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RECURSIVE FORMULAS FOR AGGREGATE CLAIM DISTRIBUTIONS

In this article we consider the distribution of aggregate claims for the family of claim distributions in which claim values are equispaced and equiprobable. Numbers of claims follow either a Poisson or negative binomial distribution. Probability generating functions and recurrence relations for the probabilities are derived. Interrelations with the class of "Stuttering" Poisson distributions are examined.

1. Introduction. Beekman and Fuelling [2] obtained a number of results for the distribution of aggregate claims for the family of claim distributions in which claim values are equispaced and equiprobable. In particular, they considered the distribution of numbers of claims to be either Poisson or negative binomial and they developed computer programs and provided extensive tabulations for the corresponding distributions of aggregate claims.

In this paper we obtain the probability generating functions (p.g.f.'s) for the distributions considered by [2] and we derive simple recurrence relationships between the probabilities. These recurrences simplify the evaluation of the corresponding probabilities, and therefore lengthy computations and tabulations can be avoided. In addition, it is pointed out that the distribution of aggregate claims where claim values are equispaced and equiprobable and numbers of claims are Poisson distributed has been studied in the literature under a variety of different names.

2. The distribution of aggregate claims. Adopting the notation used in [2], the claim values are represented by the independent random variables $\{X_i\}$ each assuming the values $1, 2, \ldots, a$ with probability $1/a$ and the corresponding p.g.f.

$$G_X(s) = \frac{s\,(1-s^a)}{a\,(1-s)} \quad \text{for } |s| < 1.$$

In addition, let $\{N(t),\ t \geqslant 0\}$ be a non-negative integer valued stochastic process independent of $\{X_i\}$ with p.g.f. $P_{N(t)}(s)$, and assume that

$$S(t) = X_1 + X_2 + \ldots + X_{N(t)}$$

represents the aggregate claims.

It is easy to show that

LEMMA. *The probability generating function of the distribution of aggregate claims is*

$$H_{S(t)}(s) = P_{N(t)}[G_X(s)] = P_{N(t)}\left[\frac{s\,(1-s^a)}{a\,(1-s)}\right].$$

When the number of claims $N(t)$ follows a Poisson distribution, we have

COROLLARY 1. *Assume that $N(t)$ has a Poisson distribution, i.e.,*

$$P[N(t)=n] = e^{-\lambda t}(\lambda t)^n/n!$$

and

$$P_{N(t)}(s) = \exp\{\lambda t(s-1)\} \quad \textit{for } n = 0, 1, 2, \ldots \textit{ and } \lambda > 0.$$

Then

$$H_{S(t)}(s) = \exp\left\{\frac{\lambda t}{a}(s-1)+\frac{\lambda t}{a}(s^2-1)+ \ldots +\frac{\lambda t}{a}(s^a-1)\right\}. \tag{1}$$

The distribution (1) has also appeared in the literature under the name *Poisson distribution of order k* (see, e.g., [4] and the references therein). It is a special case of the class of "Stuttering" Poisson distributions examined, among others, by [1], [7] and [11] with p.g.f.

$$H(s) = \exp\{\lambda_1(s-1)+\lambda_2(s^2-1) + \ldots + \lambda_k(s^k-1)\}.$$

This class is a member of the family of generalized or composed Poisson distributions ([6], p. 194).

Since equation (1) can be looked upon as the convolution of a independent random variables, an alternative representation of $S(t)$ is given by

$$S(t) = X_1(t)+2X_2(t)+ \ldots +aX_a(t),$$

where the $X_i(t)$'s, $i = 1, 2, \ldots, a$, are independent identically distributed Poisson random variables with parameter $\lambda t/a$. Consequently, the probability function of $S(t)$ can be written as

$$P\{S(t)=n\} = \sum \frac{e^{-\lambda t/a}(\lambda t/a)^{x_1+x_2+\ldots+x_a}}{x_1!x_2!\ldots x_a!},$$

where the summation is over all non-negative integers $x_1, \ldots, x_a$ such that

$$x_1+2x_2+ \ldots +ax_a = n.$$

The cumulants k_r of the distribution are given by

$$k_r = \frac{\lambda t}{a}\sum_{m=1}^{a} m^r.$$

Finally, a stochastic derivation of (1) is given by [4] and [11].

When the number of claims $N(t)$ follows a negative binomial distribution, the following corollary is immediate.

