its coordinates e_a are in $\mathscr{D}$. Let f: $K \to Z$ be an epimorphic generator of the smallest polar quotient of K including all e_a in its values. Now f is a homomorphism onto; and the conditions under which it identifies two points of K are exactly the same as the conditions under which e identifies those points. Hence Z is isomorphic with the direct product Y, and it belongs to $\mathscr{D}$.

Precisely the same type of argument shows that every subalgebra of an algebra in $\mathscr{D}$ is isomorphic with an algebra in $\mathscr{D}$, completing the proof.

5.3. Theorem. *The six properties in Theorem* 5.2 *are independent.*

Proof. Six examples are required, each having just five of the properties. We note: every pair of mappings will have an equalizer in each example, with, of course, one exception.

Four examples are trivial. The category of all cyclic groups lacks only free sums, and the category of all free groups lacks only polar quotients. The category of all abelian torsion groups in which every non-zero element has squarefree order is complete and has projective strict separators, but none of them is projectively finite. For an example lacking a generator, but complete and having projectively finite projective objects P such that for no object X do all mappings $P \to X$ lie in a proper polar subobject of X, take three sets $A \subset B \subset C$ and the six functions f among them which satisfy $f(x) = x$ identically.

Next we describe a complete category of algebras having projectively finite strict separators, but not projective ones. There are a 0-ary operation and two unary operations α, β. The 0-ary operation defines a one-element subalgebra 0 $(\alpha(0) = \beta(0) = 0)$; moreover, $x \neq 0 \Rightarrow \alpha(x) \neq 0$, $\beta(x) \neq 0$. We now describe the remaining structure, assuming $x \neq 0$, $y \neq 0$, $\alpha(\beta(x)) = \beta(\alpha(x))$; $x \neq y \Rightarrow \alpha(x) \neq \alpha(y)$, $\beta(x) \neq \beta(y)$; $\alpha^m(\beta^n(x)) = \alpha^r(\beta^s(x)) \Rightarrow r = m$, $s = n$; $\alpha^m(x) = \beta^n(y) \Rightarrow (\exists z) x = \beta^n(z)$, $y = \alpha^m(z)$. In short, if we define a partial ordering by means of $x < \alpha(x)$, $x < \beta(x)$, these algebras consist of 0 and a disjoint union of sets of integral points of the plane which are quadrants, half-planes, or planes. The class is closed under direct products and free sums (disjoint sums, except that 0 is identified). The equalizer of f, g: $X \to Y$ will consist of 0 and some set of "components" of X; so it is a subalgebra belonging to the category. To form a coequalizer, one makes the necessary identifications $f(x) \sim g(x)$; when these identifications yield components not embeddable in the set of integral points of the plane, one identifies all such components to 0. When identification yields a component embeddable in the set of all integral points of the plane but not having the correct form, one adjoins points z (with $x = \beta^n(z)$, $y = \alpha^m(z)$) as required by the last axiom. Thus the coequalizers exist, but they are not always mappings onto. Hence, one may verify, there are no non-zero projective objects, although the free algebra on one generator is a projectively finite strict separator.

The sixth example requires an artificial construction, for it must be right complete and not left complete. We describe successively three categories $\mathscr{A} \subset \mathscr{B} \subset \mathscr{C}$. The objects of $\mathscr{A}$ are P, R, and $Q_{n\alpha}$ for each integer n and each ordinal number α. The structure of $\mathscr{A}$ involves the total ordering of objects which is defined lexicographically for the $Q_{n\alpha}$, with P the least object and R the greatest object. Also we define $n(Q_{\alpha n}) = \alpha$; $j(X)$ is the greatest integer $j \leqslant n(X)/3$; $k(X)$ is the residue $n(X) - 3j(X)$. (These functions are not defined at P or R.)

Map (X, Y) is non-empty if and only if $X \leqslant Y$; and Map (X, R) has exactly one element. In particular, this defines all Map (X, R) and Map (R, Y); we specify now that the same definition applies in $\mathscr{B}$ and in $\mathscr{C}$ (each Map (X, R) has one element, Map (R, Y) is empty unless $Y = R$), and we omit R from the following description.

For $\mathscr{A}$ we use a concrete representation $\mathscr{A}^0$. Let G be a fixed abelian group having more than one element. We represent P by the set P^0 of all G-valued functions on the set Z of all integers. Every other X $(\neq R)$ is represented by a copy X^0 of the set of all G-valued functions on the set Z_j^+ of all integers $m \geqslant j = j(X)$. To simplify notation, in defining mappings we may identify the sets X^0 and Y^0 if $j(X) = j(Y)$. All the mappings in $\mathscr{A}^0$ are composed of identifications and (1) *pointwise multiplications*

by any G-valued function f on the correct domain, (2) (positive) *shifting* s^i of the functions $\varphi \in X^0$, where $s^i\varphi(m) = \varphi(m+i)$, $i \geqslant 0$, and (3) *restriction* q_n of functions φ on Z_m^+ to $\varphi \mid Z_n^+$, where $n \geqslant m$. A typical composite of these operations is written $c_{XY}s^ifq_n$; the symbol c_{XY} indicates the domain X and the range Y, and of course there are relations connecting $i, n, n(X)$ and $n(Y)$. We may omit vacuous operation symbols such as s^0.

