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Subharmonic analogues of MacLane's classes

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1. Introduction. It is natural to ask whether the classes $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{L}$ introduced by MacLane in [3] have subharmonic (s.h.) analogues, and if so what relations exist between them. We shall see that these analogues do exist and are identical as are $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{L}$. The analogues $\mathscr{A}_s$ and $\mathscr{B}_s$ of $\mathscr{A}$ and $\mathscr{B}$ are defined in the natural way, but we must take care when defining $\mathscr{C}_s$, the analogue of $\mathscr{L}$, in order to cater for the possibility of a s.h. function being locally constant, in which case it would have both a Koebe sequence of level curves and asymptotic values. The criterion we adopt is therefore somewhat different, as is seen in Definition 6.

We start off by defining what we mean by a continuum tending to the boundary of a domain.

Definition 1. If D is a domain we say that Γ is a *continuum* in D tending to the boundary of D if

$$\Gamma = \bigcup_{n=1}^{\infty} \gamma_n,$$

where

(i) γ_n is a continuum lying in D for each n,

(ii) $\gamma_{n+1} \cap \gamma_n \neq \emptyset$ for any n,

(iii) given any compact subset E of D, we can find an integer n_0 such that $E \cap \gamma_n = \emptyset$ for $n > n_0$.

If in addition $\gamma_n \to \zeta$ as $n \to \infty$ in the sense that $\operatorname{diam}\gamma_n \to 0$ as $n \to \infty$ and $d(\gamma_n, \zeta) \to 0$ as $n \to \infty$, then we say that Γ *tends to the point* ζ. Clearly by (iii) ζ must lie on the boundary of D.

Definition 2. Let D be a domain and let $\{\gamma_n\}$ be a sequence of continua in D satisfying (i) and (iii) of Definition 1 but such that $\gamma_m \cap \gamma_n = \emptyset$ for all $m, n \geq 1$ with $m \neq n$. Then $\{\gamma_n\}$ is said to be a sequence of Koebe continua, and we shall use the term *Koebe sequence* in this paper to denote such a sequence. (Thus we are dropping the condition normally imposed on Koebe sequences that the γ_n's be analytic arcs, and asking merely that they be continua.) If γ is an arc of $|z| = 1$ and

$$\max_{z \epsilon \gamma_n} \min_{z' \epsilon \gamma} |z-z'| + \max_{z' \epsilon \gamma} \min_{z \epsilon \gamma_n} |z-z'| \to 0 \quad \text{as } n \to \infty,$$

we say that $\gamma_n \to \gamma$ as $n \to \infty$.

DEFINITION 3: $\mathscr{A}_s$. Suppose that $u(z)$ is s.h. in the unit disc $|z| < 1$ and that $u(z) \to a$ as $|z| \to 1$ along some continuum Γ tending to a point ζ of $|z| = 1$ (i.e. $u(z) \to a$ as $z \epsilon \Gamma \to \zeta$).

Then we say $\zeta \epsilon A'_a$; if the set

$$A' = \bigcup_{-\infty \leqslant a \leqslant +\infty} A'_a$$

is dense on $|z| = 1$, then we say that $u \epsilon \mathscr{A}_s$.

DEFINITION 4: $\mathscr{B}_s$. Suppose that $u(z)$ is s.h. in $|z| < 1$ and that $u(z)$ is bounded above by some constant $M(\zeta)$ on a continuum Γ tending to a point ζ ($|\zeta| = 1$). Then we say that $\zeta \epsilon B'$.

If $B' \cup A'_\infty$ is dense on $|z| = 1$, we say that $u \epsilon \mathscr{B}_s$.

DEFINITION 5. For our purposes, a *subarc* of an arc γ will be a closed arc γ' lying in γ and not containing either of the end points of γ.

DEFINITION 6: $\mathscr{C}_s$. Suppose that $u(z)$ is s.h. in $|z| < 1$ and that $\{\gamma_n\}$ is a Koebe sequence tending to an arc γ of $|z| = 1$, where γ does not reduce to a point. Suppose that $u(z)$ is bounded above by M on the sequence $\{\gamma_n\}$. If these conditions always imply that for any interior point ζ of γ, $u(z)$ is bounded above in some neighbourhood $N_\delta(\zeta) = \{z\colon |\zeta - z| < \delta \cap |z| < 1\}$ of ζ in $|z| < 1$, we say that $u \epsilon \mathscr{C}_s$.

Our result is

THEOREM 1. $\mathscr{A}_s = \mathscr{B}_s = \mathscr{C}_s$.