COROLLARY 2. *Assume that $N(t)$ has a negative binomial distribution, i.e.,*

$$P\{N(t)=n\}=\frac{\Gamma(n+b)}{n!\,\Gamma(b)}\left(\frac{c}{c+t}\right)^b\left(\frac{t}{c+t}\right)^n,$$

and

$$P_{N(t)}(s)=\frac{c^b}{(c+t-ts)^b}$$

for $c>0,\ b>0$. *Then*

$$H_{S(t)}(s)=c^b\left[(c+t)-\frac{t}{a}\sum_{i=0}^{a-1}s^{i+1}\right]^{-b}. \tag{2}$$

3. Algorithms. Simple recurrences for the probabilities of the distributions (1) and (2) can be derived by utilizing a technique suggested by Kemp [7] and Papageorgiou [10].

For simplicity, let

$$f(j;\,t)=P\{S(t)=j\}.$$

Then, if we differentiate the p.g.f.

$$H_{S(t)}(s)=\sum_{j=0}^{\infty}f(j,\,t)s^j$$

once with respect to s, we obtain

$$\frac{\partial H(s)}{\partial s}=\sum_{j=0}^{\infty}jf(j,\,t)s^{j-1}.$$

For $H_{S(t)}(s)$ given by equation (1) we have

$$\frac{\partial H(s)}{\partial s}=\sum_{i=0}^{a-1}(i+1)\frac{\lambda t}{a}s^i\sum_{j=0}^{\infty}f(j,\,t)s^j$$

or, equivalently,

$$\sum_{j=0}^{\infty}jf(j,\,t)s^{j-1}=\sum_{i=0}^{a-1}\sum_{j=0}^{\infty}(i+1)\frac{\lambda t}{a}f(j,\,t)s^{i+j}. \tag{3}$$

Equating coefficients of s^j in (3) we have

$$(j+1)f(j+1,\,t)=\sum_{i=0}^{\min(a-1,j)}(i+1)\frac{\lambda t}{a}f(j-i,t),$$

which is the desired recurrence relationship and, as usual,

$$f(0,\,t)=H_{S(t)}(0)=\exp\{-\lambda t\}.$$

For $H_{S(t)}(s)$ given by equation (2) we have

$$\frac{\partial H(s)}{\partial s} = \frac{bt}{a} c^b \left[(c+t) - \frac{t}{a} \sum_{i=0}^{a-1} s^{i+1} \right]^{-b-1} \sum_{i=0}^{a-1} (i+1)s^i$$

or

$$\frac{\partial H(s)}{\partial s} \left[(c+t) - \frac{t}{a} \sum_{i=0}^{a-1} s^{i+1} \right] = \frac{bt}{a} H(s) \sum_{i=0}^{a-1} (i+1)s^i.$$

Consequently,

$$(4) \qquad (c+t) \sum_{j=0}^{\infty} j f(j, t) s^{j-1} = \frac{t}{a} \sum_{i=0}^{a-1} \sum_{j=0}^{\infty} [j + b(i+1)] f(j, t) s^{i+j}.$$

Equating coefficients of s^j in (4) we have

$$(j+1) f(j+1, t) = \frac{t}{a(c+t)} \sum_{i=0}^{\min(a-1,j)} [(j-i) + b(i+1)] f(j-i, t)$$

with

$$f(0, t) = [c/(c+t)]^b.$$

4. Applications. A procedure for calculating the aggregate claims distribution in collective risk theory mainly consists of three steps:

(a) the selection of the distribution of the number of claims $N(t)$,

(b) the selection of the distribution of individual claims X_i, which usually are assumed to have the same distribution,

(c) the computation of the distribution of the aggregate claims

$$S(t) = X_1 + X_2 + \ldots + X_{N(t)}.$$

The traditional n-fold convolution method leads to series calculations which are very difficult to compute. Some authors (see, e.g., [2]) proposed extensive tabulations for various combinations of the most commonly used distributions. However, simple recurrence relations are efficient computational tools and their usefulness in actuarial studies has been discussed by a number of authors (related references are given in [9] and [10]).

Another possible area of applications of the distributions discussed in this paper is in reliability studies. Reliability models connected with discrete distributions of order k were discussed, among others, by [3], [5], and [8], and their relevance to some telecommunications and oil pipeline systems has been indicated.

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