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TWO PROPERTIES OF FREE BOOLEAN ALGEBRAS

BY

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In this paper, based on the most elementary properties of free Boolean algebras, and using a most direct algebraic approach, we prove that every simply ordered chain (see Theorem 1), as well as every set of pairwise disjoint elements of a free Boolean algebra (see Theorem 2), is countable.

Theorem 1 appears in [3] and our proof, although simpler, is along the similar lines.

Theorem 2 appears, say, in [2], p. 57, however, our proof is independent of the measure-theoretic considerations which are usually used in the proof of that theorem.

In what follows, we let $(A, +, \cdot)$ denote a free Boolean algebra generated by a (not necessarily denumerable) set $\{g_1, g_2, \ldots\}$ of free generators g_i. Moreover, in what follows a non-zero product of finitely many g_i's or complements g_i' of g_i's is called a *monomial* of A. Thus, each g_1 or $g_1'g_2$ or $g_2'g_3'g_5$ is a monomial. Furthermore, we say, for example, that monomial $g_2'g_3'g_5$ *involves generators* g_2, g_3, g_5.

Since every non-zero element of A is a sum of finitely many monomials of A, throughout this paper, for every non-zero element x of A, we choose a *fixed disjunctive normal form* $D(x)$ (cf. [1], p. 61). For instance, for an element x of A given by

$$x = g_1+g_2 = g_1g_2+g_1g_2'+g_2 = g_1+g_2g_9+g_2g_9',$$

we may choose

$$D(x) = g_1g_2'+g_1'g_2$$

as the fixed disjunctive normal form. We recall that the monomials m_i appearing in a disjunctive normal form are *pairwise disjoint*, i.e., $m_im_j = 0$ for $i \neq j$. We observe that in A an element x may have many disjunctive normal forms. However, for our purpose, it is immaterial which disjunctive normal form is chosen for x, as long as $D(x)$ is kept fixed once and for all.

We observe also that in a finite free Boolean algebra F every non-zero element has a unique disjunctive normal form whose monomials are atoms of F.

By the *rank* of a disjunctive normal form we mean the number of its monomials, and by the *order* of a disjunctive normal from we mean the number of the factors in each monomial. Clearly, rank as well as order are always finite natural numbers.

LEMMA 1. *Let x and y be elements of the free Boolean algebra $(A, +, \cdot)$ such that $D(x)$ and $D(y)$ have the same rank and the same order. If $x \leqslant y$ (i.e. $xy = x$), then $x = y$.*

Proof. Let $D(x)$ involve the finite set $\{\ldots, g_i, \ldots\}$ of generators and $D(y)$ involve the finite set $\{\ldots, g_j, \ldots\}$ of generators. But then it is easily seen that, in view of the hypotheses of Lemma 1, in the finite free Boolean algebra F generated by $\{\ldots, g_i, \ldots\} \cup \{\ldots, g_j, \ldots\}$, the unique disjunctive normal forms of x and y have the same rank. Thus, x and y are sums of the same number of atoms of F. But then $x \leqslant y$ implies $x = y$, as desired.

THEOREM 1. *Every simply ordered chain in a free Boolean algebra is countable.*

Proof. Assume, on the contrary, that there exists an uncountable simply ordered chain $\ldots < x < y < z < \ldots$ of distinct elements $\ldots, x, y, z, \ldots$ of the free Boolean algebra $(A, +, \cdot)$. Since the set of all ranks as well as the set of all orders is denumerable, we see that in the uncountable above-given chain there must exist distinct elements u and v such that $D(u)$ and $D(v)$ have the same rank and the same order. But then, by Lemma 1, we have $u = v$ which contradicts $u \neq v$. Thus, our assumption is false and the proof of the theorem is complete.

LEMMA 2. *Let $M = \{m_i \mid i \in I\}$ be a set of pairwise disjoint monomials m_i of the free Boolean algebra $(A, +, \cdot)$ such that* $\operatorname{order}(m_i) = n$ *for every $i \in I$. Then M is a finite set.*

Proof. We use induction on n. Clearly, any set of monomials of order 0 of A is an empty set and, therefore, Lemma 2 is valid for $n = 0$.

Now, let us assume that every set of pairwise disjoint monomials of A each of order $n-1$ is finite. Let us consider the set M and let $m_k \in M$. Without loss of generality, let m_k involve generators $g_1, \ldots, g_n$. Since $m_i m_j = 0$ for every $i \neq j$, we see that in every element of M there must occur g_v or g_v' for some $v \leqslant n$. Let $M(g_v)$ be the set of all elements of M in which g_v occurs and let $M(g_v')$ be the set of all elements of M in which g_v' occurs. Clearly,

$$M = M(g_1) \cup M(g_1') \cup \ldots \cup M(g_n) \cup M(g_n'), \tag{1}$$

where the summands are not necessarily non-overlapping.

Let us observe that $M(g_i)$ as well as $M(g_i')$ for every $i \leqslant n$ is a set of pairwise disjoint monomials of A. Since g_i occurs in every element of $M(g_i)$, we see that, by discarding g_i, we may identify $M(g_i)$ with a set of pairwise disjoint monomials of A each of order $n-1$. Hence, by the induction hypothesis, $M(g_i)$ is a finite set for every $i \leqslant n$. Analogous reasoning shows that $M(g_i')$ is also a finite set for every $i \leqslant n$. But then from (1) it follows that M is a finite set, as desired.

THEOREM 2. *Every set of pairwise disjoint elements of a free Boolean algebra is countable.*

Proof. Let $H = \{x_i \mid i \in I\}$ be a set of pairwise disjoint elements of the free Boolean algebra $(A, +, \cdot)$. We prove that H is a countable set by showing that the set M_ω of all monomials occurring in $D(x_i)$ with $i \in I$ is a countable set.

Let M_n be the set of all monomials of order n occurring in $D(x_i)$ with $i \in I$. Since H is a set of pairwise disjoint elements, for every $n \in \omega$, we see that M_n is also a set of pairwise disjoint monomials of A each of order n. Thus, from Lemma 2 it follows that M_n is a finite set for every $n \in \omega$. Consequently,

$$M_\omega = \bigcup_{n \in \omega} M_n$$

is a countable set, as desired.

REFERENCES

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