2. Topological preliminaries to the proof. Suppose that $u(z)$ is s.h. on a compact set E (i.e. in some open set $O \supset E$) and let F denote the set of points $\{z\colon u(z) \geqslant K,\ z \epsilon E\}$, where K is some real number. Then since $u(z)$ is upper semi-continuous (u.s.c.), F is a closed set and therefore decomposes into closed maximal connected subsets F_a called the *components of* F, each point of F belonging to exactly one component F_a, where a runs over some index set I.

Now suppose that instead of on a compact set E, $u(z)$ is s.h. in $|z| < 1$. Let z_0 be a point in $|z| < 1$ with $u(z_0) \geqslant K$.

DEFINITION 7. The *compartment* of the set $\{z\colon u(z) \geqslant K,\ |z| < 1\}$ containing z_0 is defined to be

$$C(z_0, K, 1) = \bigcup_{r<1} C(z_0, K, r),$$

where $C(z_0, K, r)$ is the component of $\{z\colon u(z) \geqslant K,\ |z| \leqslant r\}$ containing z_0.

Since $C(z_0, K, r_1) \subset C(z_0, K, r_2)$ for $r_1 < r_2 < 1$, it follows that the compartment $C(z_0, K, 1)$ is connected.

In a similar way we can define compartments of the set $\{z: u(z) < K, |z| < 1\}$, but since this set is open there is no difficulty in defining the components, and the definitions of compartment and component give rise to the same sets.

3. Proof of Theorem 1. Following the preliminaries, we now set out to prove the theorem. The proof consists of proving the following inclusion relations:

(i) $\mathscr{A}_s \subset \mathscr{B}_s$,

(ii) $\mathscr{B}_s \subset \mathscr{C}_s$,

(iii) $\mathscr{C}_s \subset \mathscr{A}_s$,

so that, as opposed to MacLane's proof it is not necessary to show that $\mathscr{C}_s \subset \mathscr{B}_s$ and $\mathscr{B}_s \subset \mathscr{A}_s$; these two steps are combined in (iii) above.

3.1. $\mathscr{A}_s \subset \mathscr{B}_s$.

This is clear, for on any continuum Γ tending to a point ζ and on which $u(z) \to a < \infty$, $u(z)$ is bounded above so that $\zeta \in B'$.

3.2. $\mathscr{B}_s \subset \mathscr{C}_s$.

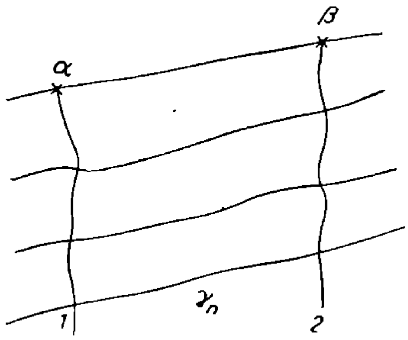

Fig. 1

Suppose that $u \in \mathscr{B}_s$ and that $\gamma_n \to \gamma$ is a Koebe sequence on which $u(z) < M' < \infty$. Let γ' be a subarc of γ.

We observe first that $\gamma' \cap A'_\infty = \emptyset$, since any continuum Γ tending to $\zeta \in \gamma'$ meets infinitely many γ_n in any neighbourhood of ζ, so that $u(z)$ cannot tend to infinity on Γ.

Thus B' is dense on γ'; choose distinct points $\alpha, \beta \in \gamma' \cap B'$ and let Γ_1, Γ_2 be continua tending to α, β respectively on which $u(z) < M''$ say; let $M = \max(M', M'')$.

Since γ' is a subarc of γ, and $\gamma_n \to \gamma$ as $n \to \infty$, it follows that Γ_1 and Γ_2 must meet γ_n for $n \geqslant n_0$ say. γ_{n_0} is a continuum, therefore compact and lying in $|z| \leqslant r_0 < 1$ say; let D_n be any domain whose boundary consists of subsets of $\Gamma_1, \Gamma_2, \gamma_{n_0}$ and γ_n for $n > n_0$. Then $u(z) < M$ on the boundary of D_n and therefore inside D_n by the maximum principle.

Let γ'' be a subarc of γ' and let z_0 be a point lying in a small neighbourhood of γ'' with $|z_0| > r_0$. Since γ_n lies finally outside any compact subset of $|z| < 1$, it follows that z_0 must lie in some domain D_n, so that $u(z_0) < M$; since z_0 is arbitrary it follows that $u(z) < M$ in a small enough neighbourhood of γ''. Since B' is dense, α and β may be chosen to lie outside γ'' and since γ' is arbitrary, γ'' is an arbitrary subarc of γ. Thus $u \in \mathscr{C}_s$.

