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Note on the existence of periodic solutions of a second order differential equation

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1. We consider the periodic boundary value problem

$$x''+f(t, x, x') = 0, \tag{1}$$

$$x(a) = x(b), \quad x'(a) = x'(b). \tag{2}$$

We assume that

$$G_1(x_1-x_2, y_1-y_2) \leqslant f(t, x_1, y_1)-f(t, x_2, y_2) \leqslant G_2(x_1-x_2, y_1-y_2), \tag{3}$$

where the functions G_1 and G_2 are defined by the formulae

$$\begin{aligned} G_1(x, y) &= \begin{cases} K_1x+L_1y, & \text{if} \quad x \geqslant 0 \text{ and } y \geqslant 0, \\ K_1x+L_2y, & \text{if} \quad x \geqslant 0 \text{ and } y \leqslant 0, \\ K_2x+L_2y, & \text{if} \quad x \leqslant 0 \text{ and } y \leqslant 0, \\ K_2x+L_1y, & \text{if} \quad x \leqslant 0 \text{ and } y \geqslant 0; \end{cases} \\ G_2(x, y) &= \begin{cases} K_2x+L_2y, & \text{if} \quad x \geqslant 0 \text{ and } y \geqslant 0, \\ K_2x+L_1y, & \text{if} \quad x \geqslant 0 \text{ and } y \leqslant 0, \\ K_1x+L_1y, & \text{if} \quad x \leqslant 0 \text{ and } y \leqslant 0, \\ K_1x+L_2y, & \text{if} \quad x \leqslant 0 \text{ and } y \geqslant 0. \end{cases} \end{aligned} \tag{4}$$

Let $\alpha(L, K)$, $\beta(L, K)$ be the distance between a zero of a non-trivial solution of

$$u''+Lu'+Ku = 0$$

and the next and preceding zeros of u', respectively. $\alpha(L, K)$ and $\beta(L, K)$ may be computed explicitly (see, for instance, [2] or [5]).

The minimum distance between consecutive zeros for any non-trivial solution of

$$u''+G_i(u, u') = 0 \quad (i = 1, 2) \tag{5_i}$$

is (see [2]) $\alpha(L_2, K_2)+\beta(L_1, K_2)$.

Shampine [5] has proved that if f is continuous on $[a, b]\times R^2$ (R denotes the real line) and satisfies (3) there with

$$0 < b-a < 2[\alpha(L_2, K_2)+\beta(L_1, K_2)] \quad \text{and} \quad K_1 > 0, \tag{6}$$

then problem (1) and (2) has at most one solution.

The purpose of the present note is to show how from mentioned Shampine uniqueness theorem and the Lasota-Opial [3] theorem one can obtain the corresponding existence theorem, that is the following

THEOREM. *If f is continuous on $[a, b]\times R^2$ and satisfies* (3) *there with* (6), *then problem* (1), (2) *has exactly one solution.*

This theorem gives, in addition, a better evaluation for $b-a$ than the corresponding Shampine existence result ([5], theorem 6). Recently, however, Bailey and Shampine [1] using quite different method have obtained the same evaluation.

2. We study first the following periodic boundary value condition

$$u(t+p) = u(t)\,, \quad u'(t+p) = u'(t)\,, \quad -\infty < t < +\infty\,. \tag{7}$$

LEMMA. *Suppose $0 < p < 2[\alpha(L_2, K_2)+\beta(L_1, K_2)]$ and $K_1 > 0$. If u such that u'' is locally summable satisfies condition* (7) *and almost everywhere the inequalities*

$$u''+G_1(u, u') \leqslant 0\,, \tag{8}$$

$$u''+G_2(u, u') \geqslant 0\,, \tag{9}$$

then $u(t) \equiv 0$.

Proof. For $u \in C^2$ satisfying (7) and everywhere (8) and (9) two cases are possible: either (i) $x(t_0) = 0$, or (ii) $x(t) \neq 0$ for all t.

Let us consider first case (i). From the Bailey, Shampine, Waltman theorem ([2], theorem 3) if follows that it is possible to apply Peixoto's theorem on differential inequalities ([4], theorem 2) to the condition

$$u(t_0+p) = u(t_0)\,, \quad u'(t_0+p) = u'(t_0) \tag{10}$$

and inequality (8) and (9). But, under our assumptions, by the mentioned Shampine theorem ([5], theorem 3) problem (5_1), (10) as well as (5_2), (10) has only the trivial solution. Hence, finally, $u(t) \equiv 0$.

Considering case (ii) we suppose $u(t) > 0$ (the proof when $u(t) < 0$ is similar). There is a point t_1 such that $u'(t_1) = 0$ and $u''(t_1) \geqslant 0$. From (8) and (4) it follows that $u''(t_1)+K_1u(t_1) \leqslant 0$ what is impossible since $K_1 > 0$. Thus $u(t) \equiv 0$.

Let now u'' be a locally summable function. Suppose u satisfies (7) and almost everywhere (8) and (9). Let r_n: $R\to[0, +\infty)$ be a C^∞ function such that $\operatorname{supp} r_n \subset [-1/n, 1/n]$, and

$$\int_{-1/n}^{1/n} r_n(t)\,dt = 1\,.$$

Then $u_n = u*r_n$ ($*$ denotes the convolution) is a C^∞ function and, by the well-known convolution properties, satisfies condition (7) and everywhere (8) and (9). The previous considerations imply $u_n(t) \equiv 0$. But $u_n \to u$ uniformly. Thus $u(t) \equiv 0$ what completes the proof of lemma.

3. Proof of theorem. Put $z(x, y)$, $g(t, z) = (y, -f(t, x, y))$, $F(z) = \{(y, \bar{y}): -G_2(x, y) \leqslant \bar{y} \leqslant -G_1(x, y)\}$. Problem (1) and (2) has now the form

$$z' = g(t, z)\,, \tag{11}$$

$$z(a) = z(b)\,. \tag{12}$$

Condition (3) may be written as follows

$$g(t, z_1) - g(t, z_2) \in F(z_1 - z_2)\,.$$

From our lemma it follows that the problem

$$z' \in F(z) \quad \text{almost everywhere}$$

and (12) has only the trivial solution. Applying the Lasota-Opial theorem [3] we deduce that there exists exactly one solution of problem (11), (12) what completes the proof of theorem.

References

[1] P. Bailey and L. F. Shampine, *Concerning periodic solutions of $y''+f(t, y, y') = 0$* (to appear).

[2] — and P. Waltman, *The first and second boundary value problems for non-linear second order differential equations*, Journ. Differential Equations 2 (1966), pp. 399-411.

[3] A. Lasota and Z. Opial, *On the existence of solutions of linear problem for ordinary differential equations*, Bull. Acad. Polon. Sci., sér. sci. math. astronom. et phys. 14 (1966), pp. 371-376.

[4] M. M. Peixoto, *Generalized convex functions and second order differential inequalities*, Bull. AMS 55 (1949), pp. 563-572.

[5] L. F. Shampine, *Some non-linear boundary value problems*, Arch. Rat. Mech. Anal. 25 (1967), pp. 123-134.

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