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AN n-DIMENSIONAL VERSION OF WAGE'S EXAMPLE

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1. Introduction. Concerning the dimension of the product spaces, Wage [10] has announced the existence of the following example, assuming the Continuum Hypothesis (CH):

Example 1 (CH). There exist two spaces X and Y such that

$$\dim(X \times Y) > \dim X + \dim Y = 0$$

and that $X \times Y$ is perfectly normal and locally compact.

The purpose of this paper is to give the following example under CH, which is the n-dimensional version of the above example, using mainly methods due to Przymusiński [6] and [8] and a recent work of the author [9].

Example 2 (CH). For every $n = 1, 2, \ldots, \infty$ there exist two spaces X and Y such that $\dim X = \dim Y = 0$ and that $X \times Y$ is perfectly normal, first-countable, and locally compact, while $\dim(X \times Y) = \operatorname{Ind}(X \times Y) = n$. Moreover, our space $X \times Y$ in the case $n = \infty$ is not countable dimensional.

Remark 1. The referee has kindly informed us that E. Pol has independently constructed similar examples for the case $n = \infty$.

We use the following notation. (C, ε) denotes the Cantor set with usual Euclidean topology ε and we also use the same letter for the standard metric on C. By Δ we denote the diagonal of C^2. For other undefined terminology we refer to Engelking [2] and Nagami [4].

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2. Constructions of special topologies. For our inductive construction of special topologies we need the following theorem which gives an n-dimensional version of Wage's metric used in [6].

THEOREM 1 ([9]). *For every $n = 1, 2, \ldots, \infty$ there exists a separable completely metrizable space $(C^2, \varrho(n))$ such that*

(i) *$\varrho(n)$ is finer than ε^2 and $\dim(C^2, \varrho(n)) = n$;*

(ii) $\varrho(n)|C^2\setminus\Delta$ *is homeomorphic to* $\varepsilon^2|C^2\setminus\Delta$ *and both* $\varrho(n)|\{t\}\times C$ *and* $\varrho(n)|C\times\{t\}$ *are homeomorphic to* (C, ε);

(iii) *there exists a collection* $\{(A_i, B_i): i = 1, \ldots, n\}$ *of n pairs of disjoint closed sets in* $(\Delta, \varrho(n)|\Delta)$ *such that if for each i a closed set* S_i *separates* A_i *and* B_i, *then the set* $\bigcap_{i=1}^{n} S_i$ *has the cardinality of continuum.*

Remark 2. The existence of the above metric topology on C^2 is one of the key points in our construction. If we could improve the above topology $\varrho(n)$ so that $\varepsilon\times(\varrho(n)|\Delta)$ is finer than $\varrho(n)$, then we can construct the n-dimensional version of the following example (also due to Wage):

Example 3 ([10]). There exist a separable metric space X and a Lindelöf space Y such that

$$\dim X+\dim Y = 0 < 1 = \dim(X\times Y).$$

Now we shall construct topologies $\tau(i)$, $i = 1, 2$, which satisfy the following conditions:

(1) $\tau = \tau(1)\times\tau(2)$ is finer than $\varrho(n)$(1).

(2) $|\mathrm{Cl}_{\varrho(n)}(A)\setminus A|_2 \leqslant \omega$ for any τ-closed set D, where $|D|_2$ denotes the 2-cardinality of D defined by Przymusiński [5].

Let $\{x_\alpha: \alpha < \omega_1\}$ be a well-ordering of C and let

$$X_\alpha = \{x_\beta: \beta < \alpha\}.$$

(Note that we assume CH.) It is easy to see that the family of all countable subsets $\{A_\alpha: \alpha < \omega_1\}$ of C^2 can be well ordered so that $A_\alpha \subset (X_\alpha)^2$.

For each x_α we define two neighbourhood bases $\{N_m(x_\alpha)_i: m < \omega\}$, $i = 1, 2$, so that the following inductive assumptions (3)-(5) are satisfied:

(3) The ε-diameter of $N_m(x_\alpha)_i$ is less than $1/m$, $N_m(x_\alpha)_i$ is a compact set consisting of countable points, and for any $y\in N_m(x_\alpha)_i$ there exists an integer k such that $N_k(y)_i \subset N_m(x_\alpha)_i$.