Multiplication is defined by the obvious rules for $c_{XY}\ldots c_{WX}$, fg, s^is^j, q_nq_m, and the commutation rules

$$q_ns^i = s^iq_{n-i}; \quad fs^i = s^i[s^{-i}(f)], \quad q_nf = [q_n(f)]q_n.$$

The mappings from X to Y, for $X = P$, are all possible mappings; specifically, Map (P, P) consists of all $c_{PP}s^if$, where $i \geqslant 0$ and $f\colon Z \to G$, Map (P, Y) for $Y > P$ consists of all $c_{PY}s^ifq_n$ for $i \geqslant 0$, $i+n = j(Y)$, $f\colon Z_n^+ \to G$. For $P < X \leqslant Y$, all multiplications and positive shiftings are allowed but restriction is allowed only when $k(X) = 0$ or $k(Y) = 2$. Thus if $k(X) = 0$, or $k(Y) = 2$, or (when $k(X) = 1$) $n(Y) = n(X)$, then Map (X, Y) consists of all $c_{XY}s^ifq_n$ such that $i \geqslant 0$, $i+n = j(Y)$, $n \geqslant j(X)$, $f\colon Z_n^+ \to G$; in the remaining cases, these conditions must hold and $i \geqslant 1$.

In $\mathscr{A}$, (i) P is projective and (ii) coequalizers exist. (i) holds because shifting is monomorphic and when $e\colon X \to Y$ involves no shift, every mapping of P to Y factors across e. As for (ii), the non-trivial coequalizers are the mappings $c_{XY}fq_n$ where $Y = Q_{m0}$, with $m \equiv 2 \pmod 3$ when $k(X)$ is 1 or 2, and $j(X) < j(Y)$; or with $m \equiv 0 \pmod 3$ when $k(X) = 0$ or $X = P$. (There are also isomorphisms and mappings to R.) $\mathscr{A}$ has many monomorphisms and P is not a generator. In $\mathscr{B}$, on the other hand, there will be no monomorphisms except the isomorphisms $c_{XX}f$ and the mappings $P \to P$.

$\mathscr{B}$ has the same objects as $\mathscr{A}$ and additional mappings $c_{XY}\lambda s^ifq_n$ as follows. $X < Y < R$; and $c_{XY}s^ifq_n$ is a mapping of $\mathscr{A}$. If $X = P$ or $k(X) = 0$, the conditions are $i = 0$, $Y = Q_{m0}$, $m \equiv 1 \pmod 3$. If $k(X)$ is 1 or 2 then for $j(Y) > j(X)$ the conditions are $i = 1$ and Y is Q_{ma} where $m \equiv 0 \pmod 3$ or $m \equiv 1 \pmod 3$ and $a > 0$. If $k(X) = 1$ there are also $c_{XY}\lambda f$ where $n(Y) = n(X)+1$. Thus every object $Y \neq P$, R is the range of some of these mappings; and note how they avoid the coequalizers of $\mathscr{A}$.

Before defining multiplication we define monic and epic mappings, which will be precisely the monomorphisms and epimorphisms of $\mathscr{B}$. A *monic* mapping is a mapping $X \to X$. The *epic* mappings are the mappings into R, the isomorphisms $c_{PP}f$, and the mappings $c_{XY}fq_n$ such that: $k(Y) = k(X) \neq 1$; or $n(Y) = n(X)$; or $X = P$, $k(Y) = 0$; or $k(X) = 1$, $k(Y) = 2$, $j(Y) > j(X)$. Now to compute a product de of mappings in $\mathscr{B}$, first ignore λ and compute the product in $\mathscr{A}$. Then insert λ in case (i) e

contains λ and d is monic, or (ii) d contains λ and e is epic, or (iii) d: $Y \to Z$ contains λ, $k(Y) = 2$, $j(Z) \geqslant j(Y)+2$, and e: $X \to Y$ satisfies $n(Y) = n(X)+1$. This multiplication is associative because all products and factors of monic mappings are monic, and the same holds for epic mappings with some exceptions which are saved by rule (iii). So the category $\mathscr{B}$ is defined.