4. The kernel of the proof consists of showing that $\mathscr{C}_s \subset \mathscr{A}_s$; this is quite a lengthy procedure and we need the following result, which is of some interest in itself.

THEOREM 2. *If $u(z)$ is s.h. in $|z| < 1$ and $u(z_0) > M$, then there exists a continuum Γ, containing z_0 and tending to $|z| = 1$, on which $u(z) \geqslant M$ and $u(z) \to M_1 \geqslant u(z_0)$ as $|z| \to 1$ along Γ.*

The proof of this result requires five subsidiary lemmas, three of which are due to M. N. M. Talpur ([4], § 1) and one of which is an immediate consequence of Lemma 3. The fifth lemma is the s.h. form of Hadamard's convexity theorem.

LEMMA 1. *Let $u(z)$ be s.h. in $|z| < R$. Then if $u(z_0) > M_0$, there is a circle C, centre z_0 and lying in $|z| < R$, such that $u(z) \geqslant M_0$ on C.*

Note. Since $u(z)$ is s.h. in $|z| < r$ for all $r < R$, we may choose C to have arbitrarily small radius.

LEMMA 2. *If $u(z)$ is s.h. in $|z| \leqslant r$, then each component of the set $\{z\colon u(z) \geqslant K\}$ contains points of modulus r.*

From this we deduce immediately

LEMMA 3. *If $u(z)$ is s.h. in $|z| < 1$, then each compartment of the set $\{z\colon u(z) \geqslant K\}$ goes to the boundary, i.e. possesses limit points of modulus 1.*

LEMMA 4. *Let $u(z)$ be s.h. in $|z| < r$ and let K be finite. Set*

$$u_r(z) = \begin{cases} u(z), & z \in C(z_0, K, r), \\ K, & \text{elsewhere in } |z| < r. \end{cases}$$

Then $u_r(z)$ is s.h. in $|z| < r$.

LEMMA 5. *Let $u(z)$ be s.h. in $|z| < 1$ and let $B(r) = \sup\limits_{|z|=r} u(z)$. Then $B(r)$ is an increasing convex function of* $\log r$, *so that if* $0 < r_0 < r < r_1 < 1$ *and $|z| = r$ it follows that*

$$u(z) \leqslant \frac{B(r_0)\log r_1/r + B(r_1)\log r/r_0}{\log r_1/r_0.}.$$

If $B(1) = \lim\limits_{|z|\to 1} [\sup u(z)]$, it follows on letting $r_1 \to 1$ that

$$u(z) \leqslant \frac{B(r_0)\log 1/r + B(1)\log r/r_0}{\log 1/r_0}.$$

We now prove Theorem 2. If $u(z)$ is a constant, then the result is trivial; we assume that $u(z)$ is non-constant in $|z| < 1$. Choose z_0, $|z_0| < 1$, and suppose that $u(z_0) > M$. Let $C(z_0, K, r)$ and $u_r(z)$ be as previously defined and set

$$M_0 = \sup_{z \in C(z_0, M, 1)} u(z).$$

We see first that $M_0 > u(z_0)$, for $u(z_0) \leqslant M_0$, since $z_0 \in C(z_0, M, 1)$ and if $u(z_0) = M_0$, then $u_r(z) \equiv M_0$, so that $C(z_0, M_0, r)$ contains the whole of $|z| < r$. Thus $u_r(z) = u(z) = u(z_0)$ for $|z| < r$; since this is true for all $r < 1$ it follows on letting $r \to 1$ that $u(z) \equiv M_0$ in $|z| < 1$, contrary to our assumption.

Next, we show that given $r_0 < 1$ there is an $M_1 = M_1(r_0)$ such that $u(z) < M_1 < M_0$, and so $u_\varrho(z) < M_1$, for $z \in C(z_0, M, 1)$ and $|z| < r_0$, and all ϱ such that $r_0 < \varrho < 1$.

To see this we use Lemma 5. By a suitable conformal map we may assume that $z_0 = 0$. Choose $\varepsilon > 0$ so that $M'' = u(0) + \varepsilon < M_0$. Then for $|z| \leqslant \delta_0$ say we have $u(z) \leqslant M''$ for otherwise the u.s.c. of $u(z)$ at $z = 0$ would be contradicted. Thus $u_\varrho(z) \leqslant M''$ for $|z| \leqslant \delta_0$ and $0 < \varrho < 1$. Thus for $|z| = r_0$, and $\delta_0 < r_0 < r_1 \leqslant \varrho < 1$ we have

$$u_\varrho(z) \leqslant \frac{M'' \log r_1/r_0 + M_0 \log r_0/\delta_0}{\log r_1/\delta_0}$$

by Lemma 5. By letting $\varrho \to 1$, we see that the same inequality holds on $C(z_0, M, 1)$ with $u(z)$ instead of $u_\varrho(z)$. Now let $r_1 \to 1$; we obtain

$$u(z) \leqslant \frac{M'' \log 1/r_0 + M_0 \log r_0/\delta_0}{\log 1/\delta_0} \quad (= M_1'(r_0) \text{ say}) \tag{1}$$

for $z \in C(z_0, M, 1)$. Since $r_0 < 1$, it follows that $M_1'(r_0) < M_0$.

