COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

ANALYTIC SEMIGROUPS IN BANACH ALGEBRAS AND A THEOREM OF HILLE

BY

TADEUSZ PYTLIK (WROCŁAW)

DEDICATED TO PROFESSOR STANISŁAW HARTMAN WHO ENCOURAGED AND INSPIRED ME

1. Introduction. Let $\mathscr{A}$ be a Banach algebra. By an *analytic semigroup* in $\mathscr{A}$ we mean a function $t \to y_t$ from the right open half plane $H = \{t \in C\colon \operatorname{Re} t > 0\}$ into $\mathscr{A}$ which is analytic and $y_s y_t = y_{s+t}$ for all $s, t \in H$.

The aim of this note is to prove the following theorem:

THEOREM 1. *Assume that an element y in a Banach algebra satisfies*

$$\|u(ny)\| = O(|n|^r) \quad as\ |n| \to \infty \tag{1}$$

where

$$u(y) = \sum_{m=1}^{\infty} \frac{i^m}{m!} y^m.$$

Then there exists an analytic semigroup $(y_t)_{\operatorname{Re} t > 0}$ in $\mathscr{A}$ over H, such that

(i) $\sup\limits_{\operatorname{Re} t \geq 1} |t|^{-k} \|y_t\| < +\infty$ *for some* $k = k(r)$;

(ii) $\lim\limits_{\substack{t \to 0 \\ t > 0}} \|y_t y^k - y^k\| = 0$.

On the other hand, in a particular case when the algebra is radical, J. Esterle ([3], Theorem 2.1) has proved the following:

THEOREM 2. *Let $\mathscr{R}$ be a radical Banach algebra and let $(y_t)_{\operatorname{Re} t > 0}$ be an analytic semigroup in $\mathscr{R}$ over H. If*

$$\sup_{\operatorname{Re} t \geq 1} e^{-|t|^\alpha} \|y_t\| < +\infty$$

for some $\alpha < 1$, then the semigroup $(y_t)_{\operatorname{Re} t > 0}$ is the zero one.

These theorems together imply that in a radical Banach algebra any element y satisfying (1) has to be nilpotent. Since the converse implication is also true, we get a characterization of nilpotent elements in radical Banach algebras. Some corollaries of this type, all closely related to a theorem of E. Hille (cf. Corollary 3) are discussed in Section 3.

2. **Proof of Theorem 1.** Let $\mathscr{A}_1$ be the Banach algebra obtained from $\mathscr{A}$ by adjoining an identity in the standard manner. Then by (1) we have

$$\|e^{i\lambda y}\|_{\mathscr{A}_1} = O(|\lambda|^r) \quad \text{as } |\lambda| \to \infty,\ \lambda \in \boldsymbol{R}. \tag{2}$$

Let $C^{r+1}(\boldsymbol{R})$ denote the space of all functions from $\boldsymbol{R}$ into $\boldsymbol{C}$ with continuous and bounded derivatives of all orders up to $r+1$, equipped with a norm

$$\|F\|_{C^{r+1}} = \max_{0 \leqslant m \leqslant r+1} \sup_{\lambda \in \boldsymbol{R}} \left| \frac{d^m}{d\lambda^m} F(\lambda) \right|.$$

It is classical that if a function F is in $C^{r+1}(\boldsymbol{R})$ and has compact support, then $\int_{-\infty}^{+\infty} |\hat{F}(\lambda)| \, |\lambda|^r d\lambda < +\infty$. Thus by (2) the integral $\int_{-\infty}^{+\infty} \hat{F}(\lambda) e^{i\lambda y} d\lambda$ is absolutely convergent and so it defines an element $F(y)$ in $\mathscr{A}_1$. We have

$$\|F(y)\|_{\mathscr{A}_1} \leqslant C \|F\|_{C^{r+1}}, \tag{3}$$

where the constant C depends only on y and on the measure of $\operatorname{supp} F$. Also

$$(F_1 \cdot F_2)(y) = F_1(y) F_2(y). \tag{4}$$

Now let F be a function from $\boldsymbol{R}$ into $[0, 1]$ which is $r+1$ times continuously differentiable and satisfies

(a) $F(0) = 0$ and $F(\lambda) > 0$ for $\lambda \neq 0$.

(b) $F(\lambda) = e^{-1/|\lambda|}$ in a neighbourhood of 0.

(c) $F(\lambda) = 1$ for $|\lambda| \geqslant 1$.

For any $t \in H$ the function $1 - F^t$ belongs to $C^{r+1}(\boldsymbol{R})$ and has support in $[-1, 1]$. Thus it operates on y in $\mathscr{A}_1$. We want to prove that the family $(y_t)_{\mathrm{Re}\, t > 0}$ defined by

$$y_t = 1 - (1 - F^t)(y)$$

has all the properties of the theorem. First observe that in fact y_t is in $\mathscr{A}$. Indeed, if φ denotes the homomorphism from $\mathscr{A}_1$ into $\boldsymbol{C}$ defined by $\varphi(x + \lambda) = \lambda$, then $\varphi(y_t) = 1 - (1 - F^t)(0) = 0$ and so $y_t \in \ker \varphi = \mathscr{A}$.