Finally, to construct $\mathscr{C}$ from $\mathscr{B}$ we adjoin formally a free sum for each family of objects of $\mathscr{B}$ which does not have exactly one element and does not contain R. (The sum of any family containing R is R.) We call the objects of $\mathscr{B}$ *prime* objects; for a free sum S of primes $\{X_\mu\}$ in C, $S \neq R$, the *prime summands* of S are the X_μ. The mappings from any prime object to a sum are, formally, all the mappings into its prime summands; thus every mapping $X \to S$ factors across a unique coordinate injection. (One easily proves, though we do not need it, that this definition, being stated in formal terms, becomes true in categorical terms; each sum $S \neq R$ determines its summands uniquely.) This completely defines $\mathscr{C}$, for the mappings whose domain is a sum are determined by their coordinates and multiplication reduces to multiplication in $\mathscr{B}$.

$\mathscr{C}$ is the required example. It is immediate that every family of objects has a free sum, that P is projectively finite, and that equalizers do not exist. In fact, every monomorphism into a prime $X > P$ except the mapping from the empty sum is isomorphic. Since every mapping from P to a sum factors across a prime summand X, and every epimorphism from a sum to X has an epimorphic coordinate (X has a largest proper polar subobject), the verification that P is projective need only be done in $\mathscr{B}$, where it is immediate. Similarly, because of the triviality of the monomorphisms, it is immediate that P is a generator.

As for coequalizers, first, they exist in $\mathscr{B}$. For the coequalizers in $\mathscr{A}$ remain coequalizers in $\mathscr{B}$, and the new problems in $\mathscr{B}$ reduce to those given by pairs of mappings $X \to Q_{n\alpha}$ differing by λ, and are solved by the identification mapping (of $\mathscr{A}^0$) to $Q_{n,\alpha+1}$. Next consider two mappings $X \to S$ in $\mathscr{C}$, where X is prime. The problem of coequalizing reduces to a previously solved problem unless the mappings take X to different prime summands of S,

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "e"'] & Y \\ Z & \end{tikzcd}$$

We may suppose $Y \leqslant Z < R$. Five cases remain. We take cases in $\mathscr{A}$ first, and we suppress pointwise multiplications, which require

only pointwise multiplication by a corrective factor. (1st) $Z = P$. Then d is $c_{PP}s^i$, e is $c_{PP}s^j$, say $j = i+h$. The coequalizer θ: $2P \to P$ has coordinates $c_{PP}s^h$ and c_{PP}. (2nd) $Y = P < Z$. Then d is $c_{PP}s^i$, e is $c_{PZ}s^u q_n$. If $i \leqslant u$ there is again a coequalizer $P+Z \to Z$ with second coordinate trivial, c_{ZZ}; but if $i = u+h > u$, we must pass to $W = Q_{3m,0}$ where $m = j(Z)+h$, and map $P+Z$ to W by $c_{PW}q_m$ and $c_{ZW}s^h$. (3rd) $P < Y$; so d is $c_{XY}s^i q_n$, e is $c_{XZ}s^u q_v$. This splits into several subcases. If $i \leqslant u$, and $j(Z) \geqslant j(Y)+u-i$, e is a multiple rd, and $Y+Z \to Z$ by r and c_{ZZ} coequalizes. If $i < u$ and $j(Z)$ is less (by b) than $j(Y)+u-i$, coequalizing requires restriction $c_{ZW}q_m$ where $m = j(Z)+b$. W will be the first object which occurs as the range of such a mapping ($Q_{3m,0}$ or $Q_{3m+2,0}$ depending on $k(Z)$). Y is mapped to W by $c_{WY}s^{u-i}q_t$ ($t = j(Y)$). Similarly, if $i > u$, Y must be mapped by restriction $c_{YW}q_t$ to an object W to which Z can be mapped by shifting $c_{ZW}s^{i-u}$; there is always a first object for which this is possible, and it yields the required coequalizer. (4th, 5th) The 2nd and 3rd cases have variants involving λ. If e is a multiple rd, r and c_{ZZ} are the coordinates of the coequalizer. If this is not the case, λ must be eliminated. Ignore λ and apply the appropriate subcase of the preceding. Either this yields a mapping $Y+Z \to W > Z$, which is then the coequalizer, or it yields a mapping $Y+Z \to Z$, which may be wrong by λ. In the latter case one must apply the identification mapping from $Z = Q_{na}$ to $Q_{n,a+1}$. This concludes all the cases of two mappings $X \to S$ in $\mathscr{C}$ for prime X. If X is a sum of primes, each prime summand of X presents a problem of the preceding type; one requires the simultaneous solution of these problems, and since the prime objects are well ordered, it is clear that the solution exists. This establishes 5.3.

We now make a series of observations about different representations of a complete category $\mathscr{C}$ as a quasi-primitive category of algebras. In 5.2 such a representation is constructed for any projectively finite projective generator P, representing P by a free algebra on one generator.

