COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 1

ON SOME CLASSES OF DISTRIBUTIONS

BY

SZYMON SZNAJDER (WROCŁAW)

Introduction. In this paper we consider the spaces H_r and K_r, $r > 0$, (for the definition see below) which, for $r = \infty$, become the spaces studied by Hasumi [1]. The general idea of the proof of our theorem 1 is the same as in [1]. In the sequel a characterisation of the functionals over the spaces H_r and K_r is given and in particular the periodic functionals over K_r are studied. We consider here only the case of one variable; the generalisation to several variables is obvious. The author would like to express his sincere thanks to Dr Z. Zieleźny for suggesting the problem and for helpful discussions as well as to Prof. C. Ryll-Nardzewski for valuable indications.

1. Let H_r, where $r > 0$, denote the space of all complex-valued C^∞-functions f on R such that, for any $k, p = 0, 1, 2, \ldots$,

$$e^{r_k|x|}|D^p f(x)| < \infty,$$

where $0 < r_k < r_{k+1} < \ldots$, $r_k \to r$, and D^p stands for d^p/dx^p. The topology in H_r is defined by the seminorms

$$q_j(f) = \sup_{x \in R} \big(e^{r_j|x|}|D^j f(x)|\big).$$

K_r, with $r > 0$, will denote the space of all complex-valued functions $\varphi(z)$, which are analytic in the strip $V_r = \{z\colon |Imz| < r\}$ and which are rapidly decreasing in any narrower strip V_{r_k} ($\{r_k\}$ is the same as before H_r) i.e., for any $k = 0, 1, \ldots$ and for any derivative $\varphi^{(n)}(z)$ there is

$$(1+|z|^2)^m|\varphi^{(n)}(z)| \to 0 \text{ if } |z| \to \infty, \quad z \in V_{r_k}.$$

The topology in the space K_r is defined by the seminorms

$$p_j(\varphi) = \sup_{z \in V_{r_j}} |z^j \varphi(z)|.$$

Note that seminorms p_j are equivalent to the seminorms

$$p_j' = \sup_{z \in V_{r_j}} |z^j D^j \varphi(z)|.$$

THEOREM 1. *The Fourier transform provides a topological isomorphism of the spaces H_r and K_r.*

Proof. If $f \in H_r$ and if $\varphi(z)$ is its Fourier transform

$$\varphi(z) = \hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-izx} f(x)\,dx,$$

then

$$z^m \varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-i)^m e^{-izx} D^m f(x)\,dx.$$

Thus, for $z \in V_{r_k}$, we have

$$|z^m \varphi(z)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{r_k|x|} e^{-r_l|x|} q_l(f)\,dx \leqslant C q_l(f)$$

with $l > \max(k, m)$.

Therefore $\varphi(z)$ is an element of K_r and the transformation $H_r \to K_r$ is continuous.

Let now $\varphi(z) \in K_r$ and let

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixu} \varphi(u)\,du$$

Using Cauchy's integral theorem, we can write, for any real v such that $v| < r$,

$$D^p f(x) = \frac{1}{\sqrt{2\pi}} D^p \Big[\int_{-\infty}^{+\infty} e^{ix(u+iv)} \varphi(u+iv)\,du \Big]$$
$$= e^{-vx} \sum_{0 \leqslant q \leqslant p} c_q \int_{-\infty}^{+\infty} e^{ixu} u^q \varphi(u+iv)\,du.$$

Therefore $D^p f(x)$ is equal to e^{-vx} multiplied by a bounded and coninuous function of x, so $f \in H_r$. The desired continuity follows from the nequality

$$\Big| \int_{-\infty}^{+\infty} e^{ixu} u^q \varphi(u+ir_k)\,du \Big| \leqslant \int_{-\infty}^{+\infty} |u+ir_k|^{-2} |u+ir_k|^{q+2} |\varphi(u+ir_k)|\,du \leqslant C p_{k+2}(\varphi)$$

which is valid for any k and $q \leqslant k$.

2. Let H_r' be the dual space of H_r. Since the space $\mathcal{D}$ is dense in H_r, H_r' is a subspace of the space $\mathcal{D}'$ of distributions of L. Schwartz. The characterisation of elements of H_r' is given in the following

THEOREM 2. *A distribution S belongs to H'_r if and only if there exist an integer k, a positive number ε, and a continuous bounded function $F(x)$ such that*

$$S = D^k[e^{(r-\varepsilon)|x|}F(x)].$$

Proof. Let S be of the desired form. Then the value of S on $f \in \mathcal{D}$,

$$\langle S, f\rangle = (-1)^k \int_{-\infty}^{+\infty} e^{(r-\varepsilon)|x|}F(x)D^k f(x)\,dx,$$

may be estimated as follows:

$$|\langle S, f\rangle| \leqslant \int_{-\infty}^{+\infty} |F(x)|\,e^{(r-\varepsilon)|x|}e^{-r_l|x|}q_l(f)\,dx \leqslant Cq_l(f),$$

where $r_l > r-\varepsilon$ and C is independent of f. Hence, by the continuity argument the formula for $\langle S, f\rangle$ holds for each $f \in H_r$. Let now $S \in H'_r$ be a functional continuous with respect to a seminorm q_j. If $f \in H_r$, then the function

$$f_1(x) = e^{r_j|x|}D^j f(x)$$

is continuous and vanishes when $|x| \to \infty$.