It is clear that the function $t \to y_t$ is analytic in H. Also the equality $y_s y_t = y_{s+t}$, $s, t \in H$, is an immediate consequence of (4). To prove (i) and (ii) we will use (3). Property (b) of the function F guarantees that

$$\left| \frac{d^m}{d\lambda^m} F(\lambda) \right| = O\left(|\lambda|^{-2m} F(\lambda)\right) \quad \text{as } |\lambda| \to 0,$$

and since $F(\lambda)$ is far from zero outside of any small neighbourhood of 0 (property (a)), together with (c) it gives

$$\left|\frac{d^m}{d\lambda^m}F(\lambda)\right| \leqslant \mathrm{const}\cdot|\lambda|^{-2m}F(\lambda) \tag{5}$$

for all $\lambda \neq 0$ and all $m = 1, 2, \ldots, r+1$. Now a routine computation and inequality (5) give

$$\left|\frac{d^m}{d\lambda^m}(1-F^t)(\lambda)\right| \leqslant \mathrm{const}\cdot\max(|t|, |t|^m)\cdot|\lambda|^{-2m}[F(\lambda)]^{\mathrm{Re}\,t}. \tag{6}$$

In particular

$$|\lambda|^{-2m}[F(\lambda)]^{\mathrm{Re}\,t} \leqslant |\lambda|^{-2m}F(\lambda) \leqslant \mathrm{const}$$

when $\mathrm{Re}\,t \geqslant 1$. Thus

$$\|1-F^t\|_{C^{r+1}} \leqslant \mathrm{const}\cdot|t|^{r+1}, \qquad \mathrm{Re}\,t \geqslant 1,$$

and (i) follows.

Finally

$$\|(y_t-1)\,y^{2r+2}\|_{\mathscr{A}_1} = \|F_t(y)\|_{\mathscr{A}_1},$$

where F_t is a function defined by

$$F_t(\lambda) = \big(1-F^t(\lambda)\big)\lambda^{2r+2}.$$

If $t \in (0, 1)$, then from (6)

$$\left|\frac{d^m}{d\lambda^m}(1-F^t)(\lambda)\right| \leqslant \mathrm{const}\cdot|t|\,|\lambda|^{-2m}.$$

Therefore $F_t \to 0$ in $C^{r+1}(\boldsymbol{R})$ when $t \to 0$ and so (ii) follows.

3. Applications. A theorem of Hille. Let $\mathscr{A}$ be a Banach algebra with a unit and let x be an arbitrary element in $\mathscr{A}$. If λ_0 is an isolated point of the spectrum $\sigma(x)$ of x, then at λ_0 the resolvent function $R(\lambda) = (x-\lambda)^{-1}$ has a Laurent expansion

$$R(\lambda) = \sum_{-\infty}^{+\infty} (\lambda-\lambda_0)^n a_n, \tag{7}$$

where

$$a_n = \frac{1}{2\pi i}\int_C (\lambda-\lambda_0)^{-n-1} R(\lambda)\,d\lambda$$

and C is a small circle about λ_0. The series (7) is absolutely convergent for λ close enough to λ_0 and by the properties of the resolvent function, it follows that a_{-1} is a non-zero idempotent in $\mathscr{A}$ and

$$a_{-n} = (x-\lambda_0)^{n-1} a_{-1} \tag{8}$$

for $n = 1, 2, \ldots$ If $a_n = 0$ for some $n < 0$, then by (8) also $a_m = 0$ for all $m < n$. In this case λ_0 is called a *pole* of x and the biggest integer n with $a_{-n} \neq 0$ is called the *order* of the pole.

COROLLARY 1. *Let $\mathscr{A}$ be a Banach algebra with unit and x an invertible element in $\mathscr{A}$. If*

$$\|x^n\| = O(|n|^r) \quad \text{as } |n| \to \infty \tag{9}$$

holds for some r, then any isolated point of the spectrum of x is a pole and the order of the pole is not greater than r.

Proof. Assume that (9) holds and that λ_0 is an isolated point in $\sigma(x)$. Let

$$R(\lambda) = \sum_{-\infty}^{+\infty} (\lambda-\lambda_0)^n a_n$$

be the Laurent expansion of $R(\lambda)$ in a neighbourhood of λ_0. We shall prove that $a_k = 0$ for some negative k.