First observation: 5.2 *gives all representations*. For, any quasi-primitive category of algebras $\mathscr{C}^0$ has a free algebra on one generator P^0, and if one repeats the proof one recovers the ground sets $X^0 \sim \mathrm{Map}\,(P, X)$ and the homomorphisms. One gets the same complete collection of algebraic operations (polynomials), for these correspond exactly to the mappings from P to its finite sums. Of course $\mathscr{C}^0$ may have been presented by a finite set of axioms and primitive operations, and 5.2 will not recover these.

Second observation: these different representations explain why Semadeni's proposed invariant definition of *free object* [10] cannot always agree with customary definitions. As Semadeni has shown, in many cases there is a natural smallest P. (And this holds in a much more general setting than the algebraic one.)

Third observation: I do not know any distinctive properties of (finitely) axiomatizable categories of algebras. A quasi-primitive category of algebras whose primitive operations are all at most n-ary, for some n, has a single left adequate projectively finite projective object (a free algebra on n generators [5]). I have circulated a "proof" of the converse, until E. G. Šulgeifer found an error in it; now I have a counterexample, and I hope to treat the problem further in another place. There is an easy reduction:

5.4. *Every quasi-primitive category of algebra with operations at most n-ary is coextensive with a quasi-primitive category of algebras whose operations are at most binary.*

To prove this, choose for P a free algebra on m generators where $2m \geqslant n$. Call the given (concrete) category $\mathscr{C}^0$; construct another concrete representation $\mathscr{C}^1$ as in 5.2 but using only the operations $a: P \to kP$ for $k = 0, 2$. ($k = 1$ can be omitted but $k = 0$ cannot.) From left adequacy of $2P$ it follows as in 4.2 that we have an isomorphic representation by a full category of algebras. (Mappings $2P \to 2P$ amount to nothing more than pairs of mappings $P \to 2P$.) For a product object X^0 of objects X^0_α, Map $(P, X) = X^1$ is the Cartesian product of the sets Map $(P, X_\alpha) = X^1_\alpha$, and the operations are defined coordinatewise. It remains to identify the subalgebras of Y^1 with the subalgebras of Y^0, i.e. to establish that classes of mappings from a free algebra P to an algebra Y which are closed under operations $a: P \to 0P$ and $a: P \to kP$, where $k > 0$ and the number of free generators of kP is at least n, are just the classes of mappings into subsets of Y which are closed under 0-ary and n-ary operations—by a routine computation.

Fourth observation: of any two projectively finite projective generators in a complete category, each is a retract of a finite sum of copies of the other.

The classification problems suggested by these remarks seem very difficult. The problem of classifying all quasi-primitive categories of algebras probably should not be posed. One can show (using an analogue of the notion of height in abelian groups) that there are more than a universe of quasi-primitive classes of algebras with two unary operations. By the first and fourth observations above, categorical isomorphism cannot reduce this diversity significantly.

From the first observation after 5.3 and Lawvere's characterization [8] of primitive categories of algebras, it follows that *a quasi-primitive category isomorphic with a primitive category is primitive.* It is central to Lawvere's work that a (concrete) primitive category of algebras is determined (up to concrete isomorphism) by the full subcategory of finitely generated free algebras. Lawvere calls this category, or rather a dually isomorphic category, an *algebraic theory*. One arrives at a central classi-

fication problem: which algebraic theories have isomorphic categories of models, i.e. determine isomorphic primitive categories of algebras?

In view of the fourth observation (about retracts), this classification problem can be reformulated in several ways. It might be useful to concentrate on the smallest rather than the largest quasi-primitive category containing the free algebras.

The characterization of algebraic theories is not very difficult. Lawvere does not treat it as a problem, but in effect he has given the solution [8].

We remark in conclusion that if one wants a characterization of quasi-primitive and primitive categories of algebras up to isomorphism rather than up to coextension, essentially one need only adjoin the condition that for each object there is a universe of isomorphic copies. There is an exception for the empty set, and although an empty algebra is usually characterized as an algebra to which no other algebra can be mapped, there is a trivial exception to that too. We omit the details.

6. Zeros. In a category $\mathscr{C}$, an object L is a *left zero* if for every object X there is exactly one mapping from L to X. A *left zero functor* is a covariant set functor assigning to each object a one-point set. Clearly any two left zero objects are isomorphic, and any two left zero functors are naturally equivalent. Right zero objects and functors are defined dually.

A mapping from an object X to a left zero object L, or more generally a natural transformation from a left zero functor F to a principal set functor Map $(X, \mathscr{C})$, distinguishes for each object Y one element of Map (X, Y), the image of $F(Y)$; in fact, F is taken by a natural equivalence onto a subfunctor of Map $(X, \mathscr{C})$, a *covariant zero ideal* of X.