Using the theorem of Hahn-Banach and the form of functionals on the space of all continuous functions vanishing at infinity we obtain

$$\langle S, f\rangle = \langle S_1, f_1\rangle = \int_{-\infty}^{+\infty} D^j f(x)\,dL(x)$$

$$= (-1)^j \int_{-\infty}^{+\infty} f(x)D^{j+1}L(x)\,dx,$$

where $L(x) = 0(e^{r_j|x|})$.

Hence $k = j+1$, $\varepsilon = r - r_{j+1}$, $F(x) = e^{-r_{j+1}|x|}\int_0^x L(x)\,dx$.

THEOREM 3. *A sequence of distributions $S_j \in H'_r$, $j = 1, 2, \ldots$ converges in the topology of H'_r to a distribution $S \in H'_r$ if and only if there exists a sequence of continuous and bounded functions $F_j(x)$ uniformly convergent to $F(x)$ and such that*

$$S_j = D^k\big(e^{(r-\varepsilon)|x|}F_j(x)\big),\quad S = D^k\big(e^{(r-\varepsilon)|x|}F(x)\big),$$

for some fixed k and some fixed $\varepsilon > 0$.

After having noted that H_r and its dual H'_r (cf. [2], p. 372) are Montel spaces, the sufficiency becomes obvious. The necessity may be proved in the same way as in theorem 2.

THEOREM 4. *The series* $\sum_{n=-\infty}^{+\infty} c_n \delta_n$ *converges in* H'_r *if and only if its coefficients satisfy, for any* n, *the inequalities*

$$|c_n| \leqslant M e^{s|n|},$$

where M *and* $s > 0$ *are constants independent of* n *and* $s < r$.

Proof. We must show that for any $f \in H_r$ the series $\sum_{n=-\infty}^{+\infty} c_n f(n)$ is convergent. To do it choose $\varepsilon > 0$ such that $s+\varepsilon < r$. Then

$$\sum_{n=-\infty}^{+\infty} |c_n f(n)| \leqslant C \sum_{n=-\infty}^{+\infty} e^{s|n|} e^{-(s+\varepsilon)|n|} < \infty.$$

If $\sum_{n=-\infty}^{+\infty} c_n \delta_n$ were convergent and if there existed increasing sequences $\{s_k\}$, $\{M_k\}$, $s_k \to r$, $M_k \to \infty$ and a subsequence $\{n_k\}$ such that for $k = 0, 1, 2, \ldots$ there is

$$|c_{n_k}| > M_k e^{s_k |n_k|},$$

then introducing the sequence of functions $\{f_k\}$, all bounded in H_r, by the formula

$$f_k(x) = e^{-s_k|n_k|} f(x - n_k),$$

where $f \in \mathscr{D}$ and $f(0) = 1$, we would have

$$\sup_{n,k} |c_n f_k(n)| \geqslant \sup_k |c_{n_k} f_k(n_k)| \geqslant \sup_k |M_k| = \infty,$$

what gives a contradiction.

3. The Fourier transform $\hat{T}$ of $T \in H'_r$ will be defined, as usually, by

$$\langle \hat{T}, \hat{f} \rangle = \langle T, \check{f} \rangle,$$

where $\check{f}(x) = f(-x)$, $f \in H_r$.

Let K'_r be the dual space of K_r. In view of theorem 1 the following is true:

COROLLARY 1. K'_r *is the space of Fourier transforms of distributions of* H'_r. *The Fourier transform provides a topological isomorphism of spaces* K'_r *and* H'_r.

Remind that the derivative $dT/dz \in K'_r$ of a functional $T \in K'_r$ is defined by the formula

$$\left\langle \frac{d}{dz} T, \varphi \right\rangle = -\left\langle T, \frac{d}{dz} \varphi \right\rangle, \qquad \varphi \in K_r.$$

THEOREM 5. *If T belongs to K_r', then there exists a functional S belonging to K_r' and such that*

$$\frac{d}{dz}S = T.$$

Proof. Note that each $\psi \in K_r$ may be represented in the form $\psi(z) = \lambda_\psi \varphi_0(z) + \varphi(z)$ with $\varphi_0, \varphi, \Phi \in K_r$, $\Phi(z) = \int_{-\infty}^{z} \varphi(t)\,dt$, and φ_0 being the same for all ψ. To show it, it suffices to put $\lambda_\psi = \int_{-\infty}^{+\infty} \psi(t)\,dt$ and to take for φ_0 any function of class K_r with the integral

$$\int_{-\infty}^{+\infty} \varphi_0(t)\,dt = 1.$$

It is easy to verify that the functional S defined by

$$\langle S, \psi \rangle = \lambda_\psi C - \langle T, \Phi \rangle$$

has the desired property.