Put

$$y = i\sum_{k=1}^{\infty} \frac{(-\lambda_0)^{-k}}{k} a_{-k-1} \tag{10}$$

(this series is absolutely convergent since $\lim_{n\to\infty} \|a_{-n}\|^{1/n} = 0$) and denote by $\mathscr{A}_0$ the closed subalgebra (without unit) in $\mathscr{A}$, generated by y.

Recall that

$$a_{-k-1} = \frac{1}{2\pi i}\int_C (\lambda-\lambda_0)^k R(\lambda)\,d\lambda, \qquad k = 1, 2, \ldots$$

Thus we may write (10) in the form

$$y = \frac{1}{2\pi}\int_C (\ln\lambda_0 - \ln\lambda) R(\lambda)\,d\lambda.$$

From this we get easily $\sigma(y) = \{0\}$ and so that $\mathscr{A}_0$ is a radical Banach algebra. Moreover for any integer n

$$u(ny) = \frac{1}{2\pi i}\int_C [e^{-n(\ln\lambda_0 - \ln\lambda)} - 1] R(\lambda)\,d\lambda$$
$$= (\lambda_0^{-n} x^n - 1) a_{-1}.$$

Comparing this last formula with (9) (note that $|\lambda_0| = 1$, which also is a consequence of (9)) we get

$$\|u(ny)\| = O(|n|^r) \quad \text{as } |n| \to \infty.$$

Therefore y satisfies the assumption of Theorem 1 and there exists an analytic semigroup $(y_t)_{\mathrm{Re}\,t>0}$ in $\mathscr{A}_0$ satisfying (i) and (ii). But $\mathscr{A}_0$ is a radical Banach algebra, thus $y_t \equiv 0$ by Theorem 2. Theorem 1 used again implies now that y is a nilpotent element, so the element

$$(\lambda_0^{-1}x-1)a_{-1} = u(y) = y\left(\sum_{m=1}^{\infty}\frac{i^m}{m!}y^{m-1}\right)$$

is also nilpotent and both have the same order. But then

$$[(\lambda_0^{-1}x-1)a_{-1}]^k = \lambda_0^{-k}a_{-k-1} = 0$$

for some k, which means that λ_0 is a pole of x.

It remains to prove that the order of λ_0 is not greater than r. Let k be the order of λ_0, so that $a_{-k-1} = 0$ but $a_{-k} \neq 0$. Then by (8)

$$x^n a_{-1} = \sum_{j=1}^{k}\binom{n}{j}\lambda_0^{n-j}a_{-j-1}$$

for any integer $n \geqslant k$. So

$$\lim_{n\to\infty} n^{-k}\|x^n a_{-1}\| = \frac{1}{k!}\|a_{-k}\| > 0$$

and so

$$\liminf_{n\to\infty} n^{-k}\|x^n\| > 0,$$

which together with (9) implies that $k \leqslant r$. This completes the proof.

Property (9) clearly implies that the spectrum $\sigma(x)$ of x is contained in the unit circle $\{\lambda \in \boldsymbol{C}: |\lambda| = 1\}$. In a particular case, when it consists only of isolated points, we have the following:

COROLLARY 2. *Let x be an element in a Banach algebra with unit. If the spectrum $\sigma(x)$ is finite and lies in the unit circle, then the following conditions are equivalent:*

(i) $\|x^n\| = O(|n|^r)$ *as* $|n| \to \infty$.

(ii) *Any point in the spectrum $\sigma(x)$ is a pole.*

(iii) *There is a polynomial P such that $P(x) = 0$.*

Proof. The implication (i) $\Rightarrow$ (ii) is just the Corollary 1. The equivalence (ii) $\Leftrightarrow$ (iii) follows from the Minimal Equation Theorem [2], VII.3.16. To get the implication (ii) $\Rightarrow$ (i) assume that the spectrum of x consists of singular points $\lambda_1, \lambda_2, \ldots, \lambda_k$ having orders $\nu_1, \nu_2, \ldots, \nu_k$

respectively. Let

$$e_j = \frac{1}{2\pi i} \int_{C_j} R(\lambda)\, d\lambda, \qquad j = 1, 2, \ldots, k,$$

where C_j is a small circle about λ_j. Then by [2], VII, Theorem 2.2, we have

$$x^n = \sum_{j=1}^{k} \sum_{m=0}^{j} \binom{n}{m} \lambda_j^{n-m} (x-\lambda_j)^m e_j$$

(for negative n the symbol $\binom{n}{m}$ is defined by $\dfrac{n(n-1)\ldots(n-m+1)}{m!}$ and $\binom{n}{0} = 1$). Since $|\lambda_j| = 1$, $j = 1, 2, \ldots, k$, we get

$$\|x^n\| \leqslant C \cdot \max_{0 \leqslant m \leqslant r} \left|\binom{n}{m}\right|$$

with $r = \max\{v_j\colon j = 1, 2, \ldots, k\}$ and $C = \max\{\|(x-\lambda_j)^m e_j\|\colon j = 1, 2, \ldots, k;$ $m = 0, 1, 2, \ldots, v_j\}$. This gives (i).