There are only a set of covariant zero ideals of X; for such an ideal I is determined by $I(X) = \{i\}$, every other $I(Y) = \{j\}$ being given by $j = fi$ for any $f\colon X \to Y$. Thus if the covariant zero functor F is proper, here is its conjugate functor F^*. F^* associates to each X a set indexing the covariant zero ideals of X, and to each $f\colon X \to Y$ the function which f induces from the zero ideals of Y to zero ideals of X.

The functor F need not be proper, e.g. for a category having a proper class of objects but no mappings between distinct objects. But when F is improper there is still a natural definition of F^*. For then there are no covariant zero ideals, since such a zero ideal I of X is a left zero functor dominated by X; so we define $F^*(X)$ to be empty for all X.

With this definition, F^* is always a proper contravariant set functor. This is trivial when every $F^*(X)$ is empty. If some $F^*(X)$ has an element I, then X dominates F^*. To see this, consider any $J \in F^*(Y)$. $J(X)$ is a singleton $\{j\}$ in Map (Y, X). We shall show $J = F^*(j)(I)$. As was noted above, for this it suffices to show agreement on Y, $J(Y) = \{ij\}$ where $I(Y) = \{i\}$. Now $J(Y)$ is at any rate some singleton $\{y\}$, and

$ijy = J(ij)(y) \in J(Y)$; so $ijy = y$. Similarly $jy \in J(X)$; so $jy = j$ and $ijy = ij$, completing the proof.

The indicated theorem is

6.1. THEOREM. *A left zero functor is reflexive if it is proper, and in any case it is the conjugate of a proper set functor. If the left zero functor is improper, there exist no covariant zero ideals.*

Proof. It remains to compute the conjugate of F^* as defined above. Let α be a natural transformation from F^* to $\mathrm{Map}(\mathscr{C}, Y)$. We wish to show that α is the evaluation taking each $I \in F^*(X)$ to $i: X \to Y$, where $I(Y) = \{i\}$. We introduce $I(X) = \{e\}$; it will suffice to compute (1) $\alpha(I)e = i$ and (2) $\alpha(I)e = \alpha(I)$.

For (1), $\alpha(I)e = I\big(\alpha(I)\big)(e) \in I(Y)$ does it. For (2), $\alpha(I)e$ is $\alpha\big(F^*(e)(I)\big)$ by naturality of α; but the zero ideal $F^*(e)(I)$ has the value $ee = e$ on X $\big(ee = I(e)(e) \in I(X)\big)$, which means it is I, completing the proof.

7. Inadequacy. Every category has some reflexive set functors (even the empty category). There are always the principal set functors [5] (and the other representable ones), and often the zero functors and their conjugates given by 6.1 and the dual theorem. We call these the *trivial* reflexive set functors. We turn next to some categories all of whose reflexive set functors are trivial.

The group Z_2, considered as a category with one object, has a non-trivial reflexive set functor, which can be described either as the sum or as the product of two principal set functors. We call this the *exceptional reflexive set functor on* Z_2.

Recall that a *principal ideal ring* is a ring in which every left ideal is principal.

7.1. THEOREM. *Let $\mathscr{A}$ be a semigroup with unit, considered as a category with one object. If $\mathscr{A}$ is the multiplicative semigroup of a principal ideal ring which has no proper divisors of zero, then there are no non-trivial reflexive set functors on $\mathscr{A}$. Excepting the exceptional reflexive set functor on Z_2, the same is true if $\mathscr{A}$ is a group.*

Proof. It is somewhat simpler to avoid chasing conjugate functors by considering $\mathscr{A}$ as embedded as an adequate subcategory of some $\mathscr{C}$ and examining $\mathrm{Map}(X, Y)$ and $\mathrm{Map}(Y, X)$, where X is the object of $\mathscr{A}$ and Y any object of $\mathscr{C}$. Let us use the notation $\mathrm{Map}^*(Y, X)$ $(\mathrm{Map}^*(X, Y))$ for the set of natural transformations from $\mathrm{Map}(A, Y)$ to $\mathrm{Map}(A, X)$ $(\mathrm{Map}(Y, A)$ to $\mathrm{Map}(X, A))$; reflexivity, or adequacy, permits us to identify these sets with $\mathrm{Map}(Y, X)$, $\mathrm{Map}(X, Y)$, respectively. Let M denote the given semigroup $\mathrm{Map}(X, X)$; and take first the case that M is a group.