Now choose z_1 and ϱ_1 with $|z_1| = r_1 > r_0$ such that $u_{\varrho_1}(z_1) > M_1$. Then $z_1 \in C(z_0, M, \varrho_1)$, which also contains z_0 and is a continuum γ_1 say. It follows from (1) and our choice of z_1 that $C(z_1, M_1, 1)$ lies in $|z| \geqslant r_0$.

Next, we repeat the above argument to obtain that given r_2 with $r_2 > r_1$, there is an $M_2 = M_2(r_2)$ such that $u(z) < M_2 < M_0'$ for $z \in C(z_1, M_1, \varrho_1)$ and $|z| < r_2$, where $M_0' = \sup_{z \in C(z_1, M_1, 1)} u(z)$. To do this we use, instead of the function $u_\varrho(z)$ which is defined relative to $C(z_0, M, 1)$, the function

$$v_\varrho(z) = \begin{cases} u(z), & z \in C(z_1, M_1, \varrho), \\ M_1, & \text{elsewhere in } |z| < 1, \end{cases}$$

which is easily seen to be s.h. in $|z| < \varrho$, $r_2 < \varrho < 1$.

It is clear that $v_\varrho(z) = u_\varrho(z)$ for $z \in C(z_1, M_1, \varrho)$, and the existence of $M_2 < M_0'$ follows since otherwise $v_\varrho(z)$, therefore $u_\varrho(z)$ and so $u(z)$, would be constant in $|z| < \varrho$ for all $\varrho < 1$ and thus in $|z| < 1$.

Now choose z_2 and ϱ_2 such that $u_{\varrho_2}(z_2) > M_2$, and denote the continuum $C(z_1, M_1, \varrho_2)$ by γ_2. Then $\gamma_1 \cap \gamma_2$ contains the point z_1, and on γ_2, $u(z) \geqslant M_1$. γ_2 lies outside $|z| < r_1$ by construction. To carry on the construction, we choose z_n and ϱ_n so that $u_{\varrho_n}(z_n) > M_n$ and set $\gamma_n = C(z_{n-1}, M_{n-1}, \varrho_n)$. Then γ_n by construction will lie in $|z| \geqslant r_{n-1}$; if we choose $r_n \to 1$, then $= \bigcup_{n=1}^{\infty} \gamma_n$ is a continuum tending to the boundary.

We obtain in this way an increasing sequence $\{M_n\}$ and a decreasing sequence $\{M_0^k\}$, with $M_0^k = \sup u(z)$, $z \in C(z_k, M_k, 1)$. If $M_0^k \equiv +\infty$, then we may choose M_n as large as we please; since $\gamma_n \to |z| = 1$ and $u(z) > M_{n-1}$ on γ_n, it follows that $u(z) \to \infty$ as Γ tends to the boundary.

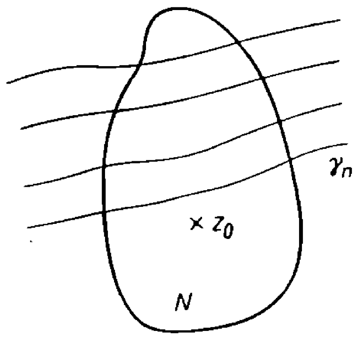

Fig. 2

If $\lim M_0^k = L < \infty$, then we must choose M_n so that $M_n \to L$ as $n \to \infty$. (It follows from the convexity argument that this is possible.) If the M_n are chosen in this way, then $u(z) \to L$ as $\Gamma \to |z| = 1$; since in both cases $\{M_n\}$ is increasing it is clear that the other conclusions of the theorem also hold.

This completes the proof of Theorem 2.

We next state another standard result which is Brelot's form of the Milloux–Schmidt inequality ([4], § 1).