(4) The $\varrho(n)$-diameter of $N_m(x_\alpha)_1\times N_m(x_\alpha)_2$ is less than $1/m$.

(5) Let $\tau(i, \alpha)$ be the topology on X_α whose local base is

$$\{N_m(x_\beta)_i: \beta < \alpha \text{ and } m < \omega\}.$$

Then for $\beta, \gamma < \alpha$

(a) if $(x_\alpha, x_\alpha)\in\mathrm{Cl}_{\varrho(n)}(A_\gamma)$, then

$$A_\gamma \cap (N_m(x_\alpha)_1\times N_m(x_\alpha)_2) \neq \emptyset \quad \text{for any } m < \omega;$$

(b) if $(x_\alpha, x_\beta)\in\mathrm{Cl}_{\varepsilon\times\tau(2,\alpha)}(A_\gamma)$, then

$$A_\gamma \cap (N_m(x_\alpha)_1\times N_m(x_\beta)_2) \neq \emptyset \quad \text{for any } m < \omega;$$

(1) The author is indebted to the referee for calling his attention to a defect of condition (1).

(c) if $(x_\beta, x_\alpha) \in \mathrm{Cl}_{\tau(1,\alpha)\times\varepsilon}(A_\gamma)$, then

$$A_\gamma \cap (N_m(x_\beta)_1 \times N_m(x_\alpha)_2) \neq \emptyset \quad \text{for any } m < \omega.$$

Suppose that two such neighbourhood bases have been constructed for each x_β such that $\beta < \alpha$. Let R_α be the collection of all A_γ such that $\gamma < \alpha$ and $(x_\alpha, x_\alpha) \in \mathrm{Cl}_{\varrho(n)}(A_\gamma)$. Let

$$S_\alpha = \{S(\beta, t):\ t < \omega,\ \beta < \alpha\}$$

be an enumeration of the collection of all A_γ such that $\gamma < \alpha$, where for every $\beta < \alpha$ the set $\{S(\beta, t):\ t < \omega\}$ is the family of all A_γ satisfying

$$(x_\alpha, x_\beta) \in \mathrm{Cl}_{\varepsilon\times\tau(2,\alpha)}(A_\gamma).$$

Finally, let

$$T_\alpha = \{T(\beta, t):\ t < \omega,\ \beta < \alpha\}$$

be an enumeration of the collection of all A_γ such that $\gamma < \alpha$, where for every $\beta < \alpha$ the set $\{T(\beta, t):\ t < \omega\}$ is the family of all A_γ satisfying

$$(x_\beta, x_\alpha) \in \mathrm{Cl}_{\tau(1,\alpha)\times\varepsilon}(A_\gamma).$$

Since $|R_\alpha \cup S_\alpha \cup T_\alpha| \leqslant \omega$, we can enumerate these families again as

$$R_\alpha = \{A_m:\ m \text{ is even}\}, \qquad S_\alpha = \{B_m:\ m \text{ is odd}\},$$

and

$$T_\alpha = \{C_m:\ m \text{ is odd}\}$$

in such a way that each member of those families is listed ω times. Using the above collection, we choose a sequence $\{(x_m, y_m)\}$ satisfying the following condition:

(6) Let G_m be a $(1/m)$-neighbourhood of (x_α, x_α) with respect to the metric $\varrho(n)$. Then there exist ε-clopen sets H_m, U_m, and V_m such that

(a) $(x_m, y_m) \in G_m$, $(x_\alpha, y_m) \in H_m \times V_m \subset G_m$, and $(x_m, x_\alpha) \in U_m \times H_m \subset G_m$,