If the set of mappings $\mathrm{Map}(X, Y)$ is empty, then $\mathrm{Map}(Y, X) = \mathrm{Map}^*(Y, X)$ has exactly one element, and Y is the left zero; similarly

if Map (Y, X) is empty then Y is the right zero. Otherwise, we have (a) for $g\colon X \to Y$ and $e \neq f$ in M, $ge \neq gf$; for there is $h\colon Y \to X$, and hg in M satisfies $hge \neq hgf$. Similarly, (b) for $g \neq g'$ in Map (X, Y) and f in M, $gf \neq g'f$; for (by adequacy) there is $h\colon Y \to X$ such that $hg \neq \neq hg'$, which implies $hgf \neq hg'f$. Then Map (X, Y), under the action of M, decomposes into orbits on each of which M is simply transitive. If there is just one orbit, Y is a principal set functor. Suppose g and g' are members of two different orbits in Map (X, Y) and there are three (or more) distinct elements, a, b, c, of M. Choose representatives g_λ of all the other orbits in Map (X, Y). We describe three different transformations α_i in Map$^*(Y, X)$. Observe that to specify each α_i we need only give its values on g, g', and all g_λ, and these values are arbitrary. Then let α_1 have the value a on each of these; let α_2 and α_3 agree with α_1 on g' and all g_λ, but $\alpha_2(g) = b$, $\alpha_3(g) = c$. Since $\mathscr{A}$ is adequate, the α_i correspond to unique mappings in Map(Y, X). Moreover, Map(Y, X) decomposes under the action of M into orbits on each of which M is simply transitive. Clearly $\alpha_1, \alpha_2, \alpha_3$ are in different orbits; choose representatives α_j of all other orbits also. There is a transformation ν in Map$^*(X, Y)$ determined by the conditions $\nu(\alpha_1) = a$, $\nu(\alpha_j) = b$ for $j \neq 1$. But ν is not induced by any member of any of the orbits in Map (X, Y), a contradiction.

If M has only two elements, the case that Map (X, Y) has two orbits is possible, as already noted. But if there are three mappings g_1, g_2, g_3 in different orbits, a slight change in the above argument leads again to a contradiction.

Now suppose M is the multiplicative semigroup of a principal ideal ring R having no divisors of zero (with $\mathscr{A}, \mathscr{C}, X, Y$ as before). There is certainly a zero transformation in Map$^*(Y, X)$, and also in Map$^*(X, Y)$; thus there are at least the corresponding zero mappings in Map (Y, X) and Map (X, Y). If there are no others, Y is a zero functor. (The left and right zero functors are conjugate in this case). Otherwise there are non-zero mappings in both Map (X, Y) and Map (Y, X). As in the previous case we conclude (a) for $g\colon X \to Y$ not zero and $0 \neq e \neq f \neq 0$ in M, $ge \neq gf$; and (b) for $g \neq g'$ in Map (X, Y) and $f \neq 0$ in M, $gf \neq g'f$. Then Map (X, Y) under the action of M decomposes into equivalence classes K_λ on each of which the semigroup of non-zero elements of M acts effectively, together with 0. (Non-zero elements g, g', of $M(X, Y)$ are in the same K_λ provided for some e and f in M, $ge = g'f \neq 0$.) Moreover, any function on Map (X, Y) into M whose restriction to each set $K_\lambda M = K_\lambda \cup \{0\}$ is a natural transformation is itself a natural transformation.

In the subcase that M is finite, the non-zero elements of M form a group, and the argument of the previous case applies with trivial modi-

fication. There is no exception for M having only two elements, $0, 1$: for now if g and g' are in different K_α, transformations α_1 taking g to 0, g' to 1, and α_2 taking g to 1, g' to 0, are unrelated under the action of M. (There is a third α_3 as before, with $\alpha_3(g) = \alpha_3(g') = 1$.)

If M is infinite, there is still only one $K_\lambda = M(X, Y) - \{0\}$. The argument is almost as before, but we give it here. Suppose g_1, g_2 are in K_1, K_2. There exists a natural transformation α_i from Map (X, Y) to M $(i = 1, 2)$ such that $\alpha_i(g_i) \neq 0$; and there exist two elements, e, f, of M different from 0 and 1. Then we define $\beta_1, \beta_2, \beta_3$ in Map$^*(Y, X)$ as follows: all three vanish on all K different from K_1 and K_2 and coincide with α_1 on K_1; on K_2, β_1 coincides with α_2, $\beta_2(x) = e\beta_1(x)$, $\beta_3(x) = f\beta_1(x)$. With $\beta_i(0) = 0$, these are homomorphisms, and may be identified with mappings in Map (Y, X). As before, the β_i lie in three different equivalence classes K_i^* in Map (Y, X); there is a transformation ν in Map* (X, Y) taking β_1 to $\alpha_1(g_1)e$, β_2 and β_3 to $\alpha_1(g_1)$, and ν is not induced by any member of Map (X, Y).