In view of the fact that the operator $(iD)^k$ acts on the Fourier transform as the operator of multiplying by ξ^k we can meaningfully define the differential operator of infinite order acting on K_r by

$$\operatorname{ch}(aiD) = \sum_{j=0}^{\infty} (-1)^j \frac{a^{2j}}{(2j)!} D^{2j}$$

due to the convergence of the corresponding series obtained by replacing iD by ξ.

Let now $T \in K_r'$. Then $\hat{T} = S$ with $S \in H_r'$, and S, in view of theorem 2, is of the form

$$S = D^k\big[\operatorname{ch}\big((r-\varepsilon)\xi\big) F(\xi)\big] = \operatorname{ch}\big((r-\varepsilon)\xi\big) G(\xi),$$

where F is a continuous and bounded function. Therefore G is the sum of derivatives of F (i.e., G is a bounded distribution), some of which are multiplied by the bounded function $\operatorname{ch}\big((r-\varepsilon)\xi\big)$. So we can write

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G},$$

and this leads to the following

THEOREM 6. *T is an element of K_r' if and only if there exists a positive number ε and a bounded distribution G such that*

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G}.$$

The distribution G used in the formula for S may be modified in a way such that $\hat{G} \in C^\infty$. In view of corollary 1 and of theorem 3 we obtain

THEOREM 7. *The sequence* $T_j \in K_r'$, $j = 1, 2, \ldots$, *converges in the topology of* K_r' *to* $T \in K_r'$ *if and only if there exist an integer* l, *a positive number* ε, *and continuous functions* $\hat{G}_j(z)$, $\hat{G}(z)$ *such that*

$$T_j = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G}_j, \qquad T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G},$$

and $\hat{G}_j(z)/(1+|z|^2)^l$ *converges to* $\hat{G}(z)/(1+|z|^2)^l$ *uniformly in any strip* V_{r_k}.
Moreover, by corollary 1 and theorem 4 we obtain

THEOREM 8. *The series* $\sum_{n=-\infty}^{+\infty} c_n e^{inz}$ *converges in* K_r' *if and only if its coefficients satisfy, for any* n, *the inequalities*

$$|c_n| \leqslant Me^{s|n|},$$

where M *and* $s < r$ *are constants independent of* n.

4. A functional T will be called periodic with the period 2π if $\tau_{2\pi} T = T$.

THEOREM 9. *If* T *of* K_r' *is periodic, then it has a representation*

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G},$$

where $\hat{G}$ *is a periodic function.*

Proof. Let $T \in K_r'$. By theorem 6 there is

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G},$$

and so it remains only to prove that $\hat{G}$ is periodic. In view of our assumption T is periodic and therefore we have

$$\operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G}(z+2\pi) = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G}(z)$$

what in terms of Fourier transforms can be put in the form

$$\operatorname{ch}\big((r-\varepsilon)\xi\big)G(\xi) = \operatorname{ch}\big((r-\varepsilon)\xi\big)e^{2\pi i\xi}G(\xi).$$

Since $\operatorname{ch}\big((r-\varepsilon)\xi\big)$ vanishes nowhere, we can conclude that

$$G(\xi) = e^{2\pi i\xi}G(\xi),$$

and this means that

$$\hat{G}(z) = \tau_{2\pi}\hat{G}(z)$$

THEOREM 10. *If* $T \in K_r'$ *is a periodic functional, then* T *is of the form*

$$T = \sum_{n=-\infty}^{+\infty} c_n e^{inz},$$

the convergence being understood in K_r'.

Proof. By theorem 9,

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big)\hat{G},$$

where $\hat{G}$ is a infinitely differentiable periodic function.

Therefore

$$\hat{G}(z) = \sum_{n=-\infty}^{+\infty} a_n e^{inz},$$

the convergence of the last series being uniform in any V_{r_k}.

Now

$$T = \operatorname{ch}\big((r-\varepsilon)iD\big) \sum_{n=-\infty}^{+\infty} a_n e^{inz} = \sum_{n=-\infty}^{+\infty} c_n e^{inz},$$

where

$$|c_n| \leqslant M e^{(r-\varepsilon)|n|}$$

which, in view of thoerem 8, guarantees the convergence.

REFERENCES

[1] M. Hasumi, *Note on the n-dimensional tempered ultra-distributions*, Tohoku Mathematical Journal, Second Series 13 (1961), p. 94-104.

[2] G. Köthe, *Topologische lineare Räume I*, Berlin-Göttingen-Heidelberg 1960.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 15. 11. 1969