(b) $H_m \supset H_{m+1}$ and $H_m \supset U_{m+1}, V_{m+1}$,

(c) $x_m \neq x_k$ and $y_m \neq y_k$ for $m \neq k$,

(d) $(x_{2m}, y_{2m}) \in A_{2m} \cap G_m$, $\quad (x_\alpha, y_{2m+1}) \in B_{2m+1} \cap (H_{2m+1} \times N_{2m+1}(x_\beta)_2)$, and $(x_{2m+1}, x_\alpha) \in C_{2m+1} \cap (N_{2m+1}(x_\beta)_1 \times H_{2m+1})$, where $B_{2m+1} = S(\beta, t)$ and $C_{2m+1} = T(\beta, t)$,

(e) when $R_\alpha \in \emptyset$, $S_\alpha = \emptyset$, or $T_\alpha = \emptyset$, we put $U_m = V_m = \emptyset$, and we do not define x_m and y_m for the corresponding m.

This choice is done in a way parallel to the one in [9], property (16). Since $x_m, y_m < x_\alpha$ by our construction, inductive assumptions can be applied so that there exist two integers $i(m)$ and $j(m)$ such that

$$N_{i(m)}(x_m)_1 \subset U_m, \qquad N_{j(m)}(y_m)_2 \subset V_m,$$

and

$$N_{i(m)}(x_m)_1 \times N_{j(m)}(y_m)_2 \subset G_m.$$

Put

$$N_m(x_\alpha)_1 = \{x_\alpha\} \cup \bigcup \{N_{i(k)}(x_k)_1 \colon k \geqslant m\}$$

and

$$N_m(x_\alpha)_2 = \{x_\alpha\} \cup \bigcup \{N_{j(k)}(y_k)_2 \colon k \geqslant m\}.$$

Then it is easy to see that $\{N_m(x_\beta)_i \colon \beta \leqslant \alpha,\ m < \omega\}$, $i = 1, 2$, satisfy all the conditions (3)-(5). Thus our inductive construction is completed. Let $\tau(i)$ be the topology whose local base is $\{N_m(x)_i\}$. Then it is easy to see that $(C, \tau(i))$ is locally compact, regular, 0-dimensional, and $\tau = \tau(1) \times \tau(2)$ is finer than $\varrho(n)$.

3. Our examples. First we need the following lemma which is essentially due to Kunen (see [8]).

LEMMA. *For every $A \subset C^2$ there exists a $\lambda < \omega_1$ such that if $\alpha \neq \beta$, $\alpha, \beta \geqslant \lambda$, and $(x_\alpha, x_\beta) \in \mathrm{Cl}_{\varepsilon^2}(A)$, then*

$$(x_\alpha, x_\beta) \in \big(\mathrm{Cl}_{\tau(1)\times\varepsilon}(A)\big) \cap \big(\mathrm{Cl}_{\varepsilon\times\tau(2)}(A)\big).$$

Proof. For any $A \subset C$ if $x \in \mathrm{Cl}_\varepsilon(A)$, then $(x, x) \in \mathrm{Cl}_{\varrho(n)}(A^2)$. Therefore, the assertion follows from Lemma 3 in [8].

Let $X = (C, \tau(1))$ and $Y = (C, \tau(2))$. Then we show that X and Y have the required properties described in Example 2.

(I) $\dim X = \dim Y = 0$ *and both X and Y are perfectly normal.*

Indeed, as is readily seen (cf. [3] and [8]), if A is $\tau(i)$-closed, then $|\mathrm{Cl}_\varepsilon(A) \backslash A| \leqslant \omega$ by our Lemma. Therefore (I) holds.

(II) $\dim(X \times Y) = n$ *and $X \times Y$ is perfectly normal.*

These facts are implied by (2) and (I). It follows from the construction of τ and our Lemma that (2) holds (cf. [8] and [9]).

(III) *Our product space $X \times Y$ also satisfies the equality* $\mathrm{Ind}(X \times Y) = n$.

This can be seen in the same fashion as Lemma 5.5 in [9].

(IV) *Our product space $X \times Y$ is not countable dimensional when $n = \infty$.*

This can be seen in the same way as Theorem 1.4 in [9].

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