Thus there is only one K_λ; that is, any non-zero g and g' in Map (X, Y) satisfy a relation $ge = g'f$, e and f non-zero members of M. It follows that a mapping h: $Y \to X$ is determined by the value hg, for any fixed non-zero g: $X \to Y$. For if $hg = h'g$ then $hge = h'ge$ for all e; but if $hg'f = h'g'f$ then $hg' = h'g'$, when $f \neq 0$. Now fix g_0, and consider the set I of all elements of M of the form hg_0, $h \in$ Map (Y, X). In the ring R, I is a left ideal, for the functions in Map* (Y, X) are clearly closed under addition and left multiplication by scalars. Then I is the set of all fx_0, f ranging over M, for some x_0 in M. There is h_0 in Map (Y, X) such that $h_0 g_0 = x_0$. Every hg_0 is then fx_0 for some f in M, which is moreover unique, since g_0 is not zero and x_0 is therefore not zero. But the function ν on Map (Y, X) to M which takes each h to this unique f is a natural transformation, since $hg_0 = fx_0$ implies $(eh)g_0 = (ef)x_0$ and $\nu(eh) = e\nu(h)$. The transformation ν is one-one from Map (X, Y) onto M: therefore Y is a principal set functor, as was to be shown.

If one wishes to generalize 7.1 to cancellation semigroups, or to arbitrary rings without zero divisors, there must be at least a longer list of exceptions. Indeed, at least for a commutative ring without zero divisors, every ring ideal yields a reflexive set functor; what happens for principal ideal rings is that all these functors are naturally equivalent.

8. Adequate and measurable. Now we shall complete the treatment of adequacy in categories of vector spaces. First, if a category $\mathscr{C}$ has an adequate subcategory $\mathscr{A}$ which can be isomorphically represented as some class of left vector spaces over a fixed division ring D, then $\mathscr{C}$ also can be so represented, by [5], 2.1. Thus the only question remaining concerns the dimensions of the spaces. (There is also a question

about uniqueness of representation, which we leave as a very easy exercise.) We shall prove, for infinite cardinals $\mathfrak{m}, \mathfrak{n}$:

8.1. THEOREM. *In a full category $\mathscr{C}$ of vector spaces over a division ring, a full subcategory containing one $\mathfrak{m}$-dimensional space is adequate if for every cardinal $\mathfrak{n}$ which is the dimension of a space in $\mathscr{C}$, every $\mathfrak{m}$-additive two-valued measure on the field of all subsets of any set is $\mathfrak{n}$-additive.*

It is not hard to show also that if $\mathscr{C}$ contains an $\mathfrak{n}$-dimensional space, and there exists a two-valued measure which is not $\mathfrak{n}$-additive but is $\mathfrak{m}$-additive for every $\mathfrak{m}$, which is the dimension of a space in the subcategory $\mathscr{A}$, then $\mathscr{A}$ is not right adequate. ($\mathscr{A}$ is left adequate as soon as it includes a twodimensional space [5], 2.2.)

8.1 is true also for $\mathfrak{m} = 2$ (by duality for finite-dimensional vector spaces) and for $\mathfrak{m} = 1$, vacuously, and $\mathfrak{m} = 3, 4, \ldots$, trivially. For $\mathfrak{m} = 0$, $\mathfrak{n} = 1$, it is false. With Theorem 7.1, we have a complete determination.

In fact we shall prove

8.1*. *Theorem* 8.1 *holds as well for free modules over any ring with unit.*

Here we do not have a complete solution of the problem.

Let us concentrate on the case $\mathfrak{m} = \aleph_0$. The cardinals $\mathfrak{n}$ for which every $\aleph_0$-additive measure is $\mathfrak{n}$-additive are called *non-measurable*, and it is known to be consistent with the axioms for set theory that every cardinal is non-measurable. Let E denote a free module on non-measurably many generators, E^* the module of all homomorphisms of E into the ground ring, and E^{**} the module of all homomorphisms of E^* into the ground ring. We call a countable sequence of elements f_n of E^* *point-finite* if, for each $x \in E$, almost all $f_n(x)$ are zero. We call an element φ of E^{**} *bounded* if for every point-finite sequence $\{f_n\}$ in E^*, almost all $\varphi(f_n)$ are zero.

By definition those elements $\hat{x}$ of E^{**} which are defined by evaluation at some $x \in E$ are bounded. Conversely,

8.2. THEOREM. *When the dimension of E is non-measurable, every bounded functional in E^{**} is an evaluation.*

The credit for 8.2 belongs largely to V. L. Klee. I was looking for a counterexample when he communicated a proof for the countable-dimensional case.

Proof of 8.2. First, represent E as the set of all functions with finite support on a set S of power $\mathfrak{n}$. Then E^* may be represented as the set of all functions on S. For subsets T of S, let χ_T denote the characteristic function of T.

Let φ be a bounded functional in E^{**}. Then

(a) There is a finite subset T of S such that for every f in E^* with finite support, $\varphi(f) = \varphi(f\chi_T)$.

Indeed, assuming the contrary, we should have f_1 with finite support such that $\varphi(f_1) \neq 0$, and successively f_{n+1} with finite support $H_{n+1} \supset \supset H_n$, such that $\varphi(f_{n+1}) \neq \varphi(f_{n+1}\chi_{H_n})$. Then the functions $f_{n+1} - f_{n+1}\chi_{H_n}$ form a point-finite sequence and φ vanishes on none of them, a contradiction.