LEMMA 6. *Let $u(z)$ be s.h. in $|z| \leqslant R$ and suppose that $u(z) \leqslant 1$ there. Let*

$$\inf_{|z|=r} u(z) \leqslant 0$$

for all r, $0 \leqslant r \leqslant R$. Then

$$u(z) \leqslant \frac{4}{\pi} \tan^{-1} \sqrt{\frac{r}{R}} \qquad (r = |z|).$$

This lemma gives almost immediately

LEMMA 7. *Let $u(z)$ be s.h. in $|z| < 1$ and let $|z_0| < 1$. Let $\{\gamma_n\}$ be a sequence of continua in $|z| < 1$ on which $u(z) < M$, and such that* $\operatorname{diam}(\gamma_n)$

$> \delta > 0$ *for all n. Suppose that any neighbourhood N of z_0 meets infinitely many of the γ_n. Then $u(z_0) \leqslant M$.*

Proof. Let $z_n \epsilon \gamma_n$ be a sequence of points tending to z_0, suppose that for $n > n_0$, $|z_n - z_0| < \frac{1}{3}(1-|z_0|)$. Let $R = \min(\delta, \frac{1}{3}(1-|z_0|))$. Let

$$B(r) = \sup_{|z| \leqslant 1-\frac{1}{3}(1-|z_0|)} u(z) < +\infty.$$

We apply Lemma 6 to the function $\frac{1}{|B(r)|}\{u(z)-M\}$ on circles of radius R centred on the points z_n, $n > n_0$, and let r_n denote $|z_n - z_0|$. Since for n large enough z_0 will lie inside these circles, we obtain that

$$u(z_0) \leqslant M + \frac{4}{\pi}\tan^{-1}\sqrt{\frac{r_n}{R}}\{B(r)-M\}.$$

Letting $z_n \to z_0$, i.e. $r_n \to 0$, we see that

$$u(z_0) \leqslant M.$$

5. Our proof of Theorem 1 is now completed by introducing the following classification of boundary points, and investigating their properties.

DEFINITION 7. Let ζ_0 be a point of $|z| = 1$, and let M be given. If for every small $\delta > 0$, the neighbourhood $N_\delta(\zeta_0) = \{|z| < 1|\} \cap \{|\zeta_0 - z| < \delta\}$ contains points belonging to a component of $u(z) < M$ which has at least one limit point of modulus 1, then we say that ζ_0 is *M-barred*. Here and subsequently 'component' in this context means a component relative to the neighbourhood.

If ζ_0 is M-barred for some M, we say that ζ_0 is *barred*.

Otherwise, we say that ζ_0 is *free*.

We now establish results on free and barred points. We first need another lemma due to Talpur ([4], § 1) namely

LEMMA 8. *Suppose that $u(z)$ is s.h. in a neighbourhood N of a continuum γ and that $u(z) \geqslant K$ for $z \epsilon \gamma$. Let z_1, z_2 be two points of γ. Then given $\varepsilon > 0$, we can find a polygonal path joining z_1 to z_2 in N such that $u(z) > K - \varepsilon$ on this path.*

Talpur proves this result with $K-1$ instead of $K-\varepsilon$. However, our result follows immediately by applying this result to the function

$$v(z) = \varepsilon u(z) + (1-\varepsilon)K.$$

The proof includes the following result, which is a deduction from a theorem of Hayman ([1], Theorem 4, p. 193) and which we shall use explicitly:

LEMMA 9. *Suppose that $u(z)$ is s.h. in $|z| < 1$ and that z_1, z_2 are two* *points in* $|z| \leqslant r_0 < 1$ *at which* $u(z) \geqslant K$. *Then we can find a* $\delta = \delta(\varepsilon, r_0)$ *such that if* $|z_1 - z_2| < \delta(\varepsilon, r_0)$, *then* z_1 *can be joined to* z_2 *by a zigzag path* $[z_1, \zeta] \cup [\zeta, z_2]$ *on which* $u(z) \geqslant K - \varepsilon$, *and such that*

$$(*) \qquad |\zeta| < r_0 \quad \text{and} \quad |z_1 - \zeta| + |z_2 - \zeta| \leqslant 2|z_1 - z_2|.$$

We show with the aid of these results

LEMMA 10. *Let* ζ *be a free point of* $|z| = 1$. *Then one of the three following possibilities must hold*:

(i) $\zeta \in A'_\infty$.

(ii) $u \notin \mathscr{C}_s$.

(iii) *There exist points in* A' *arbitrarily near* ζ.

Proof. We note first that $u(z)$ is unbounded on the radius at ζ, for if $u(z) < M$ on this radius R, then R would belong to a component of $u < M$; thus since R ends at ζ, ζ would be M-barred contrary to hypothesis.

Now if $M(\xi) = \overline{\lim}_{z \to \xi \text{ radially}} u(z)$, then $M(\xi) \to \infty$ as $\xi \to \zeta$, for if $\underline{\lim}_{\xi \to \zeta} M(\xi) < M_1 < \infty$, then every neighbourhood $N_\delta(\zeta)$ contains points ξ, where $M(\xi) < M_1 + 1$, and so an end of a radius on which $u(z) < M_1 + 2$. Thus ζ would be barred, which is again contrary to hypothesis.