We finish this section with a result which was originally proved by E. Hille [4] and later simplified by M. H. Stone [8] (see also [5], Theorem 4.10.1).

COROLLARY 3 (Theorem of Hille). *Let $\mathscr{A}$ be a Banach algebra with unit. If x is a quasi-nilpotent element in $\mathscr{A}$, then a necessary and sufficient condition that $x^{r+1} = 0$ for some $r \geqslant 0$ is that $\|(1+x)^n\| = O(|n|^r)$ as $|n| \to \infty$.*

4. Remarks. 1. It is of importance in Corollary 3 (and so in the other corollaries) to have an estimate from above of the growth of $\|(1+x)^n\|$ for both positive and negative powers. For example the operator T which acts in $L^2(0, 1)$ by

$$Tf(t) = \int_0^t f(s)\, ds$$

is not nilpotent although $\sigma(T) = 0$ and $\|(1-T)^n\| = O(n)$ as $n \to \infty$. Indeed, we have

$$T^n f(t) = \int_0^t \frac{s^{n-1}}{(n-1)!} f(t-s)\, ds, \qquad n = 1, 2, \ldots,$$

thus

$$T(1-T)^n f(t) = \int_0^t L_n(s) f(t-s)\, ds,$$

where L_n is the n-th Laguerre polynomial. Now by [9], 7.21.3,

$$e^{-s/2}|L_n(s)| \leqslant 1,$$

thus $\|(1-T)^{n+1}-(1-T)^n\| \leqslant \int_0^1 e^{s/2}\,ds \leqslant 2$ and so $\|(1-T)^n\| \leqslant 2n$.

In fact, a more strict estimate for $\|(1-T)^n\|$ holds. Namely, $\|(1-T)^n\| = O(n^{1/4})$ as $n \to \infty$. Indeed,

$$(1-T)^{n+1} f(t) = f(t) - \int_0^t L_n^{(1)}(s) f(t-s)\,ds$$

with $L_n^{(1)}(s) = -\frac{d}{ds}L_n(s)$, and one may use the estimate

$$\max_{0 \leqslant s \leqslant 1} e^{-s/2} s^{3/4} |L_n^{(1)}(s)| = O(n^{1/4})$$

(cf. [9], Theorem 7.61) instead of 7.21.3.

2. It follows directly from Theorems 1 and 2 that Corollary 3 remains true with an estimate for $u(nx)$ instead for $(1+x)^n$. This suggests some possible applications of Corollary 3. For example J. Dixmier ([1], Lemme 6) proved that if G is a group of polynomial growth then in the group algebra $L^1(G)$ any continuous and hermitian function with compact support satisfies (1) with $r = 1 + \text{growth}\, G$. Choose a net $(f_i)_{i\in I}$ of such functions with the property that $(f_i^{r+1})_{i\in I}$ form an approximate identity in $L^1(G)$. Of course $\sigma(f_i) \neq \{0\}$, but if φ is a continuous homomorphism from $L^1(G)$ into any radical Banach algebra, then we may apply Corollary 3 to the algebra $\varphi(L^1(G))$. We get then $\varphi(f_i^{r+1}) = [\varphi(f_i)]^{r+1} = 0$ and so $\varphi(L^1(G)) = \{0\}$. This shows that the group G has a weak Wiener property in the sense of [7]. This has been known since [6].

Acknowledgment. I would thank Jean Esterle who showed me the example in Remark 1.

REFERENCES

[1] J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Publ. Math. Inst. Hautes Études Sci. 6 (1960), p. 13–25.

[2] N. Dunford and J. T. Schwartz, *Linear Operators*, Part 1, Interscience Publishers, New York 1957.

[3] J. Esterle, *A complex-variable proof of the Wiener tauberian theorem*, Annales de l'Institut Fourier, Grenoble 30 (1980), p. 91–96.

[4] E. Hille, *On the theory of characters of groups and semi-groups in normed vector rings*, Proceedings of the National Academy of Sciences of the United States of America 30 (1944), p. 58–60.

[5] – and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, Volume 31, 1957.

[6] A. Hulanicki, J. W. Jenkins, H. Leptin and T. Pytlik, *Remarks on Wiener's tauberian theorem for groups with polynomial growth*, Colloquium Mathematicum 35 (1976), p. 293–304.

[7] H. Leptin, *Ideal theory in group algebras of locally compact groups*, Inventiones Mathematicae 31 (1976), p. 259–278.

[8] M. H. Stone, *On a theorem of Polyà*, Journal of Indian Mathematical Society 12 (1948), p. 1–7.

[9] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Volume 23, 1939.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
WROCŁAW, POLAND

Reçu par la Rédaction le 10. 05. 1984