It follows that φ agrees with a certain evaluation $\hat{x}$ at least on the functions with finite support. Then $\varphi - \hat{x}$ is a bounded functional vanishing on functions with finite support. It remains to show that any such functional ψ must vanish identically.

Suppose the contrary, $\psi(f) \neq 0$ for some f in E^*. Consider the values $\mu(T) = \psi(f\chi_T)$ as T varies over the subsets of S. We have

(b) For some $T, \mu(T) \neq 0$, but whenever $T = H \cup K$, with H and K disjoint, either $\mu(H)$ or $\mu(K)$ is zero.

Indeed, assuming the contrary of this, we could construct an infinite sequence of disjoint sets H_n with all $\mu(H_n)$ different from zero; then $\{f\chi_{H_n}\}$ would be a point-finite family on which ψ never vanishes.

Let T be as in (b), and define a measure ν on the subsets of T: $\nu(U) = 1$ if $\mu(U) \neq 0$, $\nu(U) = 0$ otherwise. Then ν is a countably additive two-valued measure vanishing on points. That is, (i) ν vanishes on points, since ψ vanishes on functions with finite support; (ii) ν is finitely additive, since ψ is additive; (iii) if $\{U_n\}$ is a disjoint family and each $\nu(U_n) = 0$, then $\nu(\bigcup U_n) = 0$. For, suppose the contrary. Let f_n be f times the characteristic function of the union of all U_i such that $i > n$; then $\{f_n\}$ is a point-finite sequence but $\psi(f_n) \neq 0$ for all n. This contradicts the assumption that the cardinal of S is non-measurable, and completes the proof of 8.2.

For higher cardinals, we define a *point-finite* subset of E^* as a set of functionals no infinite set of which are non-zero at any one point of E; and an $\mathfrak{m}$-*compatible* element of E^{**} as one which, on any point-finite subset of E^* of power at most $\mathfrak{m}$, agrees with an evaluation. We have

8.2*. *If the dimension of E is $\mathfrak{n}$, and every $\mathfrak{m}$-additive two-valued measure is $\mathfrak{n}$-additive* ($\mathfrak{m} \geqslant \aleph_0$), *then every $\mathfrak{m}$-compatible functional in E^{**} is an evaluation.*

For the proof, use the proof of 8.2 up to the final step (iii); then observe that the products of f with the characteristic functions of all U_n and of their union form a point-finite set of power at most $\mathfrak{m}$.

Proof of 8.1*. We know the free module F on $\mathfrak{m}$ generators is left adequate, and need only establish right adequacy. Let E be free on $\mathfrak{n}$ generators as before; let M be any module. Observe that (choosing a basis in F) the elements of Map (E, F) are just the point-finite $\mathfrak{m}$-tuples of elements of E^*. A natural transformation α from Map (E, F) to Map (M, F) associates to each such $\mathfrak{m}$-tuple an element of Map (M, F). Then for each $m \in M$, the transformation $\alpha(.)(m)$ takes point-finite $\mathfrak{m}$-tuples of ele-

ments of E^* to $\mathfrak{m}$-tuples (almost all zero) of elements of the ground ring. A simple computation using the naturality of a shows that this transformation is just termwise application of an $\mathfrak{m}$-compatible functional, which by 8.2* is evaluation at an element $a(m)$ of E. Further computation shows that the function a: $M \to E$ so defined is a homomorphism inducing a, which completes the proof.

9. Direct sums. The proofs of the following results are routine and are omitted.

Let $\{S_a\}$ be any family of semigroups with 0 and 1, and S their direct product. Then a single object X with the semigroup of endomorphisms S forms an adequate subcategory of a category which contains also objects X_a with semigroups of endomorphisms S_a, each X_a being in effect the reflexive set functor on X determined by the idempotent i_a in S which is projection on S_a. Two different X_a's are connected only by zero mappings. The object X is at once the categorical sum and product of the objects X_a. Reciprocally, the X_a's form an adequate subcategory of this category.

Any reflexive set functors F_a on some or all of the X_a have a sum F which is reflexive on the aggregate: $F(X_a) = F_a(X_a)$, or a one-element set ("zero") if no F_a is given. Every reflexive set functor on X is obtained in this way.

In particular,

9.1. *If the ring R is a finite direct sum of n total rings of endomorphisms of d_i-dimensional vector spaces over division rings D_i (each d_i finite), then the reflexive set functors on a category with one object whose endomorphisms form the multiplicative semigroup of R correspond naturally to the $\mathfrak{n}$-tuples of e_i-dimensional spaces over the D_i subject to the restriction $e_i \leqslant d_i$ if $d_i \leqslant 1$.*

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