Next, suppose that $u \in \mathscr{C}_s$; then given N_δ^M, we can find a neighbourhood $N_{\delta'}^M$, of ζ such that all components of $u < M$ for fixed M which contain points in $N_{\delta'}^{(M)}$ have compact closures in $N_\delta^{(M)}$, or else (iii) holds.

To see this, suppose that we cannot find such neighbourhoods $N_\delta^{(M)}$, $N_{\delta'}^{(M)}$. Then we recall that since ζ is a free point these components have no limit points on $|z| = 1$ for small enough δ; suppose that we can find a sequence of points $z_n \to \zeta$ such that each z_n lies in a different component K_n of $u < M$, in some fixed neighbourhood N for all n.

If in N these components lie finally outside any compact subset of $|z| < 1$, then we can select from them a sequence of Koebe continua on which $u < M$, tending to an arc of $|z| = 1$ which contains ζ (possibly as an endpoint). Choose ξ_1 and ξ_2 on this arc; then since $u \in \mathscr{C}_s$, u is bounded in a neighbourhood of the arc $[\xi_1, \xi_2]$ and so has finite asymptotic values. Since ξ_1 can be chosen arbitrarily near to ζ, it follows that (iii) holds.

If the components do not lie finally outside any compact subset of $|z| < 1$, then a subsequence $\{K_n\}$ must meet the circle $|z| = r_1$ for some $r_1 < 1$, at points $\{z'_n\}$ say. These points $\{z'_n\}$ have a limit point z_0 with $|z_0| = r_1$. We may apply Lemma 7 to deduce that $u(z_0) \leqslant M$, and from the u.s.c. of the function it follows that if δ' is small enough, $u(z) < M + 1$ in $|z - z_0| < \delta'$; thus the component of $u < M + 1$ containing z_0 also contains infinitely many K_n, and so infinitely many z_n. Since $z_n \to \zeta$, it follows that ζ is a limit point of this component of $u < M + 1$.

Unfortunately, this is not enough to show that ζ is $(M+1)$-barred, for the components are defined relative to N and as N decreases the component of $u < M+1$ may well split up into infinitely many components in a smaller neighbourhood, as indicated below (Fig. 3).

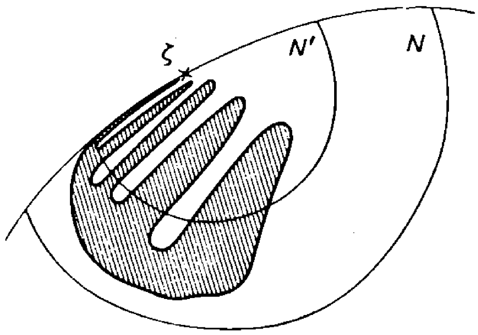

Fig. 3

To overcome this difficulty, the argument must now be repeated in some smaller neighbourhood N'; we obtain in a similar way that there exists a point of $\{|z| = 1\} \cap \delta N'$ which is a limit point of a component of $u < [(M+1)+(\frac{1}{2})^2]$ say; repeating the argument as many times as we please, we obtain a sequence of limit points $\{\xi_n\}$ of some component of $u < M + \sum_0^\infty 2^{-n} = M+2$, with $\zeta_n \to \zeta$ $(n \to \infty)$ and $|\zeta_n| = 1$ for all n. It follows that ζ is $(M+2)$-barred, contrary to hypothesis, so that if neither (i) nor (iii) hold, then given N_δ^M we can find $N_{\delta'}^M$, with the required properties.

Now choose a sequence of neighbourhoods $N_{\delta_k}^{(M+k)}$ $(k \geqslant 0)$ which are nested and whose diameter tends to zero. Then by the above we can find $\delta'_k < \delta_k$ such that any component meeting $N_{\delta'_k}^{(M+k)}$ has compact closure in $N_{\delta_k}^{(M+k)}$; we choose δ_0 so that no component of $u < M+1$ in $|z-\zeta| < \delta_0$ has a limit point on $|z| = 1$.

Choose $z_1 \in N_{\delta'_1}^{(M+1)}$ and z_2 in $N_{\delta'_2}^{(M+2)}$ such that

(a) z_1 and z_2 lie on the radius R and $|z_1| < |z_2| = r_0 < 1$.

(b) $u(z_1) \geqslant M$, $u(z_2) \geqslant M+1$.

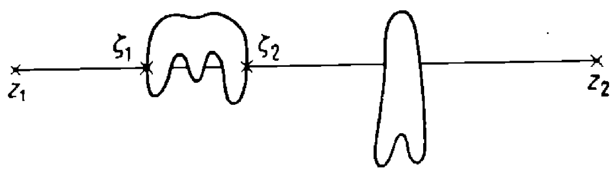

Fig. 4

Then by Lemma 9 with $\varepsilon = \frac{1}{2}$, we can find a value $\varrho > 0$ such that if $|\zeta_1|$, $|\zeta_2| \leqslant r_0 < 1$ and $u(\zeta_1)$, $u(\zeta_2) \geqslant K$, then if $|\zeta_1 - \zeta_2| < \varrho$, ζ_1 and ζ_2 can be joined by a zig-zag on which $u(z) \geqslant K - \frac{1}{2}$.

We now split the radial segment $[z_1, z_2]$ into two classes of points, those where $u(z) \geqslant M$ and those where $u(z) < M$. The latter set is the intersection of R with the set $u < M$ and so consists of open intervals of various lengths; at the end points of these intervals $u(z) \geqslant M$. The component of $u < M$ separating them is a simply connected domain whose closure lies in $N_{\delta_1}^M$, so that its boundary is a continuum, which we can surround by an open neighbourhood lying in $N_{\delta_1}^M$, and so we may apply Lemma 8 with $\varepsilon = \frac{1}{2}$ to deduce the existence of a polygonal path π lying in $N_{\delta_1}^M$ and joining ζ_1 to ζ_2, on which $u(z) \geqslant M - \frac{1}{2}$.

We now construct our asymptotic path; we consider first the 'large' intervals of R where $u(z) < M$, namely those whose length is at least $\frac{1}{2}\varrho$. These are only finite in number, and we connect their endpoints by a polygonal path as above on which $u(z) \geqslant M - \frac{1}{2}$.

On a complementary interval $[a, b]$ of R, we can find a sequence of points $a = \xi_0, \xi_1, \ldots, \xi_n = b$ such that $\frac{1}{2}\varrho \leqslant |\xi_{i+1} - \xi_i| < \varrho$ and $u(\xi_i) \geqslant M$. We then join ξ_i to ξ_{i+1} by a zig-zag which $u(z) \geqslant M - \frac{1}{2}$. Since n is finite, it follows that a, b can be joined by a polygonal path.

Combining these two sets of polygonal paths, we obtain a polygonal path π_1 joining z_1 to z_2 on which $u(z) \geqslant M - \frac{1}{2}$ and which lies in a $(\delta_1 + \varrho_1)$ neighbourhood of ζ.

We now choose z_3 to lie in $N_{\delta'_3}^{(M+3)} \cap R$ with $u(z_3) \geqslant M + 2$ and $|z_3| = r_1$, $r_0 < r_1 < 1$. Proceeding as above, we construct a polygonal path π_2 joining z_2 to z_3, lying in a $(\delta_2 + \varrho_2)$ neighbourhood of ζ and on which $u(z) \geqslant M + \frac{1}{2}$.

Continuing in this way, choosing $\delta_n \to 0$ and $\varrho_n \to 0$, we obtain a sectionally polygonal path $= \bigcup_{r=1}^{\infty} \pi_n$ ending at ζ, such that $u(z) \to \infty$ as $z \to \zeta$ on Γ. Thus $\zeta \in A'_\infty$.

LEMMA 11. *If γ is an arc of $|z| = 1$ containing no free points, then there is a value M and a subarc γ' of γ such that the set of M-barred points is dense on γ'.*

Proof. The set of all barred points is the union of the sets E_n of n-barred points for integral n, and so is a countable union. If no such arc γ' exists, then γ is the union of a countable number of nowhere dense sets and so is of the first category in itself, which is impossible. Thus one of the sets E_n must be dense on some arc γ', and the lemma is proved with $M = n$.

LEMMA 12. *If the set of M-barred points is dense on an arc γ, and $u(z)$ is unbounded near every point of γ, then there exists a continuum Γ in $|z| < 1$ tending to a point of γ and on which $u(z) \to M' > M$.*

Proof. Let ζ_1, ζ_2 be distinct M-barred points on γ, and let D_1 and D_2 be components of $u < M$ which have limit points of modulus 1 in disjoint neighbourhoods of ζ_1, ζ_2 respectively. Let K_1, K_2 be components containing limit points of modulus 1 of the restrictions of D_1, D_2 respectively to the neighbourhoods N_1, N_2 of ζ_1 and ζ_2.

Let $r_0 < 1$ be chosen so that $|z| = r_0$ meets both K_1 and K_2 and let $B(r_0) = \sup\limits_{|z|=r_0} u(z)$.

Let D be the domain bounded by $\partial K_1, \partial K_2$, $|z| = r_0$ and γ, and choose $z_0 \in D$ such that

$$\mu(z_0) > M_1 = \sup\{M, B(r_0)\}.$$

Then using Theorem 2 we construct a continuum Γ tending to $|z| = 1$ such that $u(z) \to M' > M_1$ on Γ, and $u(z) > M_1$ on Γ. Then Γ must end on γ, for otherwise it would have to cross the boundary of D and we would have $u(z) \leqslant M$ at such a point.

Further, Γ must tend to a point, for if ζ_1', ζ_2' are distinct limit points of Γ with modulus 1, we choose ζ between them so that ζ is M-barred. Γ must then meet a component of $u < M$ which has a limit point near ζ, thus contradicting the fact that $u > M_1$ on Γ.

Proof that $\mathscr{C}_s \subset \mathscr{A}_s$. Suppose that $u \notin \mathscr{A}_s$. Then there is an arc γ free of A', so that in particular $u(z)$ is unbounded near every point of γ by Littlewood's theorem ([4], p. 169). If γ' is a subarc of γ, and some $\zeta \in \gamma'$ is free, then $u \notin \mathscr{C}_s$ by Lemma 10 since γ' is free of A'.

If not, then by Lemma 11 we can find M and $\gamma'' \subset \gamma'$ such that the set of M-barred points is dense on γ''. We then apply Lemma 12 to deduce the existence of asymptotic values at points of γ'', which contradicts the assumption that γ' is free of A'.

Thus $\mathscr{C}_s \subset \mathscr{A}_s$, and the proof of Theorem 1 is complete.

We can show slightly more, namely that instead of asymptotic values on continua we can have asymptotic values on sectionally polygonal paths. This is

THEOREM 3. *Let $u(z)$ be s.h. in $|z| < 1$. Then if $u(z) \to a$ as $z \to \zeta$ ($|\zeta| = 1$) on a continuum Γ, there exists a sectionally polygonal Jordan arc π in $|z| < 1$ ending at ζ on which $u(z) \to a$.*

Proof. Let $\Gamma = \bigcup\limits_{n=1}^{\infty} \gamma_n$, where $\gamma_n \cap \gamma_{n+1} \neq \emptyset$, and choose $\varepsilon_n \to 0$. Suppose that $a < \infty$. Let

$$\mu_{n*}^{*} = \begin{cases} \sup u(z), \\ \inf z \in \gamma_n. \end{cases}$$

Then $\mu_n^* \to a$, $\mu_{n*} \to a$ and by u.s.c., γ_n lies in a domain D_n, where $u(z) < \mu_n^* + \varepsilon_n$. Let $z_n \in \gamma_{n-1} \cap \gamma_n$ $(n \geqslant 2)$. Then by Lemma 8, z_n, z_{n+1} can be joined by a polygonal path π_n in D_n on which

$$u(z) \geqslant \mu_{n*} - \varepsilon_n,$$

so that $\mu_{n*} - \varepsilon_n \leqslant u(z) \leqslant \mu_n^* + \varepsilon_n$ on π_n.

Then $\pi' = \bigcup_{n=1}^{\infty} \pi_n$ is the required sectionally polygonal path; π will end at a point provided the neighbourhoods D_n are chosen small enough, since Γ itself ends at a point.

By removing a countable number of loops, the path may be made into a sectionally polygonal Jordan arc.

Further, in view of [2], we have

THEOREM 4. *Let $u(z)$ be subharmonic in $|z| < 1$ and let $B(r) = \sup_{|z|=r} u(z)$. Suppose that*

$$\int_0^1 \log B(r)\,dr < \infty. \tag{2}$$

Then $u \in \mathscr{A}_s$.

Proof. The proof uses Lemma 1 of [2] and proceeds as ([2], Theorem 1) with the auxiliary function $v(\zeta) = \log|\Phi(\zeta)|$ instead of $\Phi(\zeta)$. Since the maximum principle holds for subharmonic functions we obtain that $u(z)$ is locally bounded near γ^*, and so lies in $\mathscr{C}_s$. The result then follows from ([2], Theorem 1)

THEOREM 5. *Given $\varepsilon > 0$, there exists $u(z)$ subharmonic in $|z| < 1$ such that*

$$B(r) < \exp \frac{\varepsilon}{(1-r)\log \dfrac{1}{1-r}} \tag{3}$$

but $u \notin \mathscr{A}_s$.

Proof. The subharmonic function $u(z)$ constructed in the proof of [2], Theorem 2, provides such